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ON THE CONVERGENCE OF SOLUTIONS OF CERTAIN NON-HOMOGENEOUS FOURTH ORDER DIFFERENTIAL EQUATIONS

ПРО ЗБІЖНІСТЬ РОЗВ'ЯЗКІВ ДЕЯКИХ НЕОДНОРІДНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ЧЕТВЕРТОГО ПОРЯДКУ

The main purpose of this paper is to give sufficient conditions for the convergence of solutions of a certain class of fourth order nonlinear differential equations with the use of Lyapunov's second method. Nonlinear functions involved are not necessarily differentiable, but a function h satisfies a certain incremental ratio that lie in the closed sub-interval of the Routh – Hurwitz interval.

Головною метою статті є наведення достатніх умов для збіжності розв'язків деякого класу нелінійних диференціальних рівнянь четвертого порядку з використанням другого методу Ляпунова. Розглядувані нелінійні функції необов'язково диференційовні, але функція h задовольняє деяке відношення приростів, що лежать у замкненому підінтервалі інтервалу Рута – Гурвіца.

1. Introduction. The convergence of solutions is very important in the theory and applications of differential equations. In the recent years, the convergence problem has been the subject of investigation by a number of authors for various forms and orders of equations (see, for example, [1 – 8]). In this connection, Afuwape [2] discussed the convergence of the solutions of the differential equations of the form

$$x^{(iv)} + a\dddot{x} + b\ddot{x} + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x})$$

with $p(t, x, \dot{x}, \ddot{x}, \dddot{x})$ is equals to $q(t) + r(t, x, \dot{x}, \ddot{x}, \dddot{x})$. During establishment of the results, Afuwape [2] assumed that h was not necessarily differentiable but satisfied an incremental ratio $\eta^{-1}\left(h(\xi+\eta) - h(\xi)\right), \eta \neq 0$, which lies in a closed subinterval I_0 of the Routh – Hurwitz interval $\left(0, \dfrac{(ab-c)\,c}{a^2}\right)$, where

$$I_0 \equiv \left[\Delta_0, \frac{K\,(ab-c)\,c}{a^2}\right], \tag{1}$$

$\Delta_0 > 0$ and $K < 1$.

In this work, we shall be concerned here with equation of the form

$$x^{(iv)} + f(\dddot{x}) + b\ddot{x} + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}), \tag{2}$$

where b is a positive constant, the functions f, g, h and p are real-valued and continuous for values of their respective arguments and dots denote differentiation with respect to t. Moreover, $f(0) = g(0) = h(0) = 0$. Using Lyapunov's second method, our results assert the existence of convergence of solutions with the functions f, g and h are not necessarily differentiable.

Definition. *Any two solutions $x_1(t), x_2(t)$ of the equation* (2) *are said to converge to each other if*

$$x_2(t) - x_1(t) \to 0, \qquad \dot{x}_2(t) - \dot{x}_1(t) \to 0, \qquad \ddot{x}_2(t) - \ddot{x}_1(t) \to 0,$$
$$\dddot{x}_2(t) - \dddot{x}_1(t) \to 0 \quad \textit{as} \quad t \to \infty.$$

2. Main results. The main results of this paper are the following.

Theorem 1. *In addition to the fundamental assumptions imposed on* f, g, h *and* p, *we assume that*

(i) *there are positive constants* a, a_0 *such that*

$$a \leq \frac{f(w_2) - f(w_1)}{w_2 - w_1} \leq a_0, \quad w_2 \neq w_1; \tag{3}$$

(ii) *there are positive constants* c, c_0 *such that*

$$c \leq \frac{g(y_2) - g(y_1)}{y_2 - y_1} \leq c_0, \quad y_2 \neq y_1, \tag{4}$$

and

$$abc > c_0^2;$$

(iii) *there are constants* $\Delta_0 > 0$, $K < 1$ *such that for any* ξ, η $(\eta \neq 0)$, *the incremental ratio for* h *satisfies*

$$\frac{(h(\xi + \eta) - h(\xi))}{\eta} \in I_0 \tag{5}$$

with I_0 *as defined* (1);

(iv) *there is a continuous function* $\phi(t)$ *such that*

$$\big|p(t, x_2, y_2, z_2, w_2) - p(t, x_1, y_1, z_1, w_1)\big| \leq$$
$$\leq \phi(t)\big\{|x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1| + |w_2 - w_1|\big\}$$

holds for arbitrary t, x_1, y_1, z_1, w_1, x_2, y_2, z_2 *and* w_2.

Then if there exists a constant D_1 *such that if*

$$\int_0^t \phi^\nu(\tau)d\tau \leq D_1 t \tag{6}$$

for some ν, *with* $1 \leq \nu \leq 2$, *then all solutions of* (2) *converge.*

Theorem 2. *Assume the conditions of Theorem* 1 *are satisfied. Let* $x_1(t)$, $x_2(t)$ *be any two solutions of* (2). *Then for each fixed* ν, $1 \leq \nu \leq 2$, *there are constants* D_2, D_3 *and* D_4 *such that for* $t_2 \geq t_1$,

$$S(t_2) \leq D_2 S(t_1) \exp\left\{ -D_3(t_2 - t_1) + D_4 \int_{t_1}^{t_2} \phi^\nu(\tau)d\tau \right\}, \tag{7}$$

where

$$S(t) = \Big\{[x_2(t) - x_1(t)]^2 + [\dot{x}_2(t) - \dot{x}_1(t)]^2 + [\ddot{x}_2(t) - \ddot{x}_1(t)]^2 + [\dddot{x}_2(t) - \dddot{x}_1(t)]^2\Big\}. \tag{8}$$

We have the following corollaries when $x_1(t) = 0$ and $t_1 = 0$.

***Corollary* 1.** *Suppose that* $p = 0$ *in* (2) *and suppose further that conditions* (i), (ii) *and* (iii) *of Theorem* 1 *hold, then the trivial solution of* (2) *is exponentially stable in the large.*

Also, if we put $\xi = 0$ in (5) with η $(\eta \neq 0)$ arbitrary, we get:

***Corollary* 2.** *If* $p = 0$ *and hypotheses* (i), (ii) *and* (iii) *of Theorem* 1 *hold for arbitrary* η $(\eta \neq 0)$, *and* $\xi = 0$, *then there exists a constant* $D_5 > 0$ *such that every solution* $x(t)$ *of* (2) *satisfies*

$$|x(t)| \le D_5, \qquad |\dot{x}(t)| \le D_5, \qquad |\ddot{x}(t)| \le D_5, |\dddot{x}(t)| \le D_5.$$

***Proof of Theorem* 2.** It is convenient here to consider (2) as the equivalent system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= w, \\ \dot{w} &= -f(w) - bz - g(y) - h(x) + p(t, x, y, z, w). \end{aligned} \tag{9}$$

Let $(x_i(t),\, y_i(t),\, z_i(t),\, w_i(t))$, $i = 1, 2$, be any two solutions of (9) such that inequalities (3), (4) and

$$\Delta_0 \le \frac{h(x_2) - h(x_1)}{x_2 - x_1} \le \frac{K(ab - c)c}{a^2}$$

are satisfied. The basic tool in the proofs of the convergence theorems will be the function

$$\begin{aligned} 2V = c^2\varepsilon(1-\varepsilon)x^2 + ac\big[(D-1) + \varepsilon\big]y^2 + 2c\big[\varepsilon + (D-1)\big]yz + \\ + \varepsilon D w^2 + b(D-1)z^2 + 2\varepsilon a D z w + \varepsilon a^2 D z^2 + \\ + \big[(1-\varepsilon)D - 1\big][az + w]^2 + \big[c(1-\varepsilon)x + by + az + w\big]^2, \end{aligned} \tag{10}$$

where $0 < \varepsilon < 1$, $ab - c > \delta > 0$, $\delta = ab\varepsilon$ and $D - 1 = \dfrac{\delta + c\varepsilon}{ab - c - \delta}$. Indeed we can rearrange the terms in (10) to obtain

$$2V = 2V_1 + 2V_2 + 2V_3,$$

where

$$\begin{gathered} 2V_1 = ac[(D-1) + \varepsilon]y^2 + 2c[\varepsilon + (D-1)]yz + b(D-1)z^2, \\ 2V_2 = \varepsilon a^2 D z^2 + 2\varepsilon a D z w + \varepsilon D w^2 \end{gathered}$$

and

$$2V_3 = c^2\varepsilon(1-\varepsilon)x^2 + [(1-\varepsilon)D - 1][az + w]^2 + [c(1-\varepsilon)x + by + az + w]^2.$$

We note that V_3 obviously positive definite. Also V_i, $i = 1, 2$, regarded as quadratic forms in y and z, z and w respectively is positive and non-negative. Let us recall that a real 2×2 matrix

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

is positive definite if and only if it is symmetric, and the elements a_1, a_4 and $a_1 a_4 - a_2 a_3$ are non-negative. Thus we can rearrange the terms in V_1 as

$$(y, z)\begin{pmatrix} ac[(D-1) + \varepsilon] & c[\varepsilon + (D-1)] \\ c[\varepsilon + (D-1)] & b(D-1) \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix},$$

from which we have as a condition for the positive definite.

Similarly, V_2 is non-negative. Hence V is positive definite. We can therefore find a constant $D_6 > 0$, such that

$$D_6\left(x^2+y^2+z^2+w^2\right) \le V. \tag{11}$$

Furthermore, by using Schwartz inequality $|y|\,|z| \le \dfrac{1}{2}\left(y^2+z^2\right)$, it can be easily obtained that

$$V \le D_7\left(x^2+y^2+z^2+w^2\right), \tag{12}$$

where D_7 is a positive constant.

Using inequalities (11) and (12), we have

$$D_6\left(x^2+y^2+z^2+w^2\right) \le V \le D_7\left(x^2+y^2+z^2+w^2\right).$$

The following result can be easily verified for $W \equiv V$.

Lemma 1. *Let the function* $W(t) = W\big(x_2-x_1, y_2-y_1, z_2-z_1, w_2-w_1\big)$ *be defined by*

$$2W = c^2\varepsilon(1-\varepsilon)\left(x_2-x_1\right)^2 + ac[(D-1)+\varepsilon]\left(y_2-y_1\right)^2 +$$
$$+2c[\varepsilon+(D-1)]\left(y_2-y_1\right)\left(z_2-z_1\right) + \varepsilon D\left(w_2-w_1\right)^2 +$$
$$+b(D-1)\left(z_2-z_1\right)^2 + 2\varepsilon aD\left(z_2-z_1\right)\left(w_2-w_1\right) + \varepsilon a^2 D\left(z_2-z_1\right)^2 +$$
$$+[(1-\varepsilon)D-1][a\left(z_2-z_1\right) + \left(w_2-w_1\right)]^2 +$$
$$+[c(1-\varepsilon)\left(x_2-x_1\right) + b\left(y_2-y_1\right) + a\left(z_2-z_1\right) + \left(w_2-w_1\right)]^2,$$

where $0 < \varepsilon < 1$ *and* $W(0,0,0,0) = 0$, *then there exist finite constants* $D_6 > 0, D_7 > 0$ *such that*

$$D_6\left\{\left(x_2-x_1\right)^2 + \left(y_2-y_1\right)^2 + \left(z_2-z_1\right)^2 + \left(w_2-w_1\right)^2\right\} \le W \le$$
$$\le D_7\left\{\left(x_2-x_1\right)^2 + \left(y_2-y_1\right)^2 + \left(z_2-z_1\right)^2 + \left(w_2-w_1\right)^2\right\}. \tag{13}$$

If we denote the function $W(t)$ by $W\big(x_2(t)-x_1(t),\, y_2(t)-y_1(t),\, z_2(t)-z_1(t),\, w_2(t)-w_1(t)\big)$, and using the fact that the solutions $\left(x_i, y_i, z_i, w_i\right)$, $i = 1, 2$, satisfy the system (9), then $S(t)$ as defined in (8) becomes

$$S(t) = \big\{[x_2(t)-x_1(t)]^2 + [y_2(t)-y_1(t)]^2 + [z_2(t)-z_1(t)]^2 + [w_2(t)-w_1(t)]^2\big\}.$$

Next we prove a result on the derivative of $W(t)$ with respect to t.

Lemma 2. *Let the hypotheses* (i), (ii) *and* (iii) *of Theorem* 1 *hold, then there exist positive finite constants* D_8 *and* D_9 *such that*

$$\frac{dW}{dt} \le -2D_8 S + D_9 S^{1/2}\,|\theta|, \tag{14}$$

where $\theta = p(t, x_2, y_2, z_2, w_2) - p(t, x_1, y_1, z_1, w_1)$.

Proof. Using the system (9), a direct computation of $\dfrac{dW}{dt}$ gives after simplification

$$\frac{dW}{dt} = -W_1 + W_2, \tag{15}$$

where

$$W_1 = c(1-\varepsilon)H(x_2,x_1)(x_2-x_1)^2 + bc\varepsilon(y_2-y_1)^2 +$$
$$+\delta D(z_2-z_1)^2 + D(F-a)(w_2-w_1)^2 +$$
$$+\big(G(y_2,y_1)-c\big)\big[c(1-\varepsilon)(x_2-x_1) + b(y_2-y_1) +$$
$$+aD(z_2-z_1) + D(w_2-w_1)\big](y_2-y_1) + H(x_2,x_1)\big[b(y_2-y_1) +$$
$$+aD(z_2-z_1) + D(w_2-w_1)\big](x_2-x_1) +$$
$$+(F(w_2,w_1)-a)\big[c(1-\varepsilon)(x_2-x_1) + b(y_2-y_1) + aD(z_2-z_1)\big](w_2-w_1),$$
$$W_2 = \theta(t)\big[c(1-\varepsilon)(x_2-x_1) + b(y_2-y_1) + aD(z_2-z_1) + D(w_2-w_1)\big]$$

with

$$F(w_2,w_1) = \frac{f(w_2)-f(w_1)}{w_2-w_1}, \quad w_2 \neq w_1,$$
$$G(y_2,y_1) = \frac{g(y_2)-g(y_1)}{y_2-y_1}, \quad y_2 \neq y_1,$$
$$H(x_2,x_1) = \frac{h(x_2)-h(x_1)}{x_2-x_1}, \quad x_2 \neq x_1.$$

Let $\chi_1 = G(y_2,y_1) - c \geq 0$ for $y_2 \neq y_1$ and $\chi_2 = F(w_2,w_1) - a \geq 0$ for $w_2 \neq w_1$. Furthermore let $H\,(x_2,x_1)$ be denote simply by H, and define

$$\sum_{i=1}^{6}\alpha_i = 1, \qquad \sum_{i=1}^{6}\beta_i = 1, \qquad \sum_{i=1}^{4}\gamma_i = 1, \qquad \sum_{i=1}^{6}\xi_i = 1,$$

where $\alpha_i > 0$, $\beta_i > 0$, $\gamma_i > 0$ and $\xi_i > 0$. Then W_1 re-arranged as

$$W_1 = W_{11} + W_{12} + W_{13} + W_{14} + W_{15} + W_{16} + W_{17} + W_{18} + W_{19} + W_{21},$$

where

$$W_{11} = \Big\{\alpha_1 c(1-\varepsilon)H\,(x_2-x_1)^2 + b\,(\beta_1 c\varepsilon + \chi_1)\,(y_2-y_1)^2 +$$
$$+\gamma_1\delta D\,(z_2-z_1)^2 + \xi_1 D\chi_2\,(w_2-w_1)^2\Big\},$$
$$W_{12} = \Big\{\beta_2 bc\varepsilon\,(y_2-y_1)^2 + \chi_1 c(1-\varepsilon)\,(x_2-x_1)\,(y_2-y_1) +$$
$$+\alpha_2 c(1-\varepsilon)H\,(x_2-x_1)^2\Big\},$$
$$W_{13} = \Big\{\beta_3 bc\varepsilon\,(y_2-y_1)^2 + \chi_1 aD\,(y_2-y_1)\,(z_2-z_1) + \gamma_2\delta D\,(z_2-z_1)^2\Big\},$$
$$W_{14} = \Big\{\beta_4 bc\varepsilon\,(y_2-y_1)^2 + \chi_1 D\,(y_2-y_1)\,(w_2-w_1) + \xi_2 D\chi_2\,(w_2-w_1)^2\Big\},$$

$$W_{15} = \left\{\alpha_3 c(1-\varepsilon)H\left(x_2-x_1\right)^2 + bH\left(x_2-x_1\right)\left(y_2-y_1\right) + \beta_5 bc\varepsilon\left(y_2-y_1\right)^2\right\},$$

$$W_{16} = \left\{\alpha_4 c(1-\varepsilon)H\left(x_2-x_1\right)^2 + aDH\left(x_2-x_1\right)\left(z_2-z_1\right) + \gamma_3\delta D\left(z_2-z_1\right)^2\right\},$$

$$W_{17} = \left\{\alpha_5 c(1-\varepsilon)H\left(x_2-x_1\right)^2 + \right.$$

$$\left. + DH\left(x_2-x_1\right)\left(w_2-w_1\right) + \xi_3 D\chi_2\left(w_2-w_1\right)^2\right\},$$

$$W_{18} = \left\{\alpha_6 c(1-\varepsilon)H\left(x_2-x_1\right)^2 + \chi_2 c(1-\varepsilon)\left(x_2-x_1\right)\left(w_2-w_1\right) + \right.$$

$$\left. + \xi_4 D\chi_2\left(w_2-w_1\right)^2\right\},$$

$$W_{19} = \left\{\beta_6 bc\varepsilon\left(y_2-y_1\right)^2 + \chi_2 b\left(y_2-y_1\right)\left(w_2-w_1\right) + \xi_5 D\chi_2\left(w_2-w_1\right)^2\right\}$$

and

$$W_{21} = \left\{\gamma_4\delta D\left(z_2-z_1\right)^2 + \chi_2 aD\left(z_2-z_1\right)\left(w_2-w_1\right) + \xi_6 D\chi_2\left(w_2-w_1\right)^2\right\}.$$

Since each W_{1i}, $i = 1, \ldots, 9$, and W_{21} are quadratic in their respective variables, then by using the fact that any quadratic of the form $Ar^2 + Brs + Cs^2$ is non-negative if $4AC - B^2 \geq 0$, it follows that

$$W_{12} \geq 0 \;\text{ if }\; \chi_1^2 \leq \frac{4\alpha_2\beta_2 b\varepsilon\Delta_0}{1-\varepsilon},$$

$$W_{13} \geq 0 \;\text{ if }\; \chi_1^2 \leq \frac{4\gamma_2\beta_3 bc\varepsilon\delta}{a^2 D},$$

$$W_{14} \geq 0 \quad\text{if}\quad \chi_1^2 \leq \frac{4\beta_4\xi_2 bc\varepsilon\chi_2}{D},$$

$$W_{15} \geq 0 \quad\text{if}\quad H \leq \frac{4\alpha_3\beta_5 c^2\varepsilon(1-\varepsilon)}{b},$$

$$W_{16} \geq 0 \quad\text{if}\quad H \leq \frac{4\alpha_4\gamma_3 c(1-\varepsilon)\delta}{a^2 D},$$

$$W_{17} \geq 0 \quad\text{if}\quad H \leq \frac{4\alpha_5\xi_3 c(1-\varepsilon)\chi_2}{D},$$

$$W_{18} \geq 0 \quad\text{if}\quad \chi_2 \leq \frac{4\alpha_6\xi_4 D\Delta_0}{c(1-\varepsilon)},$$

$$W_{19} \geq 0 \quad\text{if}\quad \chi_2 \leq \frac{4\beta_6\xi_5 c\varepsilon D}{b},$$

and

$$W_{21} \geq 0 \quad\text{if}\quad \chi_2 \leq \frac{4\gamma_4\xi_6\delta}{a^2}.$$

Thus $W_1 \geq W_{11}$ provided that above inequalities are satisfied in addition to

$$0 \leq \chi_1^2 \leq 4\min\left\{\frac{\alpha_2\beta_2 b\varepsilon\Delta_0}{1-\varepsilon}, \frac{\gamma_2\beta_3 bc\varepsilon\delta}{a^2 D}, \frac{\beta_4\xi_2 bc\varepsilon\chi_2}{D}\right\},$$

$$0 \leq \chi_2 \leq 4\min\left\{\frac{\alpha_6\xi_4 D\Delta_0}{c(1-\varepsilon)}, \frac{\beta_6\xi_5 c\varepsilon D}{b}, \frac{\gamma_4\xi_6\delta}{a^2}\right\}$$

and H lying in

$$I_0 \equiv \left[\Delta_0, \frac{K\,(ab-c)\,c}{a^2}\right],$$

where I_0 is a closed sub-interval of the Routh – Hurwitz interval $\left(0, \dfrac{(ab-c)\,c}{a^2}\right)$, and

$$K = \left(\frac{4}{ab-c}\right) \min\left\{\frac{\alpha_3\beta_5 a^2 c\varepsilon(1-\varepsilon)}{b}, \frac{\alpha_4\gamma_3(1-\varepsilon)\delta}{D}, \frac{\alpha_5\gamma_3 a^2(1-\varepsilon)\chi_2}{D}\right\} < 1.$$

On choosing $2D_8 = \min\left\{c(1-\varepsilon)\Delta_0, bc\varepsilon, \delta D, D\chi_2\right\}$, we have

$$W_1 \geq W_{11} \geq 2D_8 S \tag{16}$$

and if $D_9 = 2\max\left\{c(1-\varepsilon), b, aD, D\right\}$ then

$$W_2 \leq D_9 S^{1/2}\,|\theta|\,. \tag{17}$$

Combining (16) and (17) in (15), inequality (14) is obtained. At last the conclusion to the proof of Theorem 2 will now be given. For this purpose, let ν be any constant in the range $1 \leq \nu \leq 2$ and set $\sigma = 1 - \dfrac{\nu}{2}$, so that $0 \leq \sigma \leq \dfrac{1}{2}$. We re-write (14) in the form

$$\frac{dW}{dt} + D_8 S \leq D_9 S^{\sigma} W^*, \tag{18}$$

where

$$W^* = S^{1/2-\sigma}\left(|\theta| - D_{10}S^{1/2}\right) \tag{19}$$

with $D_{10} = \dfrac{D_8}{D_9}$. If $|\theta| < D_{10}S^{1/2}$, then $W^* < 0$. On the other hand, if $|\theta| \geq D_{10}S^{1/2}$, then the definition of W^* in (19) gives at least

$$W^* \leq S^{(1/2-\sigma)}\,|\theta|$$

and also $S^{1/2} \leq \dfrac{|\theta|}{D_{10}}$. Thus

$$S^{(1-2\sigma)/2} \leq \left[\frac{|\theta|}{D_{10}}\right]^{(1-2\sigma)}$$

and from this together with W^* follows

$$W^* \leq D_{11}\,|\theta|^{2(1-\sigma)}\,,$$

where $D_{11} = D_{10}^{(\sigma-1)}$. On using the estimate on W^* in inequality (18), we obtain

$$\frac{dW}{dt} + D_8 S \leq D_9 D_{11} S^{\sigma}\,|\theta|^{2(1-\sigma)} \leq D_{12} S^{\sigma} \phi^{2(1-\sigma)} S^{(1-\sigma)},$$

where $D_{12} > 2D_9 D_{11}$, which follows from

$$\left|p(t,x_2,y_2,z_2,w_2) - p(t,x_1,y_1,z_1,w_1)\right| \leq$$
$$\leq \phi(t)\left\{|x_2-x_1| + |y_2-y_1| + |z_2-z_1| + |w_2-w_1|\right\}.$$

In view of the fact that $\nu = 2(1-\sigma)$, we obtain

$$\frac{dW}{dt} \leq -D_8 S + D_{12}\phi^{\nu} S,$$

and on using inequalities (13), we have

$$\frac{dW}{dt} + \big(D_{13} - D_{14}\phi^{\nu}(t)\big)W \leq 0 \tag{20}$$

for some constants D_{13} and D_{14}. On integrating the estimate (20) from t_1 to t_2, $t_2 \geq t_1$, we have

$$W(t_2) \leq W(t_1) \exp\left\{ -D_{13}(t_2 - t_1) + D_{14} \int\limits_{t_1}^{t_2} \phi^{\nu}(\tau) d\tau \right\}.$$

Again, using Lemma 1, we obtain (7), with $D_2 = D_7 D_6^{-1}$, $D_3 = D_{13}$ and $D_4 = D_{14}$.

Theorem 2 is proved.

***Proof of Theorem* 1.** The proof follows from the estimate (7) and the condition (6) on $\phi(t)$. On Choosing $D_1 = D_3 D_4^{-1}$ in (6). Then, as $t = (t_2 - t_1) \to \infty, S(t) \to 0$, which proves that as $t \to \infty$,

$$x_2(t) - x_1(t) \to 0, \qquad \dot{x}_2(t) - \dot{x}_1(t) \to 0,$$
$$\ddot{x}_2(t) - \ddot{x}_1(t) \to 0, \qquad \dddot{x}_2(t) - \dddot{x}_1(t) \to 0.$$

The theorem is proved.

Remark. If $\phi(t) \equiv D_{15}$ (a constant), our results will still remain valid.

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