UDC 515.145 + 515.146

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PERIOD FUNCTIONS FOR $\mathcal{C}^0$- AND $\mathcal{C}^1$-FLOWS
ФУНКЦІЇ ПЕРІОДІВ ДЛЯ $\mathcal{C}^0$- ТА $\mathcal{C}^1$-ПОТОКІВ

Let $\mathbf{F}\colon M \times \mathbb{R} \to M$ be a continuous flow on a manifold M, $V \subset M$ be an open subset, and let $\xi\colon V \to \mathbb{R}$ be a continuous function. We say that ξ is a *period* function if $\mathbf{F}(x, \xi(x)) = x$ for all $x \in V$. Recently, for any open connected subset $V \subset M$, the author has described the structure of the set $P(V)$ of all period functions on V. Assume that $\mathbf{F}$ is topologically conjugate to some $\mathcal{C}^1$-flow. It is shown in this paper that, in this case, the period functions of $\mathbf{F}$ satisfy some additional conditions that, generally speaking, are not satisfied for general continuous flows.

Нехай $\mathbf{F}\colon M \times \mathbb{R} \to M$ — неперервний потік на многовиді M, $V \subset M$ — відкрита підмножина і $\xi\colon V \to \mathbb{R}$ — неперервна функція. Назвемо ξ функцією *періодів*, якщо $\mathbf{F}(x, \xi(x)) = x$ для всіх $x \in V$. Нещодавно для кожної відкритої зв'язної множини $V \subset M$ автором було описано структуру множини $P(V)$ всіх функцій періодів на V. Припустимо, що $\mathbf{F}$ є топологічно спряженим до деякого потоку класу $\mathcal{C}^1$. У даній роботі показано, що тоді функції періоду $\mathbf{F}$ задовольняють додаткові умови, які, взагалі кажучи, не виконуються для загальних неперервних потоків.

1. Introduction. Let $\mathbf{F}\colon M \times \mathbb{R} \to M$ be a continuous flow on a topological finite-dimensional connected manifold M. Let also Σ be the set of fixed points of $\mathbf{F}$. For $x \in M$ we will denote by o_x the orbit of x. If x is periodic, then $\mathrm{Per}(x)$ will denote the period of x.

Definition 1.1. *Let $V \subset M$ be a subset and $\xi\colon V \to \mathbb{R}$ be a continuous function. We will say that ξ is a period function or simply a P-function* (*with respect to* $\mathbf{F}$) *if* $\mathbf{F}(x, \xi(x)) = x$ *for all* $x \in V$.

The set of all P-functions on V with respect to a flow $\mathbf{F}$ will be denoted by $P(\mathbf{F}, V)$, or simply by $P(V)$. The following easy lemma explains the term *P-function*.

Lemma 1.1 [1]. 1. *For any subset $V \subset M$ the set $P(V)$ is a group with respect to the pointwise addition.*

2. *Let $x \in V$ and $\xi \in P(V)$. Then ξ is locally constant on $o_x \cap V$. In particular, if x is nonperiodic, then $\xi|_{o_x \cap V} = 0$. Suppose x is periodic, and let ω be a path component of $o_x \cap V$. Then $\xi = n_\omega \mathrm{Per}(o_x)$ for some $n_\omega \in \mathbb{Z}$ depending on ω.*

The following theorem, describing $P(V)$ for open connected subsets $V \subset M$, is a particular case of results obtained in [1]. It also extends [2] (Theorem 12) to the case of continuous flows.

Theorem 1.1 [1, 2]. *Let M be a finite-dimensional topological manifold possibly noncompact and with or without boundary, $\mathbf{F}\colon M \times \mathbb{R} \to M$ be a flow, and $V \subset M$ be an open, connected set. Suppose that $\mathrm{Int}(\Sigma) \cap V = \varnothing$. Then one of the following possibilities is realized*: *either $P(V) = \{0\}$ or $P(V) = \{n\theta\}_{n \in \mathbb{Z}}$ for some continuous function $\theta\colon V \to \mathbb{R}$ having the following properties*:

(1) *$\theta > 0$ on $V \setminus \Sigma$, so this set consists of periodic points only.*

(2) *There exists an open and everywhere dense subset $Q \subset V$ such that $\theta(x) = \mathrm{Per}(x)$ for all $x \in Q$.*

(3) *θ is constant on $o_x \cap V$ for each $x \in V$.*

(4) *Put $U = \mathbf{F}(V \times \mathbb{R})$. Then θ extends to a P-function on U and there is a circle action $\mathbf{G}\colon U \times S^1 \to U$ defined by $\mathbf{G}(x, t) = \mathbf{F}(x, t\theta(x))$, $x \in U$, $t \in S^1 = \mathbb{R}/\mathbb{Z}$. The orbits of this action coincides with the ones of* $\mathbf{F}$.

The aim of this paper is to show that if $\mathbf{F}$ is conjugate to some $\mathcal{C}^1$-flow, then P-functions of $\mathbf{F}$ have additional properties which may fail for P-functions of general continuous flows (Theorems 1.2 and 1.3).

Before formulating these results let us discuss the behaviour of P-functions under conjugations of flows.

1.1. Conjugation of flows. Let M be $\mathcal{C}^r$, $r \geq 1$, manifold and $\mathbf{F}\colon M \times \mathbb{R} \to M$ be a $\mathcal{C}^r$-flow. Then in general, $\mathbf{F}$ is generated by a $\mathcal{C}^{r-1}$-vector field

$$F(x) = \frac{\partial \mathbf{F}}{\partial t}(x,t)|_{t=0}.$$

Nonetheless, it is proved by D. Hart [3] that every $\mathcal{C}^r$-flow $\mathbf{F}$ is $\mathcal{C}^r$-conjugate to a $\mathcal{C}^r$-flow generated by a $\mathcal{C}^r$-vector field. Thus in order to study P-functions for $\mathcal{C}^r$-flows one can assume that these flows are generated by $\mathcal{C}^r$-vector fields.

Now let $h\colon M \to M$ be a homeomorphism and $\mathbf{G}\colon M \times \mathbb{R} \to M$ be the conjugate flow:

$$\mathbf{G}_t(x) = h \circ \mathbf{F}_t \circ h^{-1}(x) = h \circ \mathbf{F}(h^{-1}(x), t)).$$

Let also $V \subset M$ be an open set and $\theta\colon V \to \mathbb{R}$ be a continuous P-function for $\mathbf{F}$. Then $\theta \circ h\colon h^{-1}(V) \to \mathbb{R}$ *is a P-function for* $\mathbf{G}$. Indeed, if $x \in V$ and $y = h^{-1}(x)$, then

$$\mathbf{G}(y, \theta \circ h^{-1}(y)) = h \circ \mathbf{F}(h^{-1}(x), \theta \circ h^{-1}(x))) = h \circ h^{-1}(x) = x.$$

In other words $P(\mathbf{G}, h^{-1}V) = P(\mathbf{F}, V) \circ h$. In particular, the groups $P(\mathbf{G}, h^{-1}V)$ and $P(\mathbf{F}, V)$ are isomorphic. Thus the structure of the set of P-functions of the flow $\mathbf{F}$ depends only on its conjugate class.

Let E_k be the unit $(k \times k)$-matrix, C be a square $(k \times k)$-matrix, and $a, b \in \mathbb{R}$. Define the following matrices:

$$\mathbf{J}_p(C) = \begin{pmatrix} C & 0 & \dots & 0 \\ E_k & C & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & E_k & C \end{pmatrix}, \qquad R(a,b) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

$$\mathbf{J}_p(a \pm ib) = \mathbf{J}_p(R(a,b)).$$

For square matrices B, C it is also convenient to put $B \oplus C = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$.

Theorem 1.2. *Let $\mathbf{F}$ be a continuous flow on M and $V \subset M$ be a connected open subset such that $P(V) = \{n\theta\}_{n \in \mathbb{Z}}$ for some nonnegative P-function $\theta\colon V \to [0, +\infty)$ being strictly positive on $M \setminus \Sigma$, see* (1) *of Theorem* 1.1. *If $\mathbf{F}$ is conjugate to a $\mathcal{C}^1$-flow, then, in fact, $\theta > 0$ on all of M.*

Suppose, in addition, that $\mathbf{F}$ is generated by a $\mathcal{C}^1$-vector field F. Then for every $z \in \Sigma$ there are local coordinates in which the linear part $j^1F(z)$ of F at z is given by the following matrix:

$$\begin{pmatrix} 0 & \beta_1 \\ -\beta_1 & 0 \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} 0 & \beta_k \\ -\beta_k & 0 \end{pmatrix} \oplus 0 \oplus \ldots \oplus 0 \qquad (1.1)$$

for some $k \geq 1$ and $\beta_j \in \mathbb{R} \setminus 0$.

The proof of this theorem will be given in Section 7.

***Remark* 1.1.** Under assumptions of Theorem 1.2 suppose that θ is $\mathcal{C}^1$ and $\theta(z) \neq 0$ for some $z \in \Sigma$. In this case existence of (1.1) at z is easy to prove, c.f. [4].

Indeed, define the flow $\mathbf{G}\colon M \times \mathbb{R} \to M$ by $\mathbf{G}(x,t) = \mathbf{F}(x, t\,\theta(x))$. Then $\mathbf{G}$ is generated by the $\mathcal{C}^1$-vector field $G = \theta F$ and satisfies $\mathbf{G}_1 = \mathrm{id}_M$. Hence $\mathbf{G}$ yields a $\mathcal{C}^1$-differentiable $\mathbb{R}/\mathbb{Z} = S^1$-action on M.

Moreover, $\mathbf{G}_t(z) = z$ and so $\mathbf{G}$ yields a linear S^1-action $T_z\mathbf{G}_t$ on the tangent space T_zM. Now it follows from standard results about S^1 representations in $GL(\mathbb{R}, n)$ that the linear part of G at z in some local coordinates is given by (1.1). It remains to note that $j^1F(z) = j^1G(z)/\theta(z)$. Notice that these arguments *do not prove that the matrix* (1.1) *is non-zero.*

***Example* 1.1.** Let $\mathbf{F}\colon \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ be a continuous flow on $\mathbb{C}$ defined by

$$\mathbf{F}(z,t) = \begin{cases} e^{2\pi i\, t/|z|^2}\, z, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

The orbits of $\mathbf{F}$ are the origin $0 \in \mathbb{C}$ and the concentric circles centered at 0. Then $P(\mathbb{C}) = \{n\theta\}_{n\in\mathbb{Z}}$, where $\theta(z) = |z|^2$. Also notice that $\theta(0) = 0$ and $\theta > 0$ on $\mathbb{C} \setminus 0$. This shows that a non-zero P-function of a continuous flow $\mathbf{F}$ may vanish at its fixed points. It also implies that $\mathbf{F}$ in not conjugate to a $\mathcal{C}^1$-flow.

Our second result shows that if $\mathbf{F}$ is conjugate to a $\mathcal{C}^1$-flow, then discontinuity of P-functions at points of Σ is almost always the result of unboundedness of periods of points near Σ.

Theorem 1.3. *Let $\mathbf{F}$ be a $\mathcal{C}^1$-flow generated by a $\mathcal{C}^1$-vector field, $V \subset M$ be an open connected subset such that $P(V) = 0$, while $P(V \setminus \Sigma) = \{n\theta\}_{n\in\mathbb{Z}}$ for a certain non-zero P-function $\theta\colon V \setminus \Sigma \to \mathbb{R}$, so θ can not be continuously extended to all of V. Suppose that there exists a point $z \in V \cap \Sigma$ in which $j^1F(z)$, the linear part of F at z, is not similar to a matrix of the form* (1.1). *Then there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \subset V \setminus \Sigma$* (*consisting of periodic points*) *which converges to z and satisfies*

$$\lim_{i\to\infty} \mathrm{Per}(x_i) = \lim_{i\to\infty} \theta(x_i) = +\infty.$$

***Example* 1.2.** Let $\mathbf{F}\colon \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ be a $\mathcal{C}^\infty$-flow on $\mathbb{C}$ defined by

$$\mathbf{F}(z,t) = \begin{cases} e^{2\pi i\, t\, |z|^2}\, z, & z \neq 0, \\ 0, & z = 0, \end{cases}$$

where 0 is the origin. Then $\theta = \dfrac{1}{|z|^2}$ is a $\mathcal{C}^\infty$ P-function on $\mathbb{C} \setminus 0$ and $P(\mathbb{C} \setminus 0) = \{n\theta\}_{n\in\mathbb{Z}}$. On the other hand, $\lim\limits_{z\to 0} \theta(z) = +\infty$, whence θ can not be extended even to a continuous function on $\mathbb{C}$, so $P(\mathbb{C}) = \{0\}$.

1.2. Structure of the paper. In next section we discuss applications of P-functions to reparametrizations of flows to circle actions and also describe the relationships of Theorem 1.3 to the results of other authors.

Section 3 presents a variant of results of M. Newman, A. Dress, D. Hoffman, and L. N. Mann about lower bounds for diameters of orbits of $\mathbb{Z}_p$-actions on manifolds. Sections 4 and 5 give sufficient conditions for unboundedness of periods of $\mathcal{C}^0$- and $\mathcal{C}^1$-flows near singular points. The last two Sections 8 and 7 contain the proofs of Theorems 1.3 and 1.2 respectively.

2. *P*-functions and circle actions. In this section we discuss relation of our results to the following problem:

Problem 2.1. *Let* $\mathbf{F}$ *be a continuous flow on* M. *Do there exist a continuous* S^1*-action on* M *whose orbits coincides with the ones of* $\mathbf{F}$?

The following simple statement shows applications of P-functions to Problem 2.1.

Lemma 2.1. *Let* $\theta\colon M \to \mathbb{R}$ *be a* P*-function on all of* M. *Then the following map* $\mathbf{G}\colon M \times \mathbb{R} \to M$ *defined by* $\mathbf{G}(x,t) = \mathbf{F}(x, t\cdot\theta(x))$ *is a flow on* M *such that* $\mathbf{G}_1 = \mathrm{id}_M$, *so* $\mathbf{G}$ *factors to a circle action.*

Proof. Indeed, $\mathbf{G}_1(x) = \mathbf{F}(x, \theta(x)) = x$.

An evident necessary (but not sufficient) condition for Problem 2.1 is that every orbit of $\mathbf{F}$ is either periodic or fixed. Moreover, due to the well-known theorem of M. Newman the set Σ should be nowhere dense, see [5–8].

Suppose now that $\Sigma = \varnothing$ and all points are periodic for $\mathbf{F}$. It will be convenient to call such a flow $\mathbf{F}$ a *P-flow*. Then we have a well-defined function

$$\lambda\colon M \to (0, +\infty), \qquad \lambda(x) = \mathrm{Per}(x).$$

This function was studied by many authors. It can be shown that λ is lower semicontinuous and the set B of its continuity points is open in M, see e.g. D. Montgomery [9] and D. B. A. Epstein [10] (§ 5). Thus in the sense of Definition 1.1 λ is a P-function on B.

There are certain typical situations in which λ is discontinuous.

For instance, if λ is locally unbounded, then it can not be continuously extended to all M. Say that a P-flow $\mathbf{F}$ has property PB (resp. property PU) if λ is locally bounded (resp. locally unbounded). Equivalently, if $\mathbf{F}$ is at least $\mathcal{C}^1$, then instead of periods one can consider lengths of orbits with respect to some Riemannian metric on M.

The first example of a PU-flow was constructed by G. Reeb [11]. He produced a $\mathcal{C}^\infty$ PU-flow on a noncompact manifold. Further D. B. A. Epstein [10] constructed a *real analytic* PU-flow on a noncompact 3-manifold, D. Sullivan [12] a $\mathcal{C}^\infty$ PU-flow on a *compact* 5-manifold $S^3 \times S^1 \times S^1$, and D. B. A. Epstein and E. Vogt [13] a PU-flow on a compact 4-manifold defined by polynomial equations, with the vector field defining the flow given by polynomials, see also E. Vogt [14].

On the other hand, the following well-known example of Seifert fibrations shows that even if λ is discontinuous, then in some cases it can be continuously extended to all of M so that the obtained function is a P-function.

***Example* 2.1.** Let $D^2 \subset \mathbb{C}$ be the closed unit 2-disk centered at the origin, $S^1 = \partial D^2$ be the unit circle, and $T = D^2 \times S^1$ be the solid torus. Fix $k \geq 2$ and define the following flow on T:

$$\mathbf{F}\colon T \times \mathbb{R} \to T, \qquad \mathbf{F}(z, \tau, t) = (z e^{2\pi i t/k}, \tau e^{2\pi i t}),$$

for $(z, \tau, t) \in D^2 \times S^1 \times \mathbb{R}$. It is easy to see that every $(z, \tau) \in T$ is periodic. Moreover, $\mathrm{Per}(z,\tau) = k$ if $z \neq 0 \in D^2$, while $\mathrm{Per}(0,\tau) = 1$. Thus the function $\mathrm{Per}\colon T^2 \to \mathbb{R}$ is discontinuous on the central orbit $0 \times S^1$, but it becomes even smooth if we redefine it on $0 \times S^1$ by the value k instead of 1. This new constant function $\theta \equiv k$ is a P-function and $P(T) = \{nk\}_{n\in\mathbb{Z}}$.

Notice that in this example $\mathbf{F}$ is a suspension flow of a periodic homeomorphism $h\colon D^2 \to D^2$ being a rotation by $2\pi/k$.

More generally, let $h\colon M \to M$ be a homeomorphism of a connected manifold M such that the corresponding suspension flow $\mathbf{F}$ of h on $M \times S^1$ is a P-flow. This is possible if and only if all the points of M are periodic with respect to h. D. Montgomery [9] shown that such a homeomorphism is periodic itself. Let k be the period of h. Then the periods of orbits of $\mathbf{F}$ are bounded with k, so $\mathbf{F}$ is a PB-flow. Moreover, similarly to Example 2.1, it can be shown that $P(M) = \{nk\}_{n\in\mathbb{Z}}$.

D. B. A. Epstein [10] also proved that if M is a compact orientable 3-manifold then any $\mathcal{C}^r$ P-flow with $(1 \leq r \leq \omega)$ has property BP and even there exists a $\mathcal{C}^r$-circle action with the same orbits. In fact he shown that the structure of one-dimensional $\mathcal{C}^r$-foliations $(1 \leq r \leq \infty)$ on compact orientable 3-manifolds, possibly with boundary, is similar to Seifert fibrations described in Example 2.1.

The problem of bounded periods has its counterpart for foliations with all leaves compact. The question is whether the volumes of leaves are locally bounded with respect to some Riemannian metric, see e.g. [15 – 17]. For instance the mentioned above statements for flows can be adopted for foliations.

The results of the present paper describe the behaviour of period functions near fixed poins of $\mathcal{C}^1$-flows.

3. Diameters and lengths of orbits. ***3.1. Effective $\mathbb{Z}_p$-actions.*** We recall here results of A. Dress [8] (Lemma 3) and D. Hoffman and L. N. Mann [18] (Theorem 1) about diameters of orbits of effective $\mathbb{Z}_p$-actions.

For $x, y \in \mathbb{R}^n$ denote by $d(x, y)$ the usual Euclidean distance, and by $B_r(x)$, $r > 0$, the open ball of radius r centered at x.

Let W be an open subset of the half-space $\mathbb{R}^n_+ = \{x_n \geq 0\}$ and $x \in W$. Define the *radius r_x of convexity of W at x* as follows. If $x \in \operatorname{Int}(\mathbb{R}^n_+) \cap W$, then

$$r_x = \sup\big\{r > 0\colon B_r(x) \subset W\big\}.$$

Otherwise, $x \in \partial\mathbb{R}^n_+ \cap W$ and we put

$$r_x = \sup\big\{r > 0\colon (B_r(x) \cap \mathbb{R}^n_+) \subset W\big\}.$$

Lemma 3.1 ([8], Lemma 3)**.** *Let $U \subset \mathbb{R}^n$ be an open, relatively compact and connected subset, p be a prime, and $h\colon \overline{U} \to \overline{U}$ be a homeomorphism which induces a nontrivial $\mathbb{Z}_p$-action, that is $h \neq \mathrm{id}_{\overline{U}}$ but $h^p = \mathrm{id}_{\overline{U}}$. Define two numbers*:

$$D(U) = \max\big\{\min\{d(x, y)\colon y \in \overline{U}\}\colon x \in U\big\},$$

$$C(U) = \max\Big\{d(x, h^a(x))\colon a = 0, \ldots, p-1, \;\; x \in \overline{U} \setminus \operatorname{Int}(\overline{U})\Big\}.$$

Then $D(U) < C(U)$.

The next Lemma 3.2 is a variant of [18] (Theorem 1). It seems that in the proof of [18] (Theorem 1) the condition of connectedness of the set U, see (3.1) below, is missed, c.f. paragraph after the assumption (H) on [18, p. 345]. Therefore we recall the proof which is also applicable to manifolds with boundary.

Lemma 3.2 (c.f. [18], Theorem 1)**.** *Let $W \subset \mathbb{R}^n_+ = \{x_n \geq 0\}$ be an open subset, p be a prime, and $h\colon \overline{W} \to \overline{W}$ be a homeomorphism which induces a nontrivial $\mathbb{Z}_p$-action. Suppose $h(z) = z$ for some $z \in W$. Let also r_z be the radius of convexity of W at z and $r \in (0, r_z)$.*

If $z \in \mathrm{Int}(\mathbb{R}^n_+)$, *then there exist* $x \in \partial B_{r/2}(z)$ *and* $a = 1, \ldots, p-1$ *such that*

$$d(z, x) \leq 2d(x, h^a(x)).$$

If $z \in \partial\mathbb{R}^n_+$, *then there exist* $x \in \partial(B_{2r/3}(z) \cap W)$ *and* $a = 1, \ldots, p-1$ *such that*

$$d(z, x) \leq 4d(x, h^a(x)).$$

Proof. For simplicity denote $B_r(z)$ by B_r.

1. First suppose that $z \in \mathrm{Int}(\mathbb{R}^n_+)$. For each $s \in (0, r_z)$ put

$$U_s = B_s \cup h(B_s) \cup \ldots \cup h^{p-1}(B_s). \tag{3.1}$$

Then U_s is open, relatively compact, and h yields a nontrivial $\mathbb{Z}_p$-action on $\overline{U}$. Moreover, by assumption $h(z) = z$, therefore U_s is connected. Then by Lemma 3.1

$$D(U_s) \leq C(U_s).$$

Notice that $B_s \subset \overline{U_s}$, whence $s \leq D_s$.

On the other hand, suppose

$$d(y, h^a(y)) < r - s, \quad \text{for all} \quad y \in \partial\overline{U_s} \quad \text{and} \quad a = 1, \ldots, p-1. \tag{3.2}$$

Then in particular, $C(U_s) < r - s$ and thus

$$s \leq D(U_s) \leq C(U_s) < r - s,$$

whence $s < r/2$.

Thus if $s = r/2$, then (3.2) fails, whence there exist $y \in \partial\overline{U_{r/2}}$ and $b \in \{0, \ldots, p-1\}$ such that $d(y, h^b(y)) \geq r - r/2 = r/2$. However

$$\partial\overline{U_{r/2}} \subset \overset{p-1}{\underset{i=0}{\cup}} h^i(\partial B_{r/2}),$$

whence $y = h^c(x)$ for some $x \in \partial B_{r/2}$. Therefore at least one of the distances $d(x, y)$ or $d(x, h^b(y))$ is not less than $r/4$. In other words, $d(x, h^a(x)) \geq r/4$ for some $x \in \partial B_{r/2}$ and $a \in \{1, \ldots, p-1\}$. Then

$$d(x, z) = \frac{r}{2} = 2\frac{r}{4} \leq 2d(x, h^a(x)).$$

2. Let $z \in \partial\mathbb{R}^n_+$. For each $s \in (0, r_z)$ let $A_s = B_s \cap \mathrm{Int}(\partial\mathbb{R}^n_+)$ be the open upper half-disk centered at z, and

$$U'_s = A_s \cup h(A_s) \cup \ldots \cup h^{p-1}(A_s).$$

Then U'_s is open, relatively compact, and h yields a nontrivial $\mathbb{Z}_p$-action on $\overline{U'_s}$. Moreover, it is easy to see that U'_s is connected, whence by Lemma 3.1

$$D(U'_s) \leq C(U'_s).$$

Moreover, $B_{s/2} \subset \overline{U'_s}$. Therefore $s/2 \leq D(U'_s)$. Hence if we suppose that $C(U'_s) < r - s$, then $s/2 < r - s$ and thus $s < 2r/3$.

Put $s = 2r/3$. Then there exists $y \in \partial U'_{2r/3}$ and $b \in \{1, \ldots, p-1\}$ such that $d(y, h^b(y)) > r - 2r/3 = r/3$.

Again $\partial \overline{U'_s} \subset \bigcup\limits_{i=0}^{p-1} h^i(\partial A_s)$, whence we can find $x \in \partial A_{2r/3}$ such that $d(x, h^a(x)) >$ $> r/6$ for some $a \in \{1, \ldots, p-1\}$. Then

$$d(x, z) \leq \frac{2r}{3} = 4 \cdot \frac{r}{6} \leq 4 \cdot d(x, h^a(x)).$$

The lemma is proved.

3.2. Periodic orbits. Let F be a $\mathcal{C}^1$-vector field on a manifold M and d be any Riemannian metric on M. Then for any periodic point x of F the length $l(x)$ of its orbit can be calculated as follows:

$$l(x) = \int\limits_0^{\mathrm{Per}(x)} \big\| F(\mathbf{F}(x,t)) \big\| dt.$$

Hence

$$l(x) \leq \mathrm{Per}(x) \sup_{t \in [0, \mathrm{Per}(x)]} \big\| F(\mathbf{F}(x,t)) \big\|. \tag{3.3}$$

Also notice that

$$2 \operatorname{diam}(o_x) \leq l(x). \tag{3.4}$$

Indeed, let $y, z \in o_x$ be points for which $d(y, z) = \operatorname{diam}(o_x)$. These points divide o_x into two arcs each of which has the length $\geq d(y, z)$. This implies (3.4).

4. *P*-functions on the set of nonfixed points. Let $V \subset M$ be an open subset, $\xi \colon V \setminus \Sigma \to \mathbb{R}$ be a P-function, and $\alpha \in \mathbb{R}$. Define the following map $h_\alpha \colon V \to M$ by:

$$h_\alpha(x) = \begin{cases} \mathbf{F}(x, \alpha\, \xi(x)), & x \in V \setminus \Sigma, \\ x, & x \in \Sigma \cap V. \end{cases} \tag{4.1}$$

Then h_α is continuous on $V \setminus \Sigma$ but in general it is discontinuous at points of $\Sigma \cap V$.

The aim of this section is to establish implications between the following five conditions:

(A) The periods of periodic points in $V \setminus \xi^{-1}(0)$ are uniformly bounded above with some constant $\mathsf{C} > 0$, that is for each $x \in V$ with $\xi(x) \neq 0$ we have that $\mathrm{Per}(x) < \mathsf{C}$.

(B) Every $z \in \Sigma \cap V$ has a neighbourhood $W \subset V$ such that ξ is *regular* on $W \setminus \Sigma$, that is for every $y \in W \setminus \Sigma$ the restriction of ξ to $o_y \cap W$ is constant.

$(C)_\alpha$ The map h_α is continuous on all of V.

Let $z \in \Sigma \cap V$.

$(D)_\alpha$ Suppose $\alpha = q/p \in \mathbb{Q}$, where $q \in \mathbb{Z}$ and $p \in \mathbb{N}$. There exists a neighbourhood $W \subset V$ of z such that $h_\alpha(W) = W$, the restriction $h_\alpha \colon W \to W$ is a homeomorphism, and $h_\alpha^p = \mathrm{id}_W$.

(E) There exist $\mathsf{T} > 0$, a Euclidean metric d on some neighbourhood W of z, and a sequence $\{x_i\}_{i \in \mathbb{N}} \subset V \setminus \Sigma$ converging to z such that $\xi(x_i) \neq 0$ and

$$d(z, x_i) < \mathsf{T} \operatorname{diam}(o_{x_i} \cap W).$$

***Remark* 4.1.** By a *Euclidean* metric on W in (E) we mean a metric induced by some embedding $W \subset \mathbb{R}^n$. In fact, this condition can be formulated for arbitrary Riemannian metrics, but for technical reasons (see especially Lemma 5.1) we restrict ourselves to Euclidean ones.

***Remark* 4.2.** If $z \in \Sigma \cap V$ has a neighbourhood $W \subset V$ such that $W \setminus \Sigma$ is connected, then by (3) of Theorem 1.1 condition (B) holds for z.

Lemma 4.1. *The following implications hold true*:

$$(A) \Rightarrow (B) \vee (C)_\alpha.$$

If $\alpha \in \mathbb{Q}$, then for every $z \in \Sigma \cap V$

$$(B) \vee (C)_\alpha \Rightarrow (D)_\alpha.$$

If h is not the identity on some neighbourhood of $z \in \Sigma \cap V$, then

$$(D)_\alpha \Rightarrow (E),$$

and we can take $\mathsf{T} = 4$ in (E).

Proof. $(A) \Rightarrow (B)$. Since $z \in \Sigma \cap V$, there exists a neighbourhood W of z such that $\mathbf{F}(W \times [0, c]) \subset V$. Then W satisfies (B). Indeed, let $y \in W \setminus \Sigma$. We have to show that $\xi|_{o_y \cap W}$ is constant.

If $\xi = 0$ on $o_y \cap W$ there is nothing to prove. Therefore we can assume that $\xi(y) \neq 0$. Then, by (A), $\mathrm{Per}(y) < \mathsf{C}$, whence

$$o_y = \mathbf{F}(y \times [0, \mathrm{Per}(y)]) = \mathbf{F}(y \times [0, \mathsf{C}]) \subset \mathbf{F}(W \times [0, \mathsf{C}]) \subset V.$$

Thus $o_y \cap V = o_y$ is connected. Then, by Lemma 1.1, ξ is constant along o_y and therefore on $o_y \cap W$.

$(A) \Rightarrow (C)_\alpha$. It suffices to show that h_α is continuous at each $z \in \Sigma \cap V$. Let $V' \subset V$ be any neighbourhood of z and W be another neighbourhood of z such that $\mathbf{F}(W \times [0, c]) \subset V'$. We claim that $h_\alpha(W) \subset V'$. This will imply continuity of h_α at z.

Let $x \in W$. If $x \in \Sigma \cap W$ or $\xi(x) = 0$, then $h_\alpha(x) = x \in W \subset V'$. Otherwise, $\xi(x) \neq 0$ and x is periodic. Hence $h_\alpha(x) = \mathbf{F}(x, \tau)$ for some $\tau \in [0, \mathrm{Per}(x)] \subset [0, c]$. Therefore $h_\alpha(x) \in \mathbf{F}(W \times [0, c]) \subset V'$.

$(B) \vee (C)_\alpha \Rightarrow (D)_\alpha$. We have that $\alpha = q/p$, where $q \in \mathbb{Z}$ and $p \in \mathbb{N}$. For simplicity denote h_α by h. Since $h(z) = z$ and h is continuous, there exists a neighbourhood B of z such that $h^i(B) \subset V$ for all $i = 0, \ldots, p$. Denote

$$W = B \cup h(B) \cup \ldots \cup h^{p-1}(B).$$

We claim that W satisfies (D).

Indeed, let $x \in V$ and suppose that $h(x) \in V$ as well. Since $h(x)$ belongs to the orbit of x, then, by (B), $\xi(x) = \xi(h(x))$. Hence

$$h^2(x) = \mathbf{F}\big(h(x), \alpha\xi(h(x))\big) = \mathbf{F}\big(h(x), \alpha\xi(x)\big) =$$

$$= \mathbf{F}\big(\mathbf{F}(x, \alpha\xi(x)), \alpha \cdot \xi(x)\big) = \mathbf{F}\big(x,\, 2\alpha\xi(x)\big). \tag{4.2}$$

By induction we will get that if $h^i(x) \in V$ for all $i = 0, \ldots, j-1$, then

$$h^j(x) = \mathbf{F}\big(x,\, j\alpha\xi(x)\big).$$

In particular, $h^{kp}(x) = \mathbf{F}(x, kq\xi(x)) = x$ for any $k \in \mathbb{Z}$. By $(C)_\alpha$ we have that h is continuous on V, whence h yields a homeomorphism of W onto itself and $h^p|_W = \mathrm{id}_W$.

$(D)_\alpha \Rightarrow (E)$. Again denote h_α by h. Since h is not the identity on W, we can assume that p is a prime and thus the action of h is effective. This can be done by replacing h with h^n for some $n \in \mathbb{N}$ such that p/n is a prime.

Decreasing W we can assume that W is an open subset of the half-space $\mathbb{R}^n_+ = \{x_n \geq 0\}$. Let d be the corresponding Euclidean metric on W and r_z be the radius of convexity of W at z. Then by Lemma 3.2 for each $r \in (0, r_z)$ there exist $x_r \in W$ and $a_r \in \{1, \ldots, p-1\}$ such that

$$d(z, x_r) \leq \mathsf{S}r \leq \mathsf{T}d(x_r, h^{a_r}(x_r)) \leq \mathsf{T}\,\mathrm{diam}(o_{x_r}),$$

for some $\mathsf{S}, \mathsf{T} > 0$. In fact, $\mathsf{S} = \dfrac{1}{2}$ and $\mathsf{T} = 2$ if $z \in \mathrm{Int}(M)$, and $\mathsf{S} = \dfrac{2}{3}$ and $\mathsf{T} = 4$ if $z \in \partial M$. Notice that a_r may take only finitely many values. Therefore we can find $a \in \{1, \ldots, p-1\}$ and a sequence $\{x_{r_i}\}_{i\in\mathbb{N}}$ such that $\lim\limits_{i\to\infty} r_i = 0$ and $a_{r_i} = a$ for all $i \in \mathbb{N}$.

The lemma is proved.

5. Condition (E) for $\mathcal{C}^1$-flows. Condition (E) defined in the previous section gives some lower bound for diameters of orbits of a sequence $\{x_i\}_{i\in\mathbb{N}}$ of periodic points converging to a fixed point z. In this section it is shown that for a $\mathcal{C}^1$-flow that condition allows to estimate periods of x_i.

Let M be a $\mathcal{C}^r$, $r \geq 1$, connected, m-dimensional manifold possibly noncompact and with or without boundary. Let also F be a $\mathcal{C}^r$-vector field on M tangent to ∂M and generating a $\mathcal{C}^r$-flow $\mathbf{F}\colon M \times \mathbb{R} \to M$. Again by Σ we denote the set of fixed points of $\mathbf{F}$ which coincides with the set of *zeros* (or *singular points*) of F.

Proposition 5.1. *Let $V \subset M$ be an open subset, $\xi\colon V \setminus \Sigma \to \mathbb{R}$ be a P-function, $z \in \Sigma \cap V$, and $\{x_i\}_{i\in\mathbb{N}} \subset V \setminus \Sigma$ be a sequence of periodic points converging to z and satisfying (E). Thus $\xi(x_i) \neq 0$, and there exists $\mathsf{T} > 0$ and a Euclidean metric on some neighbourhood of z such that $\mathrm{d}(z, x_i) < \mathsf{T}\,\mathrm{diam}(o_{x_i})$. If the periods of x_i are bounded above with some $\mathsf{C} > 0$ (in particular, condition (A) holds true), then*

(e_1) *there exists $\varepsilon > 0$ such that $|\xi(x_i)| \geq \mathrm{Per}(x_i) > \varepsilon$ for all $i \in \mathbb{N}$, so the periods are bounded below as well, and*

(e_2) $j^1F(z) \neq 0$.

For the proof we need some statements. The proofs are straightforward, and we left them for the reader.

***Claim* 5.1.** *Let X be a topological space, K be a compact space, and $g\colon X \times K \to \mathbb{R}$ be a continuous function. Then the following function $\gamma\colon X \to \mathbb{R}$ defined by $\gamma(x) = \sup\limits_{y\in K} g(x, y)$ is continuous.*

***Claim* 5.2.** *Let K be a compact manifold and*

$$Q = (Q_1, \ldots, Q_n)\colon \mathbb{R}^n \times K \to \mathbb{R}^n$$

be a continuous map satisfying the following conditions:

(a) $Q(0 \times K) = 0$.

(b) *For each $k \in K$ the map $Q_k = Q(\cdot, k)\colon \mathbb{R}^n \to \mathbb{R}^n$ is $\mathcal{C}^1$.*

(c) *For every* $i, j = 1, \ldots, n$ *the partial derivative* $\dfrac{\partial Q_i}{\partial x_j} \colon \mathbb{R}^n \times K \to \mathbb{R}$ *of the* i*-th coordinate function* Q_i *of* Q *in* x_j *is continuous.*

In particular, conditions (b) *and* (c) *hold if* K *is a manifold and* Q *is a* $\mathcal{C}^1$ *map. Then there exists a continuous function* $\alpha \colon \mathbb{R}^n \to \mathbb{R}$ *such that*

$$\|Q(x,k)\| \le \|x\|\alpha(x), \quad (x,k) \in \mathbb{R}^n \times K.$$

If $j^1 Q_k(0) = 0$ *for all* $k \in K$*, then* $\alpha(0) = 0$.

Lemma 5.1. *Let* F *be a* $\mathcal{C}^1$*-vector field in* $\mathbb{R}^n$ *such that* $F(0) = 0$ *and* $(\mathbf{F}_t)$ *be the local flow of* F*. Then for every* $\mathsf{C} > 0$ *there exist a neighbourhood* W *of the origin* $0 \in \mathbb{R}^n$ *and a continuous function* $\gamma \colon W \to \mathbb{R}$ *such that*

$$\|F(\mathbf{F}(x,t))\| \le \|x\|\gamma(x)$$

for all $(x,t) \in W \times [-\mathsf{C}, \mathsf{C}]$*. If* $j^1 F(0) = 0$*, then* $\gamma(0) = 0$.

If F *is* $\mathcal{C}^2$*, then we have a usual estimation* $\|F(\mathbf{F}(x,t))\| \le \mathsf{A}\|x\|^2$ *for some* $\mathsf{A} > 0$.

Proof. Since $\mathbf{F}(0,t) = 0$ for all $t \in \mathbb{R}$, there exists a neighbourhood W of z such that for each $(x,t) \in W \times [-\mathsf{C}, \mathsf{C}]$ the point $\mathbf{F}(x,t)$ is well-defined and belongs to V, so $\mathbf{F}(W \times [-\mathsf{C}, \mathsf{C}]) \subset V$.

Moreover, $\mathbf{F}$ is $\mathcal{C}^1$ and therefore it satisfies assumptions (a) – (c) of Claim 5.2 with $K = [-\mathsf{C}, \mathsf{C}]$. Hence there exists a continuous function $\alpha \colon W \to \mathbb{R}$ such that

$$\big\|\mathbf{F}(x,t)\big\| \le \|x\|\alpha(x), \quad (x,k) \in W \times [-\mathsf{C}, \mathsf{C}].$$

Moreover, F is also $\mathcal{C}^1$ and $F(0) = 0$, whence again by Claim 5.2 (for $K = \varnothing$) there exists a continuous function $\beta \colon W \to \mathbb{R}$ such that $\|F(x)\| \le \|x\|\beta(x)$. Define $\gamma \colon W \to \mathbb{R}$ by

$$\gamma(x) = \sup_{t \in [-\mathsf{C}, \mathsf{C}]} \alpha(x)\beta(\mathbf{F}(x,t)).$$

Then by Claim 5.1 γ is continuous and

$$\|F(\mathbf{F}(x,t))\| \le \|\mathbf{F}(x,t))\|\beta(\mathbf{F}(x,t)) \le \|x\|\alpha(x)\beta(\mathbf{F}(x,t)) \le \|x\|\gamma(x).$$

Moreover, if $j^1 F(0) = 0$, then $\beta(0) = 0$. Since in addition $\mathbf{F}(0,t) = 0$, we obtain that $\gamma(0) = \sup\limits_{t \in [-\mathsf{C}, \mathsf{C}]} \alpha(0)\beta(0) = 0$ as well.

The lemma is proved.

***Proof of Proposition* 5.1.** We have to show that violating either of assumptions (e_1) or (e_2) leads to a contradiction.

By Lemma 5.1 for any $\mathsf{C} > 0$ there exist a neighbourhood W of z and a continuous function $\gamma \colon W \to \mathbb{R}$ such that

$$\|F(\mathbf{F}(x,t))\| \le d(x,z)\gamma(x), \quad (x,t) \in W \times [-\mathsf{C}, \mathsf{C}]. \tag{5.1}$$

Then

$$d(z, x_i) < \mathsf{T}\operatorname{diam}(o_{x_i}) \overset{(3.4)}{\le} \frac{\mathsf{T}}{2} l(x_i) \overset{(3.3)\vee(5.1)}{\le} \frac{\mathsf{T}}{2}\operatorname{Per}(x_i) d(z, x_i)\gamma(x_i).$$

Therefore

$$0 < \frac{2}{\mathsf{T}} < \mathrm{Per}(x_i)\gamma(x_i) \leq |\xi(x_i)|\gamma(x_i).$$

Hence if (e_1) is violated, i.e., $\lim\limits_{i\to\infty} \mathrm{Per}(x_i) = 0$, then $\lim\limits_{i\to\infty} \gamma(x_i) = +\infty$, which contradicts to continuity of γ near z.

Suppose $j^1 F(z) = 0$. Then by Lemma 5.1 $\gamma(z) = 0$, whence $\lim\limits_{i\to\infty} \gamma(x_i) = \gamma(z) = 0$. Therefore $\lim\limits_{i\to\infty} \mathrm{Per}(x_i) = \lim\limits_{i\to\infty} |\xi(x_i)| = +\infty$, which contradicts to boundedness of periods of x_i.

The proposition is proved.

6. Unboundedness of periods. Now let F be a $\mathcal{C}^1$-vector field in $\mathbb{R}^n$ such that $F(0) = 0$. We can regard F as a $\mathcal{C}^1$ map $F = (F_1, \ldots, F_n)\colon \mathbb{R}^n \to \mathbb{R}^n$. Let

$$A = \left(\frac{\partial F_i}{\partial x_j}(0)\right)_{i,j=1,\ldots,n}$$

be the Jacobi matrix of F at 0. This matrix also called the *linear part of F at* 0. By the real Jordan's normal form theorem A is similar to the matrix of the following form:

$$\mathop{\oplus}_{\sigma=1}^{s} \mathbf{J}_{q_\sigma}(a_\sigma \pm i b_\sigma)) \oplus \mathop{\oplus}_{\tau=1}^{r} \mathbf{J}_{p_\tau}(\lambda_\tau), \tag{6.1}$$

where $\lambda_\sigma \in \mathbb{R}$ and $a_\tau \pm i b_\tau \in \mathbb{C}$ are all the eigen values of A.

Theorem 1.3 is a direct consequence of the following theorem.

Theorem 6.1. *Suppose one of the following conditions holds*:

(1) *A has an eigen value λ such that $\Re(\lambda) \neq 0$.*

(2) *The matrix* (6.1) *has either a block $\mathbf{J}_q(\pm ib)$ or $\mathbf{J}_q(0)$ with $q \geq 2$.*

(3) *$A = 0$ and there exists an open neighbourhood V of 0 in $\mathbb{R}^n$ and a continuous P-function $\xi\colon V \setminus \Sigma \to \mathbb{R}$ which takes non-zero values arbitrary close to 0, that is $0 \in \overline{V \setminus \xi^{-1}(0)}$.*

Then there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \subset V \setminus \Sigma$ which converges to 0 and such that either every x_i is nonperiodic, or every x_i is periodic and $\lim\limits_{i\to\infty} \mathrm{Per}(x_i) = +\infty$.

Proof. (1) In this case by Hadamard – Perron's theorem, e.g. [19], we can find a nonperiodic orbit o of F such that $0 \in \overline{o} \setminus o$. This means that there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \subset o$ converging to 0.

(2) Consider two cases.

(a) If (6.1) has a block $\mathbf{J}_q(0)$ with $q \geq 2$, then it can be assumed that

$$A = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

(b) Suppose (6.1) has a block $\mathbf{J}_q(\pm ib)$ with $q \geq 2$. Then we can regard $\mathbb{R}^n$ as $\mathbb{C}^2 \oplus \mathbb{R}^{n-4}$, so the first two coordinates x_1 and x_2 are complex. Therefore it can be supposed that

$$A = \begin{pmatrix} ib & 0 & \ldots & 0 \\ 1 & ib & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

In both cases denote by p_1 the projection to the first (either real or complex) coordinate, i.e., $p_1(x_1, \ldots, x_n) = x_1$.

Lemma 6.1. *Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ centered at the origin 0, $\varepsilon > 0$,*

$$Y_\varepsilon = \{(x_1, \ldots, x_n) \in S^{n-1}\colon\ |x_1| \geq \varepsilon\},$$

and C_ε be the cone over Y_ε with vertex at 0.

Then for each $\mathsf{L} > 0$ there exists a neighbourhood $W = W_{\mathsf{L},\varepsilon}$ of 0 such that every

$$x \in (W \cap C_\varepsilon) \setminus 0$$

is either nonperiodic or periodic with period $\operatorname{Per}(x) > \mathsf{L}$.

Proof. Let $(\mathbf{F}_t)$ be the local flow of F. Then in general, $\mathbf{F}$ is defined only on some open neighbourhood of $\mathbb{R}^n \times 0$ in $\mathbb{R}^n \times \mathbb{R}$. Nevertheless, since $\mathbf{F}(0,t) = 0$ for all $t \in \mathbb{R}$, it follows that for each $\mathsf{L} > 0$ there exists a neighbourhood V of 0 such that $\mathbf{F}$ is defined on $V \times [-2\mathsf{L}, 2\mathsf{L}]$.

We claim that in both cases there exists $c > 0$ such that

$$\|e^{At}x - x\| \geq c|tx_1| = c|tp_1(x)|. \tag{6.2}$$

(a) In this case $e^{At}x = (x_1,\, tx_1 + x_2, \ldots)$ and we can put $c = 1$:

$$\|e^{At}x - x\| = \|(0,\, tx_1, \ldots)\| \geq |tx_1| = |tp_1(x)|.$$

(b) Now $e^{At}x = (e^{ibt}x_1,\, e^{ibt}(tx_1 + x_2), \ldots)$. Notice that we can write $e^{ibt} = 1 + t\gamma(t)$ for some smooth function $\gamma\colon \mathbb{R} \to \mathbb{C} \setminus \{0\}$. Denote $c = \min\limits_{t \in [-2\mathsf{L}, 2\mathsf{L}]} |\gamma(t)|$. Then $c > 0$ and

$$\|e^{At}x - x\| = \big\|(e^{ibt}x_1 - x_1, \ldots)\big\| \geq |t\gamma(t)x_1| = c|tp_1(x)|.$$

Since $\mathbf{F}$ is a $\mathcal{C}^1$ map and $\mathbf{F}(0,t) = 0$ for all $t \in \mathbb{R}$, it follows from Claim 5.2 that there exists a continuous function $\alpha\colon V \to [0, +\infty)$ such that

(1) $\alpha(0) = 0$,

(2) $\|\mathbf{F}(x,t) - e^{At}x\| \leq \|x\|\alpha(x)$ for all $(x,t) \in V \times [-2\mathsf{L}, 2\mathsf{L}]$.

Hence

$$\|\mathbf{F}(x,t) - x\| \geq \|e^{At}x - x\| - \|\mathbf{F}(x,t) - e^{At}x\| \geq c|tp_1(x)| - \|x\|\alpha(x)$$

for $(x,t) \in \mathbb{R}^n \times [-2\mathsf{L}, 2\mathsf{L}]$. Moreover, if $p_1(x) \neq 0$, then

$$\|\mathbf{F}(x,t) - x\| \geq c|tp_1(x)| - \|x\|\,\alpha(x) = c|p_1(x)|\left(|t| - \frac{\|x\|\,\alpha(x)}{c|p_1(x)|}\right).$$

Since $\alpha(0) = 0$, there exists a neighbourhood $W \subset V$ of 0 such that

$$\alpha(x) < c\varepsilon\mathsf{L}, \quad x \in W.$$

Now let $x \in (W \cap C_\varepsilon) \setminus 0$. Then $|x| \leq 1$ and $|p_1(x)| \geq \varepsilon$, whence

$$\frac{\|x\|\,\alpha(x)}{c|p_1(x)|} < \frac{1c\varepsilon\mathsf{L}}{c\varepsilon} = \mathsf{L}.$$

Therefore $\|\mathbf{F}(x,t) - x\| \geq c\varepsilon(|t| - \mathsf{L})$ for $t \in [-2\mathsf{L}, 2\mathsf{L}]$. In particular, $\mathbf{F}(x,t) \neq x$ for $t \in [\mathsf{L}, 2\mathsf{L}]$. It follows that x is either nonperiodic, or periodic with the period being greater than $2\mathsf{L} - \mathsf{L} = \mathsf{L}$.

The lemma is proved.

(3) Suppose that there exists a neighbourhood V of 0 such that all points in $V \setminus \Sigma$ are periodic. If the periods of points in $V \setminus \xi^{-1}(0)$ are bounded above, i.e., condition (A) holds true, then by (e_2) of Proposition 5.1 $j^1F(0) \neq 0$.

Theorem 6.1 is proved.

The following statement extends [2] (Proposition 10).

Proposition 6.1 ([2], Proposition 10)**.** *Let F be a $\mathcal{C}^1$-vector field on a manifold M, $z \in \Sigma \setminus \mathrm{Int}(\Sigma)$, $V \subset M$ be a neighbourhood of z, and $\xi \in P(V)$ be a P-function. Let also $W = \cup_{\lambda\in\Lambda} W_\lambda$ be the union of those connected components of $V \setminus \Sigma$ for which $z \in \overline{W_\lambda}$. Then each of the following conditions implies that $\xi = 0$ on $\overline{W} \cap V$:*

(a) *$j^1F(z)$ has an eigen value λ such that $\Re(\lambda) \neq 0$;*

(b) *a real normal Jordan form of $j^1F(z)$ has either a block $\mathbf{J}_q(\pm ib)$ or $\mathbf{J}_q(0)$ with $q \geq 2$;*

(c) *$j^1F(z) = 0$;*

(d) *$z \in \overline{\mathrm{Int}(\Sigma_F)} \setminus \mathrm{Int}(\Sigma)$;*

(e) *$\xi(z) = 0$.*

Proof. Suppose that ξ takes non-zero values on periodic points arbitrary close to z. First we prove that every of the assumptions (a) – (d) implies (e), and then show that (e) gives rise to a contradiction.

(a)$\vee$(b)$\vee$(c) $\Rightarrow$ (e) Suppose $j^1F(z)$ satisfies either of the conditions (a), (b) or (c). Then by assumption on ξ and Theorem 6.1 there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \subset V\setminus\Sigma$ converging to z and such that every x_i is either nonperiodic, or periodic but $\lim\limits_{i\to\infty} \mathrm{Per}(x_i) = +\infty$.

If every x_i is nonperiodic, then by Lemma 1.1 $\xi(x_i) = 0$, whence by continuity of ξ we obtain $\xi(z) = 0$ as well.

Suppose every x_i is periodic. Then $\xi(x_i) = n_i \,\mathrm{Per}(x_i)$ for some $n_i \in \mathbb{Z}$. Since $\lim\limits_{i\to\infty} \mathrm{Per}(x_i) = +\infty$ and ξ is continuous, it follows that $\lim\limits_{i\to\infty} n_i = 0$, that is $n_i = 0$ for all sufficiently large i, whence $\xi(x_i) = 0$ which implies $\xi(z) = 0$.

(d) $\Rightarrow$ (c) The assumption $z \in \overline{\mathrm{Int}(\Sigma)} \setminus \mathrm{Int}(\Sigma)$ means that there is a sequence $\{z_i\}_{i\in\mathbb{N}} \subset \mathrm{Int}(\Sigma)$ converging to z. But then $j^1F(z_i) = 0$ for all i. Since F is C^1, we obtain $j^1F(z) = 0$ as well.

(e) Suppose that $\xi(z) = 0$. Let U be a neighbourhood of z with compact closure $\overline{U} \subset V$, and $\mathsf{C} = \sup\limits_{x\in\overline{U}} |\xi(x)|$. Then the periods of points in $U \setminus \xi^{-1}(0)$ are bounded above with C, that is ξ satisfies condition (A), and therefore by Lemma 4.1 condition (E). Let $\{x_i\}_{i\in\mathbb{N}} \subset U \setminus \Sigma$ be a sequence converging to z and satisfying (E). Then by Proposition 5.1 there exists $\varepsilon > 0$ such that $|\xi(x_i)| > \varepsilon$. Since ξ is continuous, we get $|\xi(z)| \geq \varepsilon > 0$, which contradicts to the assumption $\xi(z) = 0$.

The proposition is proved.

7. Proof of Theorem 1.2. Let $\mathbf{F}$ be a flow conjugate to a $\mathcal{C}^1$-flow. Then by [3], $\mathbf{F}$ is conjugate to a flow generated by a $\mathcal{C}^1$-vector field F. As noted in Subsection 1.1, conjugation does not change the structure of the set of P-functions, therefore we can assume that $\mathbf{F}$ itself is generated by $\mathcal{C}^1$-vector field F.

Let $\theta \in P(M)$ be a nonnegative generator of $P(M)$. Put $Y = \theta^{-1}(0)$. Then Y is closed.

We claim that Y *is also open in* M. Indeed, if x is a non-fixed point of $\mathbf{F}$, then by [2] (Corollary 8) $\theta = 0$ on some neighbourhood of x. Suppose $x \in \Sigma$. Since Σ is nowhere dense in M, it follows from (1) of Proposition 6.1 that $\theta = 0$ on some neighbourhood of x as well.

As M is connected, we obtain that either $Y = \varnothing$ or $Y = M$. By Theorem 1.1 $\theta > 0$ on $M \setminus \Sigma$, whence $Y \neq M$ and therefore $Y = \varnothing$, so $\theta > 0$ on all of M.

Let $z \in \Sigma$. To establish (1.1) it suffices to prove that

(a) $j^1 F(z)$ has no eigen values λ with $\Re(\lambda) \neq 0$;

(b) a real normal Jordan form of $j^1 F(z)$ has neither a block $\mathbf{J}_q(\pm ib)$ nor $\mathbf{J}_q(0)$ with $q \geq 2$;

(c) $j^1 F(z) \neq 0$.

But if either of these conditions were violated, then it would follow from Proposition 6.1 that $\theta(z) = 0$. This contradiction completes Theorem 1.2.

8. Proof of Theorem 1.3. Let $z \in \Sigma$ be such that $j^1 F(z)$ is not similar to a matrix of the form (1.1). Then for this point z one of the conditions (1) – (3) of Theorem 6.1 holds true. Since every $x \in V \setminus \Sigma$ is periodic, it follows from Theorem 6.1 that there exists a sequence $\{x_i\}_{i \in \mathbb{N}} \subset V \setminus \Sigma$ converging to z and such $\lim\limits_{i \to \infty} \mathrm{Per}(x_i) = +\infty$.

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Received 22.02.10