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ALGEBRAIC DEPENDENCES OF MEROMORPHIC MAPPINGS IN SEVERAL COMPLEX VARIABLES*

АЛГЕБРАЇЧНА ЗАЛЕЖНІСТЬ МЕРОМОРФНИХ ВІДОБРАЖЕНЬ ДЛЯ БАГАТЬОХ КОМПЛЕКСНИХ ЗМІННИХ

In this article, some algebraic dependence theorems of meromorphic mappings in several complex variables into the complex projective spaces are given.

Наведено деякі теореми про алгебраїчну залежність мероморфних відображень для багатьох комплексних змінних на комплексні проективні простори.

1. Introduction. The theory on algebraic dependences of meromorphic mappings in several complex variables into the complex projective spaces for fixed targets is studied by Wilhelm Stoll [1]. Later, Min Ru [2] generalized Stoll's result to holomorphic curves into the complex projective spaces for moving targets and show some unicity theorems of holomorphic curves into the complex projective spaces for moving targets. As far as we know, they are the first results on the unicity problem for moving targets. We now state his remarkable results.

Let $g_0, \ldots, g_{q-1}$, $q \geq N$, be q meromorphic mappings of $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$ with reduced representations $g_j = (g_{j0} : \ldots : g_{jN}$, $0 \leqslant j \leqslant q-1$. We say that $g_0, \ldots, g_{q-1}$ are located in general position if $\det(g_{j_k l}) \not\equiv 0$ for any $0 \leqslant j_0 < j_1 < \ldots < j_N \leqslant q-1$.

Let $\mathcal{M}_n$ be the field of all meromorphic functions on $\mathbf{C}^n$. Denote by $\mathcal{R}\left(\{g_j\}_{j=0}^{q-1}\right) \subset \mathcal{M}_n$ the smallest subfield which contains $\mathbf{C}$ and all $\dfrac{g_{jk}}{g_{jl}}$ with $g_{jl} \not\equiv 0$.

Let f be a meromorphic mapping of $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$ with reduced representation $f = (f_0 : \ldots : f_N)$. We say that f is linearly nondegenerate over $\mathcal{R}\left(\{g_j\}_{j=0}^{q-1}\right)$ if $f_0, \ldots, f_N$ are linearly independent over $\mathcal{R}\left(\{g_j\}_{j=0}^{q-1}\right)$.

Let $f_t \colon \mathbf{C}^n \to \mathbf{P}^N(\mathbf{C})$, $1 \leqslant t \leqslant \lambda$, be meromorphic mappings with reduced representations $f_t := (f_{t0} : \ldots : f_{tN})$. Let $g_j \colon \mathbf{C}^n \to \mathbf{P}^N(\mathbf{C})$, $0 \leqslant j \leqslant q-1$, be moving targets located in general position with reduced representations $g_j := (g_{j0} : \ldots : g_{jN})$. Assume that $(f_t, g_j) := \sum_{i=0}^{N} f_{ti} g_{ji} \not\equiv 0$ for each $1 \leqslant t \leqslant \lambda$, $0 \leqslant j \leqslant q-1$ and $(f_1, g_j)^{-1}\{0\} = \ldots = (f_\lambda, g_j)^{-1}\{0\}$. Put $A_j = (f_1, g_j)^{-1}\{0\}$ for each $0 \leqslant j \leqslant q-1$. Assume that every analytic set A_j has the irriducible decomposition as follows $A_j = \cup_{i=1}^{t_j} A_{ji}$, $1 \leqslant t_j \leqslant \infty$. Set $A = \cup_{A_{ji} \not\equiv A_{kl}} \{A_{ji} \cap A_{kl}\}$ with $1 \leqslant i \leqslant t_j$, $1 \leqslant l \leqslant t_k$, $0 \leqslant j, k \leqslant q-1$.

Denote by $T[N+1, q]$ the set of all injective maps from $\{1, \ldots, N+1\}$ to $\{0, \ldots, q-1\}$. For each $z \in \mathbf{C}^n \setminus \{\cup_{\beta \in T[N+1,q]} \{z | g_{\beta(1)}(z) \wedge \ldots \wedge g_{\beta(N+1)}(z) = 0\} \cup A \cup \cup_{i=1}^{\lambda} I(f_i)\}$, we define $\rho(z) = \sharp\{j | z \in A_j\}$. Then $\rho(z) \leqslant N$. Indeed, suppose that $z \in A_j$ for each $0 \leqslant j \leqslant N$. Then $\sum_{i=0}^{N} f_{1i}(z) \cdot g_{ji}(z) = 0$ for each $0 \leqslant j \leqslant N$. Since

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$g_{\beta(1)}(z) \wedge \ldots \wedge g_{\beta(N+1)}(z) \neq 0$, it implies that $f_{1i}(z) = 0$ for each $0 \leqslant i \leqslant N$. This means that $z \in I(f_1)$. This is impossible.

For any positive number $r > 0$, define $\rho(r) = \sup\{\rho(z) \mid |z| \leqslant r\}$, where the supremum is taken over all $z \in \mathbf{C}^n \setminus \big\{ \cup_{\beta \in T[N+1,q]} \{z | g_{\beta(1)}(z) \wedge \ldots \wedge g_{\beta(N+1)}(z) = = 0\} \cup A \cup \cup_{i=1}^{\lambda} I(f_i)\big\}$. Then $\rho(r)$ is a decreasing function. Let

$$d := \lim_{r \to +\infty} \rho(r).$$

Then $d \leqslant N$. If for each $i \neq j$, $\dim\{A_i \cap A_j\} \leqslant n-2$, then $d = 1$.

Theorem A (see [2], Theorem 1)**.** *Let* $f_1, \ldots, f_\lambda \colon \mathbf{C} \to \mathbf{P}^N(\mathbf{C})$ *be nonconstant holomorphic curves. Let* $g_i \colon \mathbf{C} \to \mathbf{P}^N(\mathbf{C})$, $0 \leqslant i \leqslant q-1$, *be moving targets located in general position and* $T(r, g_i) = o(\max_{1 \leqslant j \leqslant \lambda} T(r, f_j))$, $0 \leqslant i \leqslant q-1$. *Assume that* $(f_i, g_j) \not\equiv 0$ *for* $1 \leqslant i \leqslant \lambda$, $0 \leqslant j \leqslant q-1$, *and* $A_j := (f_1, g_j)^{-1}\{0\} = \ldots \ldots = (f_\lambda, g_j)^{-1}\{0\}$ *for each* $0 \leqslant j \leqslant q-1$. *Denote* $\mathcal{A} = \cup_{j=0}^{q-1} A_j$. *Let* l, $2 \leqslant l \leqslant \lambda$, *be an integer such that for any increasing sequence* $1 \leqslant j_1 < \ldots < j_l \leqslant \lambda$, $f_{j_1}(z) \wedge \ldots \ldots \wedge f_{j_l}(z) = 0$ *for every point* $z \in \mathcal{A}$. *If* $q > \dfrac{dN^2(2N+1)\lambda}{\lambda - l + 1}$, *then* $f_1, \ldots, f_\lambda$ *are algebraically dependent over* $\mathbf{C}$, *i.e.,* $f_1 \wedge \ldots \wedge f_\lambda \equiv 0$ *on* $\mathbf{C}$.

Theorem B (see [2], Theorem 2)**.** *In addition to the assumption in Theorem A we assume further that* f_i, $1 \leqslant i \leqslant \lambda$, *are linearly nondegenerated. Then* $f_1, \ldots, f_\lambda$ *are algebraically dependent over* $\mathbf{C}$, *i.e.,* $f_1 \wedge \ldots \wedge f_\lambda \equiv 0$ *on* $\mathbf{C}$, *if* $q > \dfrac{dN(N+2)\lambda}{\lambda - l + 1}$.

With the same assumption on the nondegeneracy of small moving targets, it is our main purpose of the present paper to show some algebraic dependence theorems of meromorphic mappings from $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$ for moving targets in more general situations. Namely, we are going to prove the following.

Theorem 1. *Let* $f_1, \ldots, f_\lambda \colon \mathbf{C}^n \to \mathbf{P}^N(\mathbf{C})$ *be nonconstant meromorphic mappings. Let* $g_i \colon \mathbf{C}^n \to \mathbf{P}^N(\mathbf{C})$, $0 \leqslant i \leqslant q-1$, *be moving targets located in general position and* $T(r, g_i) = o(\max_{1 \leqslant j \leqslant \lambda} T(r, f_j))$, $0 \leqslant i \leqslant q-1$. *Assume that* $(f_i, g_j) \not\equiv 0$ *for* $1 \leqslant i \leqslant \lambda$, $0 \leqslant j \leqslant q-1$, *and* $A_j := (f_1, g_j)^{-1}\{0\} = \ldots = (f_\lambda, g_j)^{-1}\{0\}$ *for each* $0 \leqslant j \leqslant q-1$. *Denote* $\mathcal{A} = \cup_{j=0}^{q-1} A_j$. *Let* l, $2 \leqslant l \leqslant \lambda$, *be an integer such that for any increasing sequence* $1 \leqslant j_1 < \ldots < j_l \leqslant \lambda$, $f_{j_1}(z) \wedge \ldots \wedge f_{j_l}(z) = 0$ *for every point* $z \in \mathcal{A}$. *Then* $f_1, \ldots, f_\lambda$ *are algebraically dependent over* $\mathbf{C}$, *i.e.,* $f_1 \wedge \ldots \wedge f_\lambda \equiv 0$ *on* $\mathbf{C}$, *if* $q > \dfrac{dN(2N+1)\lambda}{\lambda - l + 1}$.

Theorem 2. *In addition to the assumption in Theorem* 1 *we assume further that* f_i, $1 \leqslant i \leqslant \lambda$, *are linearly nondegenerate over* $\mathcal{R}\left(\{g_j\}_{j=0}^{q-1}\right)$. *Then* $f_1, \ldots, f_\lambda$ *are algebraically dependent over* $\mathbf{C}$, *i.e.,* $f_1 \wedge \ldots \wedge f_\lambda \equiv 0$ *on* $\mathbf{C}$, *if* $q > \dfrac{dN(N+2)\lambda}{\lambda - l + 1}$.

Theorem 3. *Let* $f_1, \ldots, f_\lambda \colon \mathbf{C}^n \to \mathbf{P}^N(\mathbf{C})$ *be nonconstant meromorphic mappings. Let* $g_i \colon \mathbf{C}^n \to \mathbf{P}^N(\mathbf{C})$, $0 \leqslant i \leqslant q-1$, *be moving targets located in general position such that* $T(r, g_i) = o(\max_{1 \leqslant j \leqslant \lambda} T(r, f_j))$, $0 \leqslant i \leqslant q-1$, *and* $(f_i, g_j) \not\equiv 0$ *for* $1 \leqslant i \leqslant \lambda$, $0 \leqslant j \leqslant q-1$. *Let* $\varkappa$ *be a positive integer or* $\varkappa = \infty$ *and* $\overline{\varkappa} = \min\{\varkappa, N\}$. *Assume that the following conditions are satisfied*:

(i) $\min\{\varkappa, \nu_{(f_1, g_j)}\} = \ldots = \min\{\varkappa, \nu_{(f_\lambda, g_j)}\}$ *for each* $0 \leqslant j \leqslant q-1$,

(ii) $\dim\{z | (f_1, g_i)(z) = (f_1, g_j)(z) = 0\} \leqslant n-2$ *for each* $0 \leqslant i < j \leqslant q-1$,

(iii) *there exists an integer number* l, $2 \leqslant l \leqslant \lambda$, *such that for any increasing sequence* $1 \leqslant j_1 < \ldots < j_l \leqslant \lambda$, $f_{j_1}(z) \wedge \ldots \wedge f_{j_l}(z) = 0$ *for every point* $z \in \in \cup_{i=0}^{q-1}(f_1, g_i)^{-1}\{0\}$.

Then

(i) *If* $q > \dfrac{N(2N+1)\lambda - (\overline{\varkappa}-1)(\lambda-1)}{\lambda - l + 1}$, *then* $f_1, \ldots, f_\lambda$ *are algebraically dependent over* $\mathbf{C}$, *i.e.,* $f_1 \wedge \ldots \wedge f_\lambda \equiv 0$ *on* $\mathbf{C}$;

(ii) *if* f_i, $1 \leqslant i \leqslant \lambda$, *are linearly nondegenerate over* $\mathcal{R}\{g_j\}_{j=0}^{q-1}$ *and*

$$q > \frac{N(N+2)\lambda - (\overline{\varkappa}-1)(\lambda-1)}{\lambda - l + 1},$$

then $f_1, \ldots, f_\lambda$ *are algebraically dependent over* $\mathbf{C}$;

(iii) *if* f_i, $1 \leqslant i \leqslant \lambda$, *are linearly nondegenerate over* $\mathbf{C}$, g_i, $0 \leqslant i \leqslant q-1$, *are constant mappings and* $(q - N - 1)((\lambda-1)(\overline{\varkappa}-1) + q(\lambda - l + 1)) \leqslant qN\lambda$, *then* $f_1, \ldots, f_\lambda$ *are algebraically dependent over* $\mathbf{C}$.

2. Basic notions and auxiliary results from Nevanlinna theory. ***2.1.*** We set $\|z\| = \left(|z_1|^2 + \ldots + |z_n|^2\right)^{1/2}$ for $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$ and define

$$B(r) := \{z \in \mathbf{C}^n \colon \|z\| < r\}, \qquad S(r) := \{z \in \mathbf{C}^n \colon \|z\| = r\}, \quad 0 < r < \infty.$$

Define

$$v_{n-1}(z) := \left(dd^c\|z\|^2\right)^{n-1}$$

and

$$\sigma_n(z) := d^c\log\|z\|^2 \wedge \left(dd^c\log\|z\|^2\right)^{n-1} \quad \text{on} \quad \mathbf{C}^n \setminus \{0\}.$$

2.2. Let F be a nonzero holomorphic function on a domain Ω in $\mathbf{C}^n$. For a set $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers, we set $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $\mathcal{D}^\alpha F = = \dfrac{\partial^{|\alpha|}F}{\partial^{\alpha_1}z_1 \ldots \partial^{\alpha_n}z_n}$. We define the map $\nu_F \colon \Omega \to \mathbf{Z}$ by

$$\nu_F(z) := \max\left\{m : \mathcal{D}^\alpha F(z) = 0 \ \text{ for all } \ \alpha \ \text{ with } \ |\alpha| < m\right\}, \quad z \in \Omega.$$

We mean by a divisor on a domain Ω in $\mathbf{C}^n$ a map $\nu \colon \Omega \to \mathbf{Z}$ such that, for each $a \in \Omega$, there are nonzero holomorphic functions F and G on a connected neighbourhood $U \subset \Omega$ of a such that $\nu(z) = \nu_F(z) - \nu_G(z)$ for each $z \in U$ outside an analytic set of dimension $\leqslant n-2$. Two divisors are regarded as the same if they are identical outside an analytic set of dimension $\leqslant n-2$. For a divisor ν on Ω we set $|\nu| := \overline{\{z \colon \nu(z) \neq 0\}}$, which is a purely $(n-1)$-dimensional analytic subset of Ω or empty.

Take a nonzero meromorphic function φ on a domain Ω in $\mathbf{C}^n$. For each $a \in \Omega$, we choose nonzero holomorphic functions F and G on a neighbourhood $U \subset \Omega$ such that $\varphi = \dfrac{F}{G}$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leqslant n-2$, and we define the divisors ν_φ, ν_φ^∞ by $\nu_\varphi := \nu_F$, $\nu_\varphi^\infty := \nu_G$, which are independent of choices of F and G and so globally well-defined on Ω.

2.3. For a divisor ν on $\mathbf{C}^n$ and for a positive integer M or $M = \infty$, we define the counting function of ν by

$$\nu^{(M)}(z) = \min\{M, \nu(z)\},$$

$$n(t) = \begin{cases} \displaystyle\int_{|\nu|\,\cap B(t)} \nu(z) v_{n-1} & \text{if } \ n \geq 2, \\ \displaystyle\sum_{|z| \leqslant t} \nu(z) & \text{if } \ n = 1. \end{cases}$$

Similarly, we define $n^{(M)}(t)$.

Define

$$N(r,\nu) = \int_1^r \frac{n(t)}{t^{2n-1}} dt, \quad 1 < r < \infty.$$

Similarly, we define $N(r,\nu^{(M)})$ and denote them by $N^{(M)}(r,\nu)$ respectively.

Let $\varphi\colon \mathbf{C}^n \to \mathbf{C}$ be a meromorphic function. Define

$$N_\varphi(r) = N(r,\nu_\varphi), \qquad N_\varphi^{(M)}(r) = N^{(M)}(r,\nu_\varphi).$$

For brevity we will omit the character ${}^{(M)}$ if $M = \infty$.

2.4. Let $f\colon \mathbf{C}^n \to \mathbf{P}^N(\mathbf{C})$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \ldots : w_N)$ on $\mathbf{P}^N(\mathbf{C})$, we take a reduced representation $f = (f_0 : \ldots : f_N)$, which means that each f_i is a holomorphic function on $\mathbf{C}^n$ and $f(z) = \big(f_0(z) : \ldots : f_N(z)\big)$ outside the analytic set $\{f_0 = \ldots = f_N = 0\}$ of codimension ≥ 2. Set $\|f\| = \big(|f_0|^2 + \ldots + |f_N|^2\big)^{1/2}$.

The characteristic function of f is defined by

$$T(r,f) = \int_{S(r)} \log\|f\|\sigma_n - \int_{S(1)} \log\|f\|\sigma_n.$$

Let a be a meromorphic mapping of $\mathbf{C}^n$ into $\mathbf{P}^N(\mathbf{C})$ with reduced representation $a = (a_0 : \ldots : a_N)$. We define

$$m_{f,a}(r) = \int_{S(r)} \log\frac{\|f\|\|a\|}{|(f,a)|}\sigma_n - \int_{S(1)} \log\frac{\|f\|\|a\|}{|(f,a)|}\sigma_n,$$

where $\|a\| = \big(|a_0|^2 + \ldots + |a_N|^2\big)^{1/2}$.

If $f,\, a\colon \mathbf{C}^n \to \mathbf{P}^N(\mathbf{C})$ are meromorphic mappings such that $(f,a) \not\equiv 0$, then the first main theorem for moving targets in value distribution theory (see [3]) states

$$T(r,f) + T(r,a) = m_{f,a}(r) + N_{(f,a)}(r).$$

2.5. Let φ be a nonzero meromorphic function on $\mathbf{C}^n$, which are occationally regarded as a meromorphic map into $\mathbf{P}^1(\mathbf{C})$. The proximity function of φ is defined by

$$m(r,\varphi) := \int_{S(r)} \log\max\,(|\varphi|,1)\sigma_n.$$

2.6. As usual, by the notation $''\|\, P''$ we mean the assertion P holds for all $r \in$ $\in [0,\infty)$ excluding a Borel subset E of the interval $[0,\infty)$ with $\displaystyle\int_E dr < \infty$.

2.7. **The First Main Theorem for general position** [1, p. 326]**.** *Let* $f_i\colon \mathbf{C}^n \to \mathbf{P}^N(\mathbf{C})$, $1 \leqslant i \leqslant \lambda$, *be meromorphic mappings located in general position. Assume that* $1 \leqslant \lambda \leqslant N+1$. *Then*

$$N(r, \mu_{f_1\wedge\ldots\wedge f_\lambda}) + m(r, f_1 \wedge \ldots \wedge f_\lambda) \leqslant \sum_{1\leqslant i\leqslant\lambda} T(r, f_i) + O(1).$$

Let V be a complex vector space of dimension $N \geq 1$. The vectors $\{v_1, \ldots, v_k\}$ are said to be in general position if for each selection of integers $1 \leqslant i_1 < \ldots < i_p \leqslant k$ with $p \leqslant N$, then $v_{i_1} \wedge \ldots \wedge v_{i_p} \neq 0$. The vectors $\{v_1, \ldots, v_k\}$ are said to be in special position if they are not in general position. Take $1 \leqslant p \leqslant k$. Then $\{v_1, \ldots, v_k\}$ are said to be in p-special position if for each selection of integers $1 \leqslant i_1 < \ldots < i_p \leqslant k$, the vectors $v_{i_1}, \ldots, v_{i_p}$ are in special position.

2.8. **The Second Main Theorem for general position** ([1, p. 320], Theorem 2.1)**.** *Let* M *be a connected complex manifold of dimension* m. *Let* A *be a pure* $(m-1)$*-dimensional analytic subset of* M. *Let* V *be a complex vector space of dimension* $n+1 > 1$. *Let* p *and* k *be integers with* $1 \leqslant p \leqslant k \leqslant n+1$. *Let* $f_j\colon M \to P(V)$, $1 \leqslant j \leqslant k$, *be meromorphic mappings. Assume that* $f_1, \ldots, f_k$ *are in general position. Also assume that* $f_1, \ldots, f_k$ *are in* p*-special position on* A. *Then we have*

$$\mu_{f_1\wedge\ldots\wedge f_k} \geq (k - p + 1)\nu_A.$$

2.9. **The Second Main Theorem for moving target.** ***2.9.1*** ([4], Theorem 3.1). *Let* $f\colon \mathbf{C}^n \to \mathbf{P}^N(\mathbf{C})$ *be a meromorphic mapping. Let* $\{a_1, \ldots, a_q\}$, $q \neq 2$, *be a set of* q *meromorphic mappings of* $\mathbf{C}^n$ *into* $\mathbf{P}^N(\mathbf{C})$ *in general position such that* f *is linearly nondegenerate over* $\mathcal{R}\left(\{a_j\}_{j=1}^q\right)$. *Then*

$$\frac{q}{N+2}T(r,f) \leqslant \sum_{i=1}^{q} N^{(N)}_{(f,a_i)}(r) + O\left(\max_{0\leqslant i\leqslant q-1} T(r,a_i)\right) + o\big(T(r,f)\big).$$

2.9.2 ([5], Corollary 1)**.** *Let* $f\colon \mathbf{C}^n \to \mathbf{P}^N(\mathbf{C})$ *be a meromorphic mapping. Let* $\mathcal{A} = \{a_1, \ldots, a_q\}$, $q \geq 2N+1$, *be a set of* q *meromorphic mappings of* $\mathbf{C}^n$ *into* $\mathbf{P}^N(\mathbf{C})$ *located in general position such that* $(f, a_i) \not\equiv 0$ *for each* $1 \leqslant i \leqslant q$. *Then*

$$\frac{q}{2N+1}T(r,f) \leqslant \sum_{i=1}^{q} N^{(N)}_{(f,a_i)}(r) + O\left(\max_{1\leqslant i\leqslant q} T(r,a_i)\right) + O\big(\log^+ T(r,f)\big).$$

3. Proofs of main theorems. ***3.1. Proof of Theorem 1.*** It suffices to prove Theorem 1 in the case of $\lambda \leqslant N+1$.

Assume that $f_1 \wedge \ldots \wedge f_\lambda \not\equiv 0$. We denote by $\mu_{f_1\wedge\ldots\wedge f_\lambda}$ the divisor associated with $f_1 \wedge \ldots \wedge f_\lambda$. Denote by $N(r, \mu_{f_1\wedge\ldots\wedge f_\lambda})$ the counting function associated with the divisor $\mu_{f_1\wedge\ldots\wedge f_\lambda}$. We now prove the following.

Claim **3.1.1.** *For every* $1 \leqslant t \leqslant \lambda$, *we have*

$$\sum_{j=1}^{q} \min\{N, \nu_{(f_t,g_j)}(z)\} \leqslant \frac{dN}{\lambda - l + 1}\mu_{f_1\wedge\ldots\wedge f_\lambda}(z) + qN\sum_{\beta} \mu_{g_{\beta(1)}\wedge\ldots\wedge g_{\beta(N+1)}}(z)$$

for each $z \notin A \cup \cup_{i=1}^{\lambda} I(f_i)$, *where the sum is over all injective maps* $\beta\colon \{1, 2, \ldots, N+1\} \to \{1, 2, \ldots, q\}$.

We now prove Claim 3.1.1.

For each regular point $z_0 \in \mathcal{A} \setminus (A \cup \cup_{i=1}^{\lambda} I(f_i) \cup \cup_{\beta \in T[N+1,q]} \{z | g_{\beta(1)}(z) \wedge \ldots \ldots \wedge g_{\beta(N+1)}(z) = 0\})$, let S be an irriducible analytic subset of $\mathcal{A}$ containing z_0. Since $z_0 \notin A$ and $A = \cup_{A_{ij} \not\equiv A_{kl}} \{A_{ji} \cap A_{kl}\}$, where A_{ji} are the irriducible components of $A_j = (f_1, g_j)^{-1}\{0\}$, it implies that S is a pure $(n-1)$-dimentional analytic subset and hence, S is only contained in at most d sets of A_j. Thus $\nu_{(f_t,g_j)}(z_0) \neq 0$ at most d indices. We have

$$\sum_{j=1}^{q} \min\{N, \nu_{(f_t,g_j)}(z_0)\} \leqslant dN.$$

For each increasing sequence $1 \leqslant j_1 < \ldots < j_l \leqslant \lambda$, we have

$$f_{j_1}(z) \wedge \ldots \wedge f_{j_l}(z) = 0 \quad \forall z \in S.$$

This implies that the fimily $\{f_1, \ldots, f_\lambda\}$ is in l-special position on S. By the Second Main Theorem for general position [1, p. 320] (Theorem 2.1), we have

$$\mu_{f_1 \wedge \ldots \wedge f_\lambda}(z) \geq (\lambda - (l-1))\nu_S.$$

By the properties of divisor, we have

$$\mu_{f_1 \wedge \ldots \wedge f_\lambda}(z_0) \geq \lambda - l + 1.$$

Hence

$$\sum_{j=1}^{q} \min\{N, \nu_{(f_t,g_j)}(z_0)\} \leqslant dN \leqslant \frac{dN}{\lambda - l + 1} \mu_{f_1 \wedge \ldots \wedge f_\lambda}(z_0).$$

If $z_0 \in \cup_{\beta \in T[N+1,q]} \{z | g_{\beta(1)}(z) \wedge \ldots \wedge g_{\beta(N+1)}(z) = 0\}$, then we have

$$\sum_{j=1}^{q} \min\{N, \nu_{(f_t,g_j)}(z_0)\} \leqslant qN \leqslant qN \sum_{\beta \in T[N+1,q]} \mu_{g_{\beta(1)} \wedge \ldots \wedge g_{\beta(N+1)}}(z_0).$$

From the above cases and by the properties of divisor, for each $z \notin A \cup \cup_{i=1}^{\lambda} I(f_i)$, we have

$$\sum_{j=1}^{q} \min\{N, \nu_{(f_t,g_j)}(z)\} \leqslant$$

$$\leqslant \frac{dN}{\lambda - l + 1} \mu_{f_1 \wedge \ldots \wedge f_\lambda}(z) + qN \sum_{\beta \in T[N+1,q]} \mu_{g_{\beta(1)} \wedge \ldots \wedge g_{\beta(N+1)}}(z).$$

Claim 3.1.1 is proved.

The above assertions and The First Main Theorem for general position [1, p. 326], yield that

$$\sum_{j=1}^{q} N_{(f_t,g_j)}^{(N)}(r) \leqslant$$

$$\leqslant \frac{dN}{\lambda - l + 1} N(r, \mu_{f_1 \wedge \ldots \wedge f_\lambda}) + qN \sum_{\beta \in T[N+1,q]} N(r, \mu_{g_{\beta(1)} \wedge \ldots \wedge g_{\beta(N+1)}}) \leqslant$$

$$\leqslant \frac{dN}{\lambda - l + 1} \sum_{i=1}^{\lambda} T(r, f_i) + qN \sum_{\beta \in T[N+1,q]} \sum_{i=1}^{N+1} T(r, g_{\beta(i)}) =$$

$$= \frac{dN}{\lambda - l + 1} \sum_{i=1}^{\lambda} T(r, f_i) + o\left(\max_{1 \leqslant i \leqslant \lambda} \{T(r, f_i)\} \right).$$

Thus, by summing them up, we have

$$\sum_{t=1}^{\lambda} \sum_{j=1}^{q} N^{(N)}_{(f_t, g_j)}(r) \leqslant \frac{dN\lambda}{\lambda - l + 1} \sum_{i=1}^{\lambda} T(r, f_i) + o\left(\max_{1 \leqslant i \leqslant \lambda} \{T(r, f_i)\} \right). \qquad (1)$$

By using the Second Main Theorem for moving targets [5] (Corollary 1), it implies that

$$\sum_{t=1}^{\lambda} \frac{q}{2N+1} T(r, f_t) \leqslant \frac{dN\lambda}{\lambda - l + 1} \sum_{i=1}^{\lambda} T(r, f_i) + o\left(\max_{1 \leqslant i \leqslant \lambda} \{T(r, f_i)\} \right).$$

Letting $r \to +\infty$, we get $q \leqslant \dfrac{dN(2N+1)\lambda}{\lambda - l + 1}$. This is a contradiction. Thus, the family $\{f_1, \ldots, f_\lambda\}$ is algebraically dependent over $\mathbf{C}^n$, i.e., $f_1 \wedge \ldots \wedge f_\lambda = 0$.

Theorem 1 is proved.

***3.2. Proof of Theorem* 2.** From (1), we have

$$\sum_{t=1}^{\lambda} \sum_{j=1}^{q} N^{(N)}_{(f_t, g_j)}(r) \leqslant \frac{dN\lambda}{\lambda - l + 1} \sum_{i=1}^{\lambda} T(r, f_i) + o\left(\max_{1 \leqslant i \leqslant \lambda} \{T(r, f_i)\} \right).$$

By using the Second Main Theorem for moving targets [4] (Theorem 3.1), it implies that

$$\sum_{t=1}^{\lambda} \frac{q}{N+2} T(r, f_t) \leqslant \frac{dN\lambda}{\lambda - l + 1} \sum_{i=1}^{\lambda} T(r, f_i) + o\left(\max_{1 \leqslant i \leqslant \lambda} \{T(r, f_i)\} \right).$$

Letting $r \to +\infty$, we get $q \leqslant \dfrac{dN(N+2)\lambda}{\lambda - l + 1}$. This is a contradiction. Thus, the family $\{f_1, \ldots, f_\lambda\}$ is algebraically dependent over $\mathbf{C}^n$, i.e. $f_1 \wedge \ldots \wedge f_\lambda = 0$.

Theorem 2 is proved.

***3.3. Proof of Theorem* 3.** It suffices to prove Theorem 3 in the case of $\lambda \leqslant N+1$. Assume that $f_1 \wedge \ldots \wedge f_\lambda \not\equiv 0$.

We now prove the following.

***Claim* 3.3.1.** *For any* $\lambda - 1$ *moving targets* $g_{j_1}, \ldots, g_{j_{\lambda-1}} \in \{g_j\}_{j=1}^q$, *there exists* $g_{j_0} \notin \{g_{j_1}, \ldots, g_{j_{\lambda-1}}\}$ *such that*

$$\det \begin{pmatrix} (f_1, g_{j_1}) & \ldots & (f_\lambda, g_{j_1}) \\ \vdots & \vdots & \vdots \\ (f_1, g_{j_{\lambda-1}}) & \ldots & (f_\lambda, g_{j_{\lambda-1}}) \\ (f_1, g_{j_0}) & \ldots & (f_\lambda, g_{j_0}) \end{pmatrix} \not\equiv 0.$$

We now prove Claim 3.3.1.

Suppose on contrary. Without loss of generality, we assume $g_{j_1} = g_1, \ldots, g_{j_{\lambda-1}=}g_{\lambda-1}$. Then

$$\operatorname{rank}\begin{pmatrix} (f_1, g_1)(z) & \ldots & (f_\lambda, g_1)(z) \\ \vdots & \vdots & \vdots \\ (f_1, g_{N+1})(z) & \ldots & (f_\lambda, g_{N+1})(z) \end{pmatrix} \leqslant \lambda - 1$$

for each $z \in \mathbf{C}^n$. By $f_1 \wedge \ldots \wedge f_\lambda \not\equiv 0$, there exists $z_0 \in \mathbf{C}^n$ such that $f_1(z_0) \wedge \ldots \ldots \wedge f_\lambda(z_0) \neq 0$ and $z_0 \notin \{g_1 \wedge \ldots \wedge g_{N+1}\}^{-1}(0)$. On the other hand, we have

$$\begin{pmatrix} (f_1, g_1)(z_0) & \ldots & (f_\lambda, g_1)(z_0) \\ \vdots & \vdots & \vdots \\ (f_1, g_{N+1})(z_0) & \ldots & (f_\lambda, g_{N+1})(z_0) \end{pmatrix} =$$

$$= \begin{pmatrix} g_{10}(z_0) & \ldots & g_{1N}(z_0) \\ \vdots & \vdots & \vdots \\ g_{N+10}(z_0) & \ldots & g_{N+1N}(z_0) \end{pmatrix} \begin{pmatrix} f_{10}(z_0) & \ldots & f_{\lambda 0}(z_0) \\ \vdots & \vdots & \vdots \\ f_{1N}(z_0) & \ldots & f_{\lambda N}(z_0) \end{pmatrix}.$$

Since the family $\{g_j\}_{j=1}^q$ is located in general position, is implies that the matrix

$$\begin{pmatrix} f_{10}(z_0) & \ldots & f_{\lambda 0}(z_0) \\ \vdots & \vdots & \vdots \\ f_{1N}(z_0) & \ldots & f_{\lambda N}(z_0) \end{pmatrix}$$

is of rank $\leqslant \lambda - 1$. This is a contradiction.

The Claim 3.3.1 is proved.

We now consider $\lambda - 1$ moving targets $g_1, \ldots, g_{\lambda-1}$. Then, by Claim 3.3.1, there exists g_{j_0} with $j_0 > \lambda - 1$ such that

$$\det\begin{pmatrix} (f_1, g_1) & \ldots & (f_\lambda, g_1) \\ \vdots & \vdots & \vdots \\ (f_1, g_{\lambda-1}) & \ldots & (f_\lambda, g_{\lambda-1}) \\ \\ (f_1, g_{j_0}) & \ldots & (f_\lambda, g_{j_0}) \end{pmatrix} \not\equiv 0.$$

Without loss of generality, we may assume that $j_0 = \lambda$.

Now we put $\mathcal{A} := \cup_{j=1}^{\lambda}(f_1, g_j)^{-1}\{0\}$, $A = \cup_{1 \leqslant i < j \leqslant \lambda}\big((f_1, g_i)^{-1}\{0\} \cap (f_1, g_j)^{-1}\{0\}\big)$.

We now show the following.

***Claim* 3.3.2.** *For each* $1 \leqslant t \leqslant \lambda$, *we have*

$$\sum_{i=1}^{\lambda}\left(\min\left\{\varkappa, \nu_{(f_t, g_i)}(z)\right\} + (\lambda - l)\min\left\{1, \nu_{(f_t, g_i)}(z)\right\}\right) +$$

$$+ \sum_{i=\lambda+1}^{q}(\lambda - l + 1)\min\{1, \nu_{(f_t, g_i)}(z)\} \leqslant$$

$$\leqslant \mu_{\tilde{f}_1 \wedge \ldots \wedge \tilde{f}_\lambda}(z) + (\lambda(\varkappa + \lambda - l) + (q - \lambda)(\lambda - l + 1))\mu_{g_1 \wedge \ldots \wedge g_\lambda}(z) \qquad (2)$$

for every $z \in \mathbf{C}^n \setminus (A \cup \cup_{i=1}^{\lambda} I(f_i))$, *where* $\tilde{f}_i := ((f_i, g_1) : \ldots : (f_i, g_\lambda))$ *for each* $1 \leqslant i \leqslant \lambda$. *Furthermore we have*

$$\sum_{i=1}^{\lambda} \Big(N^{(\varkappa)}_{(f_t,g_i)}(r) + (\lambda - l) N^{(1)}_{(f_t,g_i)}(r) \Big) +$$

$$+ \sum_{i=\lambda+1}^{q} (\lambda - l + 1) N^{(1)}_{(f_t,g_i)}(r) \leqslant$$

$$\leqslant \sum_{i=1}^{\lambda} T(r, f_i) + o\left(\max_{1 \leqslant i \leqslant \lambda} \{T(r, f_i)\} \right). \tag{3}$$

We now prove Claim 3.3.2.

By the properties of divisor, we only consider three cases for regular points.

***Case* 1.** Let $z_0 \in \mathcal{A} \setminus (A \cup \cup_{i=1}^{\lambda} I(f_i) \cup \{z | g_1 \wedge \ldots \wedge g_\lambda(z) = 0\})$ be a regular point of $\mathcal{A}$. Then z_0 is only a zero of one of the meromorphic functions $\{(f_t, g_j)\}_{j=1}^{\lambda}$. Without loss of generality, we may assume that z_0 is a zero of (f_t, g_1). Let S be an irriducible analytic subset of $\mathcal{A}$, containing z_0. Then the pure dimension of S is $n-1$. Suppose that U is an open neighbourhood of z_0 in $\mathbf{C}^n$ such that $U \cap \{\mathcal{A} \setminus S\} = \varnothing$. Choose a holomorphic function h on $\mathbf{C}^n$ such that $\nu_h = \min\{\varkappa, \nu_{(f_t,g_1)}\}$ if $z \in S$ and $\nu_h = 0$ if $z \notin S$. Then $(f_i, g_1) = a_i h$, $1 \leqslant i \leqslant \lambda$, where a_i are holomorphic functions. Since the matrix $\begin{pmatrix} (f_1, g_2)(z) & \ldots & (f_\lambda, g_2)(z) \\ \vdots & \vdots & \vdots \\ (f_1, g_\lambda)(z) & \ldots & (f_\lambda, g_\lambda)(z) \end{pmatrix}$ is of rank $\leqslant \lambda - 1$ for each $z \in \mathbf{C}^n$, it implies that there exist holomorphic functions $b_1, \ldots, b_\lambda$ such that there is at least $b_i \not\equiv 0$ and

$$\sum_{i=1}^{\lambda} b_i (f_i, g_j) = 0, \quad 2 \leqslant j \leqslant \lambda.$$

Without loss of generality, we may assume that the set of common zeros of $\{b_i\}_{i=1}^{\lambda}$ is an analytic subset of codimension ≥ 2. Then there exist an index i_1, $1 \leqslant i_1 \leqslant \lambda$ such that $S \not\subset b_{i_1}^{-1}\{0\}$. We can assume that $i_1 = \lambda$. Then for each $z \in (U \cap S) \setminus b_\lambda^{-1}\{0\}$, we have

$$\tilde{f}_1(z) \wedge \ldots \wedge \tilde{f}_\lambda(z) = \tilde{f}_1(z) \wedge \ldots \wedge \tilde{f}_{\lambda-1}(z) \wedge \left(\tilde{f}_\lambda(z) + \sum_{i=1}^{\lambda-1} \frac{b_i}{b_\lambda} \tilde{f}_i(z) \right) =$$

$$= \tilde{f}_1(z) \wedge \ldots \wedge \tilde{f}_{\lambda-1}(z) \wedge (V(z) h(z)) =$$

$$= h(z) \Big(\tilde{f}_1(z) \wedge \ldots \wedge \tilde{f}_{\lambda-1}(z) \wedge V(z) \Big),$$

where $V(z) := \left(a_\lambda + \sum_{i=1}^{\lambda-1} \frac{b_i}{b_\lambda} a_i, 0, \ldots, 0 \right)$.

By assumption, for any increasing sequence $1 \leqslant j_1 < \ldots < j_l \leqslant \lambda - 1$, we have $f_{j_1} \wedge \ldots \wedge f_{j_l} \equiv 0$ on S. Then

$$\operatorname{rank}(f_{j_1}(z),\ldots,f_{j_l}(z)) = \operatorname{rank}\begin{pmatrix} f_{j_1 0}(z) & \ldots & f_{j_l 0}(z) \\ \vdots & \vdots & \vdots \\ f_{j_1 N}(z) & \ldots & f_{j_l N}(z) \end{pmatrix} \leqslant l-1 \quad \forall z \in S.$$

On the other hand

$$\begin{pmatrix} (f_{j_1}, g_1)(z) & \ldots & (f_{j_l}, g_1)(z) \\ \vdots & \vdots & \vdots \\ (f_{j_1}, g_\lambda)(z) & \ldots & (f_{j_l}, g_\lambda)(z) \end{pmatrix} =$$

$$= \begin{pmatrix} g_{10}(z) & \ldots & g_{1N}(z) \\ \vdots & \vdots & \vdots \\ g_{\lambda 0}(z) & \ldots & g_{\lambda N}(z) \end{pmatrix} \begin{pmatrix} f_{j_1 0}(z) & \ldots & f_{j_l 0}(z) \\ \vdots & \vdots & \vdots \\ f_{j_1 N}(z) & \ldots & f_{j_l N}(z) \end{pmatrix}.$$

Hence

$$\operatorname{rank}\left(\tilde{f}_{j_1}(z),\ldots,\tilde{f}_{j_l}(z)\right) =$$

$$= \operatorname{rank}\begin{pmatrix} (f_{j_1}, g_1)(z) & \ldots & (f_{j_l}, g_1)(z) \\ \vdots & \vdots & \vdots \\ (f_{j_1}, g_\lambda)(z) & \ldots & (f_{j_l}, g_\lambda)(z) \end{pmatrix} \leqslant l-1 \quad \forall z \in S.$$

Therefore, $\tilde{f}_{j_1} \wedge \ldots \wedge \tilde{f}_{j_l} \equiv 0$ on S. This implies that the family $\{\tilde{f}_1, \ldots \tilde{f}_{\lambda-1}\}$ is in l-special position on S, and $\{\tilde{f}_1, \ldots \tilde{f}_{\lambda-1}, V\}$ is in $(l+1)$-special position on S. By using The Second Main Theorem for general position [1, p. 320] (Theorem 2.1), we have

$$\mu_{\tilde{f}_1 \wedge \ldots \wedge \tilde{f}_{\lambda-1} \wedge V}(z) \geq (\lambda - l)\nu_S \quad \forall z \in S.$$

Hence

$$\mu_{\tilde{f}_1 \wedge \ldots \wedge \tilde{f}_\lambda}(z) \geq \nu_h(z) + (\lambda - l)\nu_S =$$

$$= \min\{\varkappa, \nu_{(f_t, g_1)}(z)\} + (\lambda - l)\nu_S, \forall z \in (U \cup S) \setminus b_{i_1}^{-1}\{0\}.$$

By the properties of divisors, we have

$$\mu_{\tilde{f}_1 \wedge \ldots \wedge \tilde{f}_\lambda}(z_0) \geq \min\{\varkappa, \nu_{(f_t, g_1)}(z_0)\} + \lambda - l.$$

This implies that

$$\sum_{i=1}^{\lambda} \left(\min\{\varkappa, \nu_{(f_t, g_i)}(z_0)\} + (\lambda - l)\min\{1, \nu_{(f_t, g_i)}(z_0)\}\right) +$$

$$+ \sum_{i=\lambda+1}^{q} (\lambda - l + 1)\min\{1, \nu_{(f_t, g_i)}(z_0)\} =$$

$$= \min\{\varkappa, \nu_{(f_t, g_1)}(z_0)\} + \lambda - l \leqslant$$

$$\leqslant \mu_{\tilde{f}_1 \wedge \ldots \wedge \tilde{f}_\lambda}(z_0) + (\lambda(\varkappa + \lambda - l) + (q - \lambda)(\lambda - l + 1))\mu_{g_1 \wedge \ldots \wedge g_\lambda}(z_0).$$

***Case* 2.** Let $z_0 \in \mathcal{A} \setminus (A \cup \cup_{i=1}^{\lambda} I(f_i) \cup \{z | g_1 \wedge \ldots \wedge g_\lambda(z) = 0\})$ be a regular point of $\mathcal{A}$. Then z_0 is only a zero of $(f_t, g_i), i > \lambda$. By the assumption, we have the family $\{\tilde{f}_1, \ldots, \tilde{f}_\lambda\}$ is in l-special position on an irriducible analytic subset of codimension 1 of $\mathcal{A}$ which containing z_0. By using The Second Main Theorem for general position [1, p. 320] (Theorem 2.1), we have

$$\mu_{\tilde{f}_1 \wedge \ldots \wedge \tilde{f}_\lambda}(z_0) \geq \lambda - l + 1.$$

Hence

$$\sum_{i=1}^{\lambda} \big(\min\{\varkappa, \nu_{(f_t,g_i)}(z_0)\} + (\lambda - l) \min\{1, \nu_{(f_t,g_i)}(z_0)\}\big) +$$

$$+ \sum_{i=\lambda+1}^{q} (\lambda - l + 1) \min\{1, \nu_{(f_t,g_i)}(z_0)\} =$$

$$= (\lambda - l + 1) \min\{1, \nu_{(f_t,g_i)}(z_0)\} =$$

$$= \lambda - l + 1 \leqslant \mu_{\tilde{f}_1 \wedge \ldots \wedge \tilde{f}_\lambda}(z_0) + (\lambda(\varkappa + \lambda - l) +$$

$$+ (q - \lambda)(\lambda - l + 1))\mu_{g_1 \wedge \ldots \wedge g_\lambda}(z_0)).$$

***Case* 3.** Assume that $z_0 \in (g_1 \wedge \ldots \wedge g_\lambda)^{-1}\{0\}$. Then

$$\sum_{i=1}^{\lambda} \big(\min\{\varkappa, \nu_{(f_t,g_i)}(z_0)\} + (\lambda - l) \min\{1, \nu_{(f_t,g_i)}(z_0)\}\big) +$$

$$+ \sum_{i=\lambda+1}^{q} (\lambda - l + 1) \min\{1, \nu_{(f_t,g_i)}(z_0)\} \leqslant$$

$$\leqslant \lambda(\varkappa + (\lambda - l)) + (q - \lambda)(\lambda - l + 1) \leqslant$$

$$\leqslant \mu_{\tilde{f}_1 \wedge \ldots \wedge \tilde{f}_\lambda}(z_0) + (\lambda(\varkappa + \lambda - l) + (q - \lambda)(\lambda - l + 1))\mu_{g_1 \wedge \ldots \wedge g_\lambda}(z_0)).$$

From the above cases and by the properties of divisors, for each $z \notin A \cup_{i=1}^{\lambda} I(f_i)$, we have

$$\sum_{i=1}^{\lambda} \big(\min\{\varkappa, \nu_{(f_t,g_i)}(z)\} + (\lambda - l) \min\{1, \nu_{(f_t,g_i)}(z)\}\big) +$$

$$+ \sum_{i=\lambda+1}^{q} (\lambda - l + 1) \min\{1, \nu_{(f_t,g_i)}(z)\} \leqslant$$

$$\leqslant \mu_{\tilde{f}_1 \wedge \ldots \wedge \tilde{f}_\lambda}(z) + (\lambda(\varkappa + \lambda - l) + (q - \lambda)(\lambda - l + 1))\mu_{g_1 \wedge \ldots \wedge g_\lambda}(z).$$

The first assertion of Claim 3.3.2 is proved.

By the assumption and definiton of the characteristic function, for each $1 \leqslant j \leqslant \lambda$, we have

$$T(r, \tilde{f_j}) \leqslant T(r, f_j) + o\left(\max_{1 \leqslant j \leqslant \lambda} \{T(r, f_i)\}\right).$$

By The First Main Theorem for general position [1, p. 326], it implies that

$$\sum_{i=1}^{\lambda} \left(N^{(\varkappa)}_{(f_t, g_i)}(r) + (\lambda - l) N^{(1)}_{(f_t, g_i)}(r)\right) + \sum_{i=\lambda+1}^{q} (\lambda - l + 1) N^{(1)}_{(f_t, g_i)}(r) \leqslant$$

$$\leqslant N(r, \mu_{\tilde{f}_1 \wedge \ldots \wedge \tilde{f}_\lambda})(r) + \Big(\lambda(\varkappa + \lambda - l) + (q - \lambda)(\lambda - l + 1)\Big) N_{\mu_{g_1 \wedge \ldots \wedge g_\lambda}}(r) \leqslant$$

$$\leqslant \sum_{i=1}^{\lambda} T(r, \tilde{f_i}) + \Big(\lambda(\varkappa + \lambda - l) + (q - \lambda)(\lambda - l + 1)\Big) \sum_{i=1}^{\lambda} T(r, g_i) + O(1) \leqslant$$

$$\leqslant \sum_{i=1}^{\lambda} T(r, f_i) + o\left(\max_{1 \leqslant i \leqslant \lambda} \{T(r, f_i)\}\right).$$

The second assertion of Claim 3.3.2 is proved.

Thus, for any increasing sequence $1 \leqslant i_1 < \ldots < i_{\lambda-1} \leqslant q$, we have

$$\sum_{j=1}^{\lambda-1} \left(N^{(\varkappa)}_{(f_t, g_{i_j})}(r) + (\lambda - l) N^{(1)}_{(f_t, g_{i_j})}(r)\right) + \sum_{i \in I} (\lambda - l + 1) N^{(1)}_{(f_t, g_i)}(r) \leqslant$$

$$\leqslant \sum_{i=1}^{\lambda} T(r, f_i) + o\left(\max_{1 \leqslant i \leqslant \lambda} \{T(r, f_i)\}\right),$$

where $I = \{1, 2, \ldots, q\} \setminus \{i_1, \ldots, i_{\lambda-1}\}$.

Thus, by summing-up them over all sequences $1 \leqslant i_1 < \ldots < i_{\lambda-1} \leqslant q$, we have

$$\sum_{i=1}^{q} ((\lambda - 1) N^{(\varkappa)}_{(f_t, g_i)}(r) + ((\lambda - 1)(\lambda - l) +$$

$$+ (q - \lambda + 1)(\lambda - l + 1)) N^{(1)}_{(f_t, g_i)}(r)) \leqslant$$

$$\leqslant q \sum_{i=1}^{\lambda} T(r, f_i) + o\left(\max_{1 \leqslant i \leqslant \lambda} \{T(r, f_i)\}\right).$$

Since $\overline{\varkappa} N^{(N)}_f(r) \leqslant N N^{(\varkappa)}_f(r) \ \ \forall \varkappa$, we have

$$\sum_{i=1}^{q} \Big((\lambda - 1) \overline{\varkappa} N^{(N)}_{(f_t, g_i)}(r) + ((\lambda - 1)(\lambda - l) +$$

$$+ (q - \lambda + 1)(\lambda - l + 1)) N^{(N)}_{(f_t, g_i)}(r)\Big) \leqslant$$

$$\leqslant qN\sum_{i=1}^{\lambda}T(r,f_i)+o\left(\max_{1\leqslant i\leqslant\lambda}\{T(r,f_i)\}\right).$$

This implies that

$$\sum_{i=1}^{q}\Big((\lambda-1)\overline{\varkappa}+(\lambda-1)(\lambda-l)+(q-\lambda+1)(\lambda-l+1)\Big)N^{(N)}_{(f_t,g_i)}(r)\leqslant$$

$$\leqslant qN\sum_{i=1}^{\lambda}T(r,f_i)+o\left(\max_{1\leqslant i\leqslant\lambda}\{T(r,f_i)\}\right).$$

Thus, by summing them up over all t $(1\leqslant t\leqslant\lambda)$, we have

$$\sum_{i=1}^{q}\sum_{t=1}^{\lambda}\Big((\lambda-1)\overline{\varkappa}+(\lambda-1)(\lambda-l)+(q-\lambda+1)(\lambda-l+1)\Big)N^{(N)}_{(f_t,g_i)}(r)\leqslant$$

$$\leqslant qN\lambda\sum_{i=1}^{\lambda}T(r,f_i)+o\left(\max_{1\leqslant i\leqslant\lambda}T(r,f_i)\right). \tag{4}$$

We now prove the assertions of Theorem 3.

i) By applying the Second Main Theorem for moving targets [5] (Corollary 1) to the left-hand side of (4), it implies that

$$\frac{q}{2N+1}\sum_{i=1}^{\lambda}\Big((\lambda-1)\overline{\varkappa}+(\lambda-1)(\lambda-l)+(q-\lambda+1)(\lambda-l+1)\Big)T(r,f_i)\leqslant$$

$$\leqslant qN\lambda\sum_{i=1}^{\lambda}T(r,f_i)+o\left(\max_{1\leqslant i\leqslant\lambda}\{T(r,f_i)\}\right).$$

Letting $r\to+\infty$, we have

$$q\leqslant\lambda-1+\frac{(2N+1)N\lambda-(\lambda-1)\overline{\varkappa}-(\lambda-1)(\lambda-l)}{\lambda-l+1}=$$

$$=\frac{(2N+1)N\lambda-(\lambda-1)(\overline{\varkappa}-1)}{\lambda-l+1}.$$

This is a contradiction. Thus, we have $f_1\wedge\ldots\wedge f_\lambda\equiv 0$.

ii) By applying the Second Main Theorem for moving targets [4] (Theorem 3.1) to the left-hand side of (4), it implies that

$$\frac{q}{N+2}\sum_{i=1}^{\lambda}((\lambda-1)\overline{\varkappa}+(\lambda-1)(\lambda-l)+(q-\lambda+1)(\lambda-l+1))T(r,f_i)\leqslant$$

$$\leqslant qN\lambda\sum_{i=1}^{\lambda}T(r,f_i)+o\left(\max_{1\leqslant j\leqslant\lambda}\{T(r,f_i)\}\right).$$

Letting $r \to +\infty$, we have

$$q \leqslant \lambda - 1 + \frac{(N+2)N\lambda - (\lambda-1)\varkappa - (\lambda-1)(\lambda-l)}{\lambda - l + 1} =$$

$$= \frac{(N+2)N\lambda - (\lambda-1)(\varkappa-1)}{\lambda - l + 1}.$$

This is a contradiction. Thus, we have $f_1 \wedge \ldots \wedge f_\lambda \equiv 0$.

iii) By applying the Second Main Theorem for hyperplanes in general position [6, p. 304] to the left-hand side of (4), it implies that

$$(q-N-1)\sum_{i=1}^{\lambda}((\lambda-1)\varkappa + (\lambda-1)(\lambda-l) + (q-\lambda+1)(\lambda-l+1))T(r,f_i) \leqslant$$

$$\leqslant qN\lambda \sum_{i=1}^{\lambda} T(r,f_i) + o\left(\max_{1\leqslant j\leqslant\lambda}\{T(r,f_i)\}\right).$$

Letting $r \to +\infty$, we have

$$(q-N-1)((\lambda-1)(\varkappa-1) + q(\lambda-l+1)) \leqslant qN\lambda.$$

This is a contradiction. Thus, we have $f_1 \wedge \ldots \wedge f_\lambda \equiv 0$.

Theorem 3 is proved.

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