We generalize the concepts of semicommutative, skew Armendariz, Abelian, reduced, and symmetric left ideals and study the relations between them.

1. Introduction. Throughout this paper $R$ denotes an associative ring with identity $1$ and $\alpha$ denotes a nonzero and nonidentity endomorphism of a given ring with $\alpha(1) = 1$, and $1$ denotes identity endomorphism, unless specified otherwise.

We write $R[x]$, for the polynomial ring, moreover, $R[x, \alpha] = \{ \sum_{i=0}^{n} a_i x^i \mid n \geq 0, a_i \in R \}$ becomes a ring under the following operation:

$$f(x) = \sum_{i=0}^{n} a_i x^i, \quad g(x) = \sum_{j=0}^{m} b_j x^j \in R[x, \alpha], \quad f(x)g(x) = \sum_{k=0}^{m+n} \left( \sum_{i+j=k} a_i \alpha^i (b_j) \right) x^k.$$

The ring $R[x, \alpha]$ is called the skew polynomial extension of $R$.

In [4], Baer-rings are introduced as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. According to Clark [5], a ring $R$ is said to be quasi-Baer ring if the right annihilator of every right ideal of $R$ is generated (as a right ideal) by an idempotent. A ring $R$ is called right principally quasi-Baer ring if the right annihilator of a principally right ideal of $R$ is generated (as a right ideal) by an idempotent. Finally, a ring $R$ is called right principally projective ring if the right annihilator of an element of $R$ is generated by an idempotent [4].

For an endomorphism $\alpha$ of ring $R$, Hong, Kim, and Kowak [7] called $R$ an $\alpha$-skew Armendariz ring if whenever polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x, \alpha]$, $f(x)g(x) = 0$ then $a_i \alpha^i b_j = 0$ for each $i$ and $j$. Some properties of Armendariz rings have been studied in [9–11].

In [2], the notions of $\alpha$-Abelian, $\alpha$-semicommutative, $\alpha$-reduced, $\alpha$-symmetric and $\alpha$-Armendariz rings have been introduced which generalize Abelian, semicommutative, reduced, symmetric and Armendariz rings. Aghayev et al. defined a ring $R$ is called $\alpha$-Abelian if, for any $a, b \in R$, and any idempotent $e \in R$, $ea = ae$ and $ab = 0$ if and only if $a \alpha(b) = 0$ and $R$ is called $\alpha$-semicommutative if, for any $a, b \in R$, $ab = 0$ implies $aRb = 0$ and $ab = 0$ if and only if $a \alpha(b) = 0$. A ring $R$ is called $\alpha$-reduced if, for any $a, b \in R$, $ab = 0$ implies $aR \cap Rb = 0$ and $ab = 0$ if and only if $a \alpha(b) = 0$. A ring $R$ is called $\alpha$-symmetric if, for any $a, b, c \in R$, $abc = 0$ implies $acb = 0$ and $ab = 0$ if and only if $a \alpha(b) = 0$.

They proved that $\alpha$-semicommutative, $\alpha$-reduced, $\alpha$-symmetric and $\alpha$-Armendariz rings are $\alpha$-Abelian. For a right principally projective ring $R$, they also proved the following conditions on $\alpha$-reduced of a ring $R$ are equivalent:

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In this paper we introduce the concepts of $\alpha$-Abelian, $\alpha$-semicommutative, $\alpha$-reduced, $\alpha$-symmetric and $\alpha$-skew Armendariz left ideals and investigate their properties. Moreover, we prove that if there exists a classical right quotient ring $Q$ of a ring $R$ consisting of central elements and $I$ is $\alpha$-semicommutative left ideal of $R$, then $QI$ is $\alpha$-semicommutative left ideal of $Q(R)$.

Similarly we prove that if $I$ is a left ideal of a ring $R$ and $\Delta$ is a multiplicatively closed subset of $R$ consisting of central elements and $I$ is $\alpha$-semicommutative left ideal of $R$, then $\Delta^{-1}I$ is $\alpha$-semicommutative left ideal of $\Delta^{-1}R$.

2. Semicommutative and skew Armendariz ideals. In this section the notion of an $\alpha$-Abelian left ideals is introduced as a generalization of Abelian left ideals. We recall that a left ideal $I$ of $R$ is called Abelian if for any $a, b \in R$ and any idempotent $e \in R$, $ea - ae \in r_R(I)$. Now we have the following definition.

**Definition 2.1.** A left ideal $I$ of $R$ is called $\alpha$-Abelian if, for any $a, b \in R$ and any idempotent $e \in R$, we have the following conditions:

1) $ea - ae \in r_R(I)$,
2) $ab \in r_R(I)$ if and only if $a\alpha(b) \in r_R(I)$.

So a left ideal $I$ is Abelian if and only if it is 1-Abelian. The following example shows that there exists an Abelian left ideal, but it is not $\alpha$-Abelian left ideal.

**Example 2.1.** Let $R$ be the ring $\mathbb{Z} \oplus \mathbb{Z}$ with the usual componentwise operation. It is clear that $R$ is an Abelian ring. Let $\alpha : R \rightarrow R$ be defined by $\alpha((a, b)) = (b, a)$. Then $(1, 0)(0, 1) = 0$, but $(1, 0)\alpha((0, 1)) \neq 0$. Hence $R$ is not $\alpha$-Abelian. If ideal $I = R$ then $r_R(I) = 0$ and then $I$ is an Abelian left ideal, but it is not an $\alpha$-Abelian left ideal.

**Definition 2.2.** A left ideal $I$ of $R$ is called semicommutative if, for any $a, b \in R$, $ab \in r_R(I)$ then $aRb \subseteq r_R(I)$.

**Definition 2.3.** A left ideal $I$ of $R$ is called $\alpha$-semicommutative if, for any $a, b \in R$ we have the following conditions:

1) $ab \in r_R(I)$ then $aRb \subseteq r_R(I)$,
2) $ab \in r_R(I)$ if and only if $a\alpha(b) \in r_R(I)$.

So a left ideal $I$ is semicommutative if and only if it is 1-semicommutative.

In general the reverse implication in the above definition does not hold by the following example which also shows that there exist an endomorphism $\alpha$ of a ring $R$ and left ideal $I$ of $R$ such that $I$ is semicommutative but is not $\alpha$-semicommutative.

**Example 2.2.** Let $\mathbb{Z}_2$ be the ring of integers modulo 2 and consider a ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the usual addition and multiplication. If $I = \mathbb{Z}_2 \oplus 0$ be a left ideal of $R$ then $r_R(I) = 0 \oplus \mathbb{Z}_2$. Now, let $\alpha : R \rightarrow R$ be defined by $\alpha((a, b)) = (b, a)$. Then $\alpha$ is an automorphism of $R$. It is clear that $I$ is semicommutative left ideal. For $a = (1, 0)$ and $b = (0, 1) \in R$, $ab = (0, 0) \in r_R(I)$ but $a\alpha(b) = (1, 0) \notin r_R(I)$.

**Lemma 2.1.** If the left ideal $I$ of $R$ is $\alpha$-semicommutative, then $I$ is $\alpha$-Abelian.
Proof. If $e$ is an idempotent in $R$, then $e(1-e) = 0 \in r_R(I)$. Since $I$ is $\alpha$-semicommutative, we have $\alpha(e(1-e)) = 0 \in r_R(I)$ for any $\alpha \in R$ and so $ea - eae \in r_R(I)$. Similarly, $(1-e)e = 0 \in r_R(I)$. Since $I$ is $\alpha$-semicommutative $(1-e)ae = 0 \in r_R(I)$. So $ae - eae \in r_R(I)$. Therefore, $ae - eae \in r_R(I)$. Thus $I$ is $\alpha$-Abelian.

Let $I$ be a left ideal of $R$.

The following example shows that the condition $\alpha(1) = 1$ in Lemma 2.1 is not superfluous.

Example 2.3. Let $Z$ be the ring of integers. Consider a ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \bigg| a, b, c \in \mathbb{Z} \right\}.$$ 

If $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \bigg| b \in \mathbb{Z} \right\}$ be an right ideal of $R$ then $l_R(I) = \left\{ \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \bigg| b, c \in \mathbb{Z} \right\}$. Let $\alpha : R \to R$ be defined by $\alpha \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$.

For $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ and $B = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \in R$, if $AB \in l_R(I)$ then we obtain $aa' = 0$, and so $a = 0$ or $a' = 0$. This implies $AR\alpha(B) \subseteq l_R(I)$ and thus $I$ is $\alpha$-semicommutative. Note that $\alpha(1) \neq 1$ and $I$ is not Abelian.

Definition 2.4. A left ideal $I$ of $R$ is called $\alpha$-skew Armendariz if the following conditions are satisfied:

1) for any $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x, \alpha], f(x)g(x) \in r_{R[x, \alpha]}(I[x])$

implies $a_i \alpha^i(b_j) \in r_R(I)$,

2) $ab \in r_R(I)$ if and only if $\alpha a \in r_R(I)$.

We introduce an $\alpha$-skew Armendariz left ideal in the following example.

Example 2.4. Let $R$ be an $\alpha$-skew Armendariz ring and consider

$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bigg| a, b \in R \right\}.$$ 

It is clear that $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \bigg| b \in R \right\}$ is the left ideal of $S$. Let $f(x) = A_0 + A_1 x + \ldots + A_n x^n$ and $g(x) = B_0 + B_1 x + \ldots + B_m x^m \in S[x, \alpha], A_i = \begin{pmatrix} a_i & a_{i1} \\ 0 & a_{0i} \end{pmatrix}, B_j = \begin{pmatrix} b_{0j} & b_{1j} \\ 0 & b_{0j} \end{pmatrix}$ for $i = 0, \ldots, n, j = 0, \ldots, m$ such that $f(x)g(x) \in r_{R[x, \alpha]}(I[x])$. Let

$$f(x) = \begin{pmatrix} \alpha_0(x) & \alpha_1(x) \\ 0 & \alpha_0(x) \end{pmatrix}, \quad g(x) = \begin{pmatrix} \beta_0(x) & \beta_1(x) \\ 0 & \beta_0(x) \end{pmatrix},$$

$$\alpha_0(x) = a_{00} + a_{01} x + \ldots + a_{0n} x^n, \quad \beta_0(x) = b_{00} + b_{01} x + \ldots + b_{0m} x^m.$$ 

Since $f(x)g(x) \in r_{R[x, \alpha]}(I[x])$ thus for any $h(x) = \begin{pmatrix} 0 & \gamma(x) \\ 0 & 0 \end{pmatrix} \in I[x], \gamma(x) = \gamma_0 + \gamma_1 x + \ldots + \gamma_t x^t, \gamma(x)f(x)g(x) = 0$. Thus $\gamma(x)\alpha_0(x)\beta_0(x) = 0$. Since $I$ is $\alpha$-skew Armendariz left ideal hence $\gamma_0 \alpha^k(\alpha_0 \alpha^i(b_{0j})) = 0$ for all $k = 0, \ldots, t, i = 0, \ldots, n$ and $j = 0, \ldots, m$. If set $k = 0$, then $\gamma_0(\alpha_0 \alpha^i(b_{0j})) = 0$. Since $\gamma_0 \in R$ is arbitrary, thus $\begin{pmatrix} 0 & \gamma_0 \\ 0 & 0 \end{pmatrix} \in I$. Therefore $a_0 \alpha^i(b_{0j}) \in r_R(I)$, and hence $A_i \alpha^i(B_j) \in r_R(I)$. Now we consider
Proposition 2.1. If $I$ is an $\alpha$-skew Armendariz left ideal of $R$ and for some $a, b, c \in R$ and some integer $n \geq 1$, $ab \in r_{R}(I)$ and $ac^{n}\alpha^{n}(b) \in r_{R}(I)$, then $acb \in r_{R}(I)$.

Proof. Consider $f(x) = a(1 - cx), g(x) = (1 + cx + \ldots + c^{n-1}x^{n-1})b \in R[x, \alpha]$, $f(x)g(x) = ab - ac^{n}\alpha^{n}(b)x^{n} \in r_{R[x, \alpha]}(I[x])$. Since $I$ is an $\alpha$-skew Armendariz left ideal of $R$, so $acb \in r_{R}(I)$. Proposition 2.1 is proved.

Next, we show that every $\alpha$-skew Armendariz left ideal of $R$ is an $\alpha$-Abelian left ideal.

Proposition 2.2. If $I$ is an $\alpha$-skew Armendariz left ideal of $R$, then $I$ is an $\alpha$-Abelian left ideal.

Proof. Assume that $I$ is an $\alpha$-skew Armendariz left ideal of $R$. Consider $e = e^{2} \in R$ and let $a = e, b = (1 - e), c = er(1 - e)$ with $r \in R$. Then clearly $ab \in r_{R}(I)$ and $e^{2} = 0 \in r_{R}(I)$ and hence $ae^{2}\alpha^{2}(b) \in r_{R}(I)$ and by Proposition 2.1, $acb \in r_{R}(I)$. So $re - er \in r_{R}(I)$. Let $a_{1} = 1 - e, b_{1} = e$ and $c_{1} = (1 - e)re$, we also have $a_{1}b_{1}c_{1} \in r_{R}(I)$. So $re - er \in r_{R}(I)$. Then $re - er \in r_{R}(I)$.

Proposition 2.2 is proved.

Theorem 2.1. Let $R$ be a ring and $I, J$ be left ideals of $R$. If $I \subseteq J$ and $J/I$ is an $\alpha$-skew Armendariz left ideal of $R/I$, then $J$ is an $\alpha$-skew Armendariz left ideal of $R$.

Proof. Let $f(x) = \sum_{i=0}^{n}a_{i}x^{i}$ and $g(x) = \sum_{j=0}^{m}b_{j}x^{j} \in R[x, \alpha]$ such that $f(x)g(x) \in r_{R[x, \alpha]}(J[x])$. Then $\sum_{i=0}^{n}a_{i}x^{i}\sum_{j=0}^{m}b_{j}x^{j} \in r_{R/I[x, \alpha]}(J/I[x])$. Thus $a_{i}\alpha^{i}(\bar{b}_{j}) \in r_{R/I}(J/I)$. Hence $a_{i}\alpha^{i}(b_{j}) \in r_{R}(J)$. Therefore $J$ is an $\alpha$-skew Armendariz left ideal of $R$.

Theorem 2.1 is proved.

The following is an immediate corollary of Theorem 2.1.

Corollary 2.1. Let $R$ be a ring and $I$ an left ideal of $R$. If $R/I$ is an $\alpha$-skew Armendariz ring then $R$ is an $\alpha$-skew Armendariz ring.

A ring $R$ is called locally finite if every finite subset of $R$ generates a finite semigroup multiplicatively. Finite rings are clearly locally finite and the algebraic closure of a finite field is locally finite but it is not finite.

Proposition 2.3. Let $R$ be a locally finite ring and $I$ be an $\alpha$-skew Armendariz left ideal of $R$. Then $I$ is an $\alpha$-semicommutative left ideal of $R$.

Proof. Let $ab \in r_{R}(I)$ with $a, b \in R$. For any $r \in R$, since $R$ is locally finite there exist integers $m, k \geq 1$ such that $r^{m} = r^{m+k}$. So we obtain inductively $r^{m} = r^{m}r^{k} = r^{2k} = \ldots = r^{m}r^{mk} = r^{m(k+1)}$, put $h = k + 1$ then $r^{m} = r^{mh}$ with $h \geq 2$. Notice that $r^{(h-1)m} = r^{(h-2)m}r^{m} = r^{(h-2)m}r^{mh} = r^{2(h-2)m} = (r^{(h-1)m})^{2}$. Hence $r^{(h-1)m}$ is an idempotent and so by Proposition 2.2, $ar^{(h-1)m} - r^{(h-1)m}a \in r_{R}(I)$ and $abr^{(h-1)m} - r^{(h-1)m}ab \in r_{R}(I)$. Thus
$r^{(h-1)m}ab \in r_R(I)$. On the other hand by Proposition 2.2, $ar^{(h-1)m} - r^{(h-1)m}a \in r_R(I)$, so $ar^{(h-1)m}b - r^{(h-1)m}ab \in r_R(I)$, and hence $ar^{(h-1)m}b \in r_R(I)$. Since $I$ is an $\alpha$-skew Armendariz left ideal of $R$ so $ar^{(h-1)m}a \alpha^{(h-1)m}(b) \in r_R(I)$, and by Proposition 2.1, we imply that $arb \in r_R(I)$ for all $r \in R$.

Proposition 2.3 is proved.

Let $\alpha$ be an endomorphism of a ring $R$ and $M_n(R)$ be the $(n \times n)$-matrix over ring $R$ and $\bar{\alpha}: M_n(R) \rightarrow M_n(R)$ defined by $\bar{\alpha}(a_{ij}) = (\alpha(a_{ij}))$. Then $\bar{\alpha}$ is an endomorphism of $M_n(R)$. It is obvious that, the restriction of $\bar{\alpha}$ to $D_n(R)$ is an endomorphism of $D_n(R)$, where $D_n(R)$ is the diagonal $(n \times n)$-matrix ring over $R$. We also denote $\bar{\alpha} : D_n(R)$ by $\bar{\alpha}$.

**Proposition 2.4.** Let $\alpha$ be an endomorphism of a ring $R$. Then $D_n(I)$ is an $\alpha$-skew Armendariz left ideal of $D_n(R)$ if $I$ is an $\alpha$-skew Armendariz left ideal for any $n$.

**Proof.** Let $f(x) = A_0 + A_1x + \ldots + A_px^p$ and $g(x) = B_0 + B_1x + \ldots + B_qx^q \in D_n(R)[x, \bar{\alpha}]$ satisfying $f(x)g(x) \in r_{D_n(R)[x, \bar{\alpha}]}(D_n(I)[x])$, where

$$A_i = \begin{pmatrix} a^{(i)}_{11} & 0 & \ldots & 0 \\ 0 & a^{(i)}_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & a^{(i)}_{nn} \end{pmatrix} \quad \text{and} \quad B_j = \begin{pmatrix} b^{(j)}_{11} & 0 & \ldots & 0 \\ 0 & b^{(j)}_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & b^{(j)}_{nn} \end{pmatrix}.$$

Then from $f(x)g(x) \in r_{D_n(R)[x, \bar{\alpha}]}(D_n(I)[x])$, it follows that

$$\left( \sum_{i=0}^{p} a^{(i)}_{ss} x^i \right) \left( \sum_{j=0}^{q} b^{(j)}_{ss} x^j \right) \in r_{R[x, \alpha]}(I[x]),$$

for each $1 \leq s \leq n$. Since $I$ is an $\alpha$-skew Armendariz left ideal of $R$, then $a^{(i)}_{ss} \alpha^i(b^{(j)}_{ss}) \in r_R(I)$ for any $1 \leq i \leq p$ and $1 \leq j \leq q$. Therefore

$$A_i \alpha^i(B_j) = \begin{pmatrix} a^{(i)}_{11} \alpha^i(b^{(j)}_{11}) & 0 & \ldots & 0 \\ 0 & a^{(i)}_{22} \alpha^i(b^{(j)}_{22}) & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & a^{(i)}_{nn} \alpha^i(b^{(j)}_{nn}) \end{pmatrix} \in r_{D_n(R)}(D_n(I)).$$

Thus it shows that $D_n(I)$ is an $\alpha$-skew Armendariz left ideal of $D_n(R)$.

Proposition 2.4 is proved.

Every endomorphism $\sigma$ of rings $R$ and $S$ can be extended to the endomorphism of rings $R[x]$ and $S[x]$ defined by $\sum_{i=0}^{m} a_ix^i \rightarrow \sum_{j=0}^{m} \sigma(a_i)x^i$, which we also denote by $\sigma$.

**Proposition 2.5.** Let $\sigma : R \rightarrow S$ be a ring isomorphism, $I_1$ be an ideal of $R$ and $I_2$ be an ideal of $S$ with $\sigma(I_1) = I_2$. If $I_2$ is an $\sigma \alpha \sigma^{-1}$-skew Armendariz left ideal of ring $S$, then $I_1$ is an $\alpha$-skew Armendariz left ideal of $R$.

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and hence triangular skew matrices over a ring \(V\) are introduced. A ring \(A\) is an \(\alpha\)-Armendariz left ideal of ring \(R\) if and only if \(\theta(a)\alpha(b) = 0\) for any \(a, b \in A\). The proof of this statement is similar to the proof of Proposition 2.5.

Corollary 2.2. Suppose that \(\alpha\) is an endomorphism of a ring \(R\), \(\theta: V_n(R) \rightarrow \frac{R[x]}{(x^n)}\) be a ring isomorphism, \(I_1\) is a left ideal of \(V_n(R)\) and \(I_2\) is a left ideal of \(\frac{R[x]}{(x^n)}\). If \(I_2\) is an \(\alpha\)-Armendariz left ideal of \(\frac{R[x]}{(x^n)}\) and \(\theta(I_1) = I_2\), then \(I_1\) is an \(\alpha\)-Armendariz left ideal of \(V_n(R)\).

Proof. Assume that \(I_2\) is an \(\alpha\)-Armendariz left ideal of \(\frac{R[x]}{(x^n)}\) and define

\[
\theta: V_n(R) \rightarrow \frac{R[x]}{(x^n)}
\]

by

\[
\theta(r_0 I_n + r_1 V + \ldots + r_{n-1} V^{n-1}) = r_0 + r_1 x + \ldots + r_{n-1} x^{n-1} + (x^n).
\]

Now we have \(I_1\) is a \(\theta^{-1}\alpha\theta\)-skew Armendariz left ideal of \(V_n(R)\) and that

\[
\theta^{-1}\alpha\theta(I_1) = \theta^{-1}\alpha\theta(r_0 I_n + r_1 V + \ldots + r_{n-1} V^{n-1}) = \alpha(r_0 I_n + r_1 V + \ldots + r_{n-1} V^{n-1}),
\]

which means that \(I_1\) is an \(\alpha\)-Armendariz left ideal of \(V_n(R)\).

Corollary 2.2 is proved. Recall that a ring is reduced if it has no nonzero nilpotent elements. In [2], \(\alpha\)-reduced ring is introduced. A ring \(R\) is \(\alpha\)-reduced, if for any \(a, b \in R\)

1) \(ab = 0\) implies \(aR \cap Rb = 0\),
2) \(ab = 0\) if and only if \(a\alpha(b) = 0\).

In this work we define reduced and \(\alpha\)-reduced left ideals.

Definition 2.5. A left ideal \(I\) of \(R\) is called reduced, if for any \(a, b \in R\), \(ab \in r_R(I)\), then \(aR \cap Rb \subseteq r_R(I)\).

Definition 2.6. A left ideal \(I\) of \(R\) is called \(\alpha\)-reduced, if for any \(a, b \in R\), we have the following conditions:
1) \( ab \in r_R(I) \) then \( aR \cap Rb \subseteq r_R(I) \).
2) \( ab \in r_R(I) \) if and only if \( \alpha(b) \in r_R(I) \).

So the left ideal \( I \) is reduced if and only if it is 1-reduced.

### Lemma 2.2
If \( I \) is an \( \alpha \)-reduced left ideal of \( R \), then \( I \) is an \( \alpha \)-semicommutative.

**Proof.** Suppose \( ab \in r_R(I) \) for any \( a, b \in R \). Since \( I \) is an \( \alpha \)-reduced left ideal of \( R \) then \( aR \cap Rb \subseteq r_R(I) \). Because \( aR \subseteq aR \cap Rb \), then \( aR \subseteq r_R(I) \). Therefore \( I \) is an \( \alpha \)-semicommutative.

Now by Lemma 2.2 we have the following lemma.

### Lemma 2.3
If \( I \) is an \( \alpha \)-reduced left ideal of \( R \), then \( I \) is \( \alpha \)-Abelian.

**Proposition 2.6.** Let \( \alpha \) be an endomorphism of a ring \( R \) and \( I \) be an \( \alpha \)-reduced left ideal of \( R \). Then \( I \) is an \( \alpha \)-skew Armendarize left ideal.

**Proof.** Let \( f(x) = \sum_{i=0}^{n} a_i x^i \) and \( g(x) = \sum_{j=0}^{m} b_j x^j \in R[\alpha] \) such that \( f(x)g(x) \in r_R(I[\alpha]) \). Then for each \( h \in I \), \( h\left( \sum_{i+j=1} a_i \alpha^i(b_j) \right) = 0 \). Thus \( \sum_{i+j=1} a_i \alpha^i(b_j) \in r_R(I) \) for \( i = 0, \ldots, m + n \). So \( ha_0b_0 = 0 \). Thus \( ha_0b_1b_0 = 0 \), since \( I \) is \( \alpha \)-semicommutative and \( h(a_1 \alpha(b_0) + a_0 b_1) = 0 \). Multiplying by \( b_0 \) on the right we have \( h(a_1 \alpha(b_0) + a_0 b_1)b_0 = 0 \). So we have \( h(a_1 \alpha(b_0) b_0) = 0 \). Thus \( h(a_1 \alpha^2(b_0)) = 0 \), and then \( h(a_1 \alpha(b_0)) = 0 \), since \( I \) is \( \alpha \)-reduced. Thus \( h(a_0 b_1) = 0 \). Assume that \( s \geq 1 \) and \( h(a_i \alpha^i(b_j)) = 0 \) for all \( i \) and \( j \) with \( i + j \leq s \). Note that \( h(a_0 b_{s+1} + a_1 \alpha(b_0) + \ldots + a_{s+1} \alpha^{s+1}(b_0)) = 0 \), where \( a_i \) and \( b_j \) are 0 if \( i > n \) and \( j > m \).

Multiplying by \( \alpha^s(b_0) \) on the right yields

\[
ha_0 b_{s+1} \alpha^s(b_0) + a_1 \alpha(b_0) \alpha^s(b_0) + \ldots + a_{s+1} \alpha^{s+1}(b_0) \alpha^s(b_0) = 0.
\]

Since \( I \) is \( \alpha \)-semicommutative and \( h(a_1 \alpha^i(b_0)) = 0 \) for \( i \leq s \), it follows that \( h(a_i \alpha^i(b_0)) = 0 \). Thus \( h(a_1 \alpha^{s+1}(b_0)) = h(a_{s+1} \alpha^s(b_0) \alpha^s(b_0)) = 0 \), which implies \( h(a_{s+1} \alpha^{s+1}(b_0)) = 0 \) by assumption. So

\[
h(a_0 b_{s+1} + a_1 \alpha(b_0) + \ldots + a_s \alpha^s(b_1)) = 0.
\]

Analogously, multiplying by \( \alpha^{s-1}(b_1) \) on the right yields

\[
h(a_0 b_{s+1} \alpha^{s-1}(b_1) + a_1 \alpha(b_0) \alpha^{s-1}(b_1) + \ldots + a_s \alpha^s(b_1) \alpha^{s-1}(b_1)) = 0.
\]

The similar argument as the above reveals that \( h(a_s \alpha^s(b_1) \alpha^{s-1}(b_1)) = 0 \). Thus \( h(a_s \alpha^s(b_1)) = 0 \).

Continuing this process, we have \( ha_s \alpha^s(b_1) = \ldots = ha_1 \alpha(b_0) = ha_0 b_{s+1} = 0 \). So we prove that \( h \alpha^i(b_j) = 0 \) for all \( i \) and \( j \) with \( i + j \leq s + 1 \). By the induction principle, \( h \alpha^i(b_j) = 0 \) for every \( i \) and \( j \).

Proposition 2.6 is proved.

**Definition 2.7.** A left ideal \( I \) of \( R \) is called symmetric, if for any \( a, b, c \in R \), \( abc \in r_R(I) \), then \( acb \in r_R(I) \).

**Definition 2.8.** A left ideal \( I \) of \( R \) is called \( \alpha \)-symmetric, if for any \( a, b, c \in R \),

1) \( abc \in r_R(I) \) then \( acb \in r_R(I) \),
2) \( ab \in r_R(I) \) if and only if \( \alpha(b) \in r_R(I) \).

So the left ideal \( I \) is symmetric if and only if it is 1-symmetric.

**Proposition 2.7.** If \( I \) is an \( \alpha \)-symmetric left ideal of \( R \), then \( I \) is an \( \alpha \)-semicommutative.

**Proof.** Suppose \( ab \in r_R(I) \), for any \( a, b \in R \). Thus \( abr \in r_R(I) \), for any \( r \in R \). So \( abr \in r_R(I) \), since \( I \) is \( \alpha \)-symmetric. Therefore \( I \) is an \( \alpha \)-semicommutative.

Now by Proposition 2.7 we have the following corollary.
**Corollary 2.3.** If $I$ is an $\alpha$-symmetric left ideal of $R$, then $I$ is an $\alpha$-Abelian.

There exists an $\alpha$-Abelian right ideal which are also $\alpha$-semicommutative, $\alpha$-reduced and $\alpha$-symmetric.

**Example 2.5.** Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Z}^{2 \times 2}$ the full $(2 \times 2)$-matrix ring over $\mathbb{Z}$,

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathbb{Z}^{2 \times 2} \mid a \equiv d \pmod{2}, b \equiv 0 \pmod{2} \right\}$$

and

$$I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in \mathbb{Z}^{2 \times 2} \mid b \equiv 0 \pmod{2} \right\}$$

be the right ideal of $R$. We have

$$l_R(I) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in \mathbb{Z}^{2 \times 2} \mid d \equiv 0 \pmod{2}, b \equiv 0 \pmod{2} \right\}.$$

We define $\alpha \left( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}. 0, 1$ are only idempotents in $R$ and for any $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in R$ and $B = \begin{pmatrix} c & e \\ 0 & h \end{pmatrix} \in R$, $AB \in l_R(I)$ if and only if $ac = 0$. Since $\mathbb{Z}$ is domain we have $a = 0$ or $c = 0$. If $a = 0$, then

$$A\alpha(B) = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \alpha \begin{pmatrix} c & e \\ 0 & h \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & dh \end{pmatrix} \in l_R(I).$$

If $c = 0$, then

$$A\alpha(B) = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \alpha \begin{pmatrix} 0 & e \\ 0 & h \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & dh \end{pmatrix} \in l_R(I).$$

On the other hand if $A\alpha(B) \in l_R(I)$ therefore $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \alpha \begin{pmatrix} c & e \\ 0 & h \end{pmatrix} \in l_R(I)$, then $\begin{pmatrix} ac & bh \\ 0 & dh \end{pmatrix} \in l_R(I)$. So $ac = 0$ and similarly we have $AB \in l_R(I)$. Therefore $I$ is $\alpha$-Abelian right ideal of $R$. Now we show that $I$ is $\alpha$-semicommutative right ideal of $R$. For any $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, B = \begin{pmatrix} c & e \\ 0 & h \end{pmatrix}$ and $C = \begin{pmatrix} g & k \\ 0 & m \end{pmatrix} \in R$, let $AB \in l_R(I)$ thus $ac = 0$ and so $acg = agc = 0$, since $a, c, g \in \mathbb{Z}$. We have

$$ACB = \begin{pmatrix} age & age + akh + bhm \\ 0 & dmh \end{pmatrix} = \begin{pmatrix} 0 & age + akh + bhm \\ 0 & dmh \end{pmatrix} \in l_R(I).$$

$I$ is $\alpha$-symmetric right ideal of $R$, since $ABC \in l_R(I)$ iff $acg = 0$, iff $acg = 0$. Therefore $ACB \in l_R(I)$. Now we show that $I$ is $\alpha$-reduced right ideal of $R$. Let $AB \in l_R(I)$, then $ac = 0$. Thus $a = 0$. Thus $a = 0$. 

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or $c = 0$. Now if $X \in AR \cap RB$, then there exist $K = \begin{pmatrix} k_1 & k_2 \\ 0 & k_3 \end{pmatrix} \in R$ and $G = \begin{pmatrix} g_1 & g_2 \\ 0 & g_3 \end{pmatrix} \in R$, such that $X = AK = GB$. But $AK = \begin{pmatrix} ak_1 & ak_2 + bk_3 \\ 0 & dk_3 \end{pmatrix}$ and $GB = \begin{pmatrix} 0 & g_1c + g_2h \\ 0 & g_3h \end{pmatrix}$. Thus $g_1c = ak_1$. If $a = 0$, then $X = \begin{pmatrix} 0 & bk_3 \\ 0 & dk_3 \end{pmatrix} \in l_R(I)$ and if $c = 0$, then $X = \begin{pmatrix} 0 & g_2h \\ 0 & g_3h \end{pmatrix} \in l_R(I)$.

Therefore $I$ is $\alpha$-reduced right ideal of $R$.

Recall that $r_R(\bigoplus I_i) = \bigcap r_R(I_i)$. Now we have the next proposition.

**Proposition 2.8.** For any index set $\Gamma$, if $I_i$ is an $\alpha$-Abelian left ideal of $R$ for each $i \in \Gamma$, then $\bigoplus_{i \in \Gamma} I_i$ is an $\alpha$-Abelian left ideal of $R$.

**Theorem 2.2.** Suppose that $I$ is left ideal a ring $R$ and $\Delta$ is a multiplicatively closed subset of $R$ consisting of central regular elements. We have the following conditions:

1. If $I$ is an $\alpha$-semicommutative left ideal of $R$, then $\Delta^{-1}I$ is an $\alpha$-semicommutative left ideal of $\Delta^{-1}R$.
2. If $I$ is an $\alpha$-symmetric left ideal of $R$, then $\Delta^{-1}I$ is an $\alpha$-symmetric left ideal of $\Delta^{-1}R$.
3. If $I$ is an $\alpha$-reduced left ideal of $R$, then $\Delta^{-1}I$ is an $\alpha$-reduced left ideal of $\Delta^{-1}R$.

**Proof.** We employ the method used in the proof of [8] (Proposition 3.1). For instance, we prove (1). Let $\beta \gamma \in r_{\Delta^{-1}R}(\Delta^{-1}I)$ with $\beta = u^{-1}a$, $\gamma = v^{-1}b$, $u, v \in \Delta$ and $a, b \in R$. Since $\Delta$ is contained in the center of $R$, we have $0 = \Delta^{-1}I\beta \gamma = \Delta^{-1}Iu^{-1}av^{-1}b = \Delta^{-1}Iab(uv)^{-1}.$

So $Iab = 0$. It follows that $arb \in r_R(I)$ for all $r \in R$, since $I$ is an $\alpha$-semicommutative left ideal of $R$. Now for $\delta = w^{-1}r$ with $w \in \Delta$ and $r \in R$, $\Delta^{-1}I\beta \delta \gamma = \Delta^{-1}Iarb(uwv)^{-1} = 0$. Thus $\beta \delta \gamma \in r_{\Delta^{-1}R}(\Delta^{-1}I)$. Now suppose that $\beta \gamma \in r_{\Delta^{-1}R}(\Delta^{-1}I)$. Therefore $0 = \Delta^{-1}I\beta \gamma = \Delta^{-1}Iu^{-1}av^{-1}b = \Delta^{-1}Iab(uwv)^{-1}$ iff $Iab = 0$, iff $Iab(b) = 0$, iff $\Delta^{-1}Iab(b)(uwv)^{-1} = 0$, iff $\beta \alpha(\gamma) \in r_{\Delta^{-1}R}(\Delta^{-1}I)$, since $I$ is an $\alpha$-semicommutative left ideal of $R$ and $\alpha$ is endomorphism of $R$ and $\alpha(\gamma) = v^{-1}\alpha(b)$. Hence $\Delta^{-1}I$ is an $\alpha$-semicommutative left ideal of $\Delta^{-1}R$.

Theorem 2.2 is proved.

A ring of $R$ is called right Ore if given $a, b \in R$ with $b$ regular there exist $a_1, b_1 \in R$ with $b_1$ regular such that $ab_1 = ba_1$. It is a well-known fact that $R$ is a right Ore ring if and only if there exists a classical right quotient ring of $R$.

**Theorem 2.3.** Suppose that there exists a classical right quotient $Q$ of a ring $R$ consisting of central elements. We have the following conditions:

1. $I$ is an $\alpha$-semicommutative left ideal of $R$ if and only if $QI$ is an $\alpha$-semicommutative left ideal of $Q$.
2. $I$ is an $\alpha$-symmetric left ideal of $R$ if and only if $QI$ is an $\alpha$-symmetric left ideal of $Q$.
3. $I$ is an $\alpha$-left reduced ideal of $R$ if and only if $QI$ is an $\alpha$-left reduced ideal of $Q$.

**Proof.** For instance, we prove (1). Let $\beta \gamma \in r_Q(QI)$ with $\beta = u^{-1}a$, $\gamma = v^{-1}b, u, v \in R$ and $a, b \in R$. Since $Q$ is contained in the center of $R$, we have $0 = QI\beta \gamma = QIu^{-1}av^{-1}b = QIab(uwv)^{-1}$, so $Iab = 0$. It follows that $arb \in r_R(I)$ for all $r \in R$, since $I$ is an $\alpha$-semicommutative ideal of $R$. Now for $\delta = w^{-1}r$ with $w \in R$ and $r \in R, QI\beta \delta \gamma = QIarb(uwv)^{-1} = 0$. Thus $\beta \delta \gamma \in r_Q(QI)$. Now suppose that $\beta \gamma \in r_Q(QI)$. Therefore $0 = QI\beta \gamma = QIu^{-1}av^{-1}b = QIab(uwv)^{-1}$ iff $Iab = 0$, iff $Iab(b) = 0$, iff $QIab(b)(uwv)^{-1} = 0$, iff $\beta \alpha(\gamma) \in r_Q(QI)$, since $I$ is an $\alpha$-semicommutative ideal of $R$ and $\alpha$ is endomorphism of $R$ and $\alpha(\gamma) = v^{-1}\alpha(b)$. Hence $QI$ is an $\alpha$-semicommutative left ideal of $Q$.

Theorem 2.3 is proved.
Let \( \alpha \) be an automorphism of a ring \( R \). Suppose that there exists a classical right quotient \( Q \) of a ring \( R \). Then for any \( b^{-1}a \in Q \), where \( a, b \in R \) with \( b \) regular the induced map \( \alpha : Q(R) \to Q(R) \) defined by \( \alpha(b^{-1}a) = (\alpha(b))^{-1}\alpha(a) \) is also an automorphism.

**Proposition 2.9.** Suppose that there exists a classical right quotient \( Q \) of a ring \( R \) consisting of central elements. If \( I \) is \( \alpha \)-semicommutative left ideal of \( R \), then \( I \) is \( \alpha \)-skew Armendariz left ideal of \( R \) if and only if \( QI \) is \( \alpha \)-skew Armendariz left ideal of \( Q \).

**Proof.** Suppose that \( I \) is \( \alpha \)-skew Armendariz. Let \( f(x) = s_0^{-1}a_0 + s_1^{-1}a_1x + \ldots + s_m^{-1}a_mx^m \) and \( g(x) = t_0^{-1}b_0 + t_1^{-1}b_1x + \ldots + t_n^{-1}b_nx^n \) such that \( f(x)g(x) \in r_{QI[x,M]}(QI[x]) \). Let \( C \) be a left denominator set. There exist \( s, t \in C \) and \( a_i, b_j \in R \) such that \( s_i^{-1}a_i = s^{-1}a_i t^{-1}b_j = t^{-1}b_j \) for \( i = 0, 1, \ldots, m \) and \( j = 0, 1, \ldots, n \). Then \( s^{-1}(a_0 + a_1x + \ldots + a_mx^m)t^{-1}(b_0 + b_1x + \ldots + b_nx^n) \in r_{QI[x,M]}(QI[x]) \). It follows that \( (a_0 + a_1x + \ldots + a_mx^m)t^{-1}(b_0 + b_1x + \ldots + b_nx^n) \in r_{QI[x,M]}(QI[x]) \). Thus \( (a_0t^{-1} + a_1(t^{-1}))^{-1}x + \ldots + a_m(\alpha(m)(t^{-1}))^{-1}x^n(b_0 + b_1x + \ldots + b_nx^n) \in r_{QI[x,M]}(QI[x]) \).

For \( a_i(\alpha(t))^{-1} \), \( i = 0, 1, \ldots, n \), there exist \( t' \in C \) and \( a''_i \in R \) such that \( a''_i(\alpha(t))^{-1} = t'^{-1}a''_i \). Then \( t'^{-1}a''_i \alpha(t''_i) \in r_{QI}(QI) \). So \( a''_i(\alpha(t''_i)^{-1})^{-1} \alpha(t''_i) = (t'^{-1}a''_i)^{-1} \in r_{QI}(QI) \). Similarly we have \( (s_i^{-1}a''_i x, s_i^{-1}a''_i x, s_i^{-1}a''_i x, \ldots, s_i^{-1}a''_i x) \in r_{QI[x,M]}(QI[x]) \).

Let \( \beta \gamma \in r_{QI}(QI) \) with \( \beta = u^{-1}a, \gamma = v^{-1}b, u, v \in R \) and \( a, b \in R \). Therefore \( 0 = QI\beta\gamma \) is an automorphism of \( R \) and \( Q \) is contained in the center of \( R \). Thus \( QI \) is \( \alpha \)-skew Armendariz. The converse is clear.

Proposition 2.9 is proved.


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