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FRACTIONAL CALCULUS OF A UNIFIED MITTAG-LEFFLER FUNCTION

The main aim of the paper is to introduce an operator in the space of Lebesgue measurable real or complex functions \( L(a, b) \). Certain properties of the Riemann–Liouville fractional integrals and differential operators associated with the function \( E_{\alpha, \beta, \lambda, \mu, \rho, p}^{\gamma, \delta}(cz; s, r) \) are studied and the integral representations are obtained. Some properties of a special case of this function are also studied by means of fractional calculus.

1. Introduction, definitions and preliminaries. The Mittag-Leffler function has been studied by many researchers either in context with obtaining new properties or by introducing a new generalization and then deriving its properties [9, 11, 13]. Recently, we [7] have also studied various properties of our newly introduced generalization of Mittag-Leffler function in the form

\[
E_{\alpha, \beta, \lambda, \mu, \rho, p}^{\gamma, \delta}(cz; s, r) = \sum_{n=0}^{\infty} \frac{[\gamma]^n}{\Gamma(\alpha(n + \rho - 1))} \frac{(\lambda)_n}{(\rho)_n} f(t) dt,
\]

where \( \alpha, \beta, \gamma, \lambda, \rho \in \mathbb{C}, \ Re(\alpha, \beta, \gamma, \lambda, \rho) > 0; \delta, \mu, p, c > 0 \) and \( (\gamma)_n = \frac{\Gamma(\gamma + qn)}{\Gamma(\gamma)} \) is the generalized Pochhammer symbol [8]. In particular, if \( q \in \mathbb{N} \), it takes the form

\[
(\gamma)_q = q^n \prod_{r=1}^{q} \left( \frac{\gamma + r - 1}{q} \right).
\]

If \( p = 1, \rho = 1, r = 0, s = 1, q = 1, c = 1, \) then (1.1) yields the generalization due to Shukla and Prajapati [11]. Here, we also introduce an operator denoted and defined by

\[
\mathcal{E}_{\alpha, \beta, \lambda, \mu, \rho, p, \omega, a}^{\gamma, \delta}(f)(x) = \int_{a}^{x} (x-t)^{\beta-1} E_{\alpha, \beta, \lambda, \mu, \rho, p}^{\gamma, \delta}(\omega(x-t)^{\alpha}; s, r) f(t) dt,
\]

where \( \alpha, \beta, \gamma, \lambda, \rho, \omega \in \mathbb{C}; \ Re(\alpha, \beta, \gamma, \lambda, \rho) > 0; \delta, \mu, p > 0 \), and \( x > a \).

We enlist the following definitions and well-known formulas for studying the properties of the Riemann–Liouville (R–L) fractional integrals and differential operators associated with our generalization (1.1) as well as as the operator (1.2).

The space \( L(a, b) \) of (real or complex valued) Lebesgue measurable functions [4, 10] is given by

\[
L(a, b) = \left\{ f : ||f|| = \int_{a}^{b} |f(t)| dt < \infty \right\}.
\]
For \( f(x) \in L(a, b), \mu \in \mathbb{C}, \) and \( \text{Re}(\mu) > 0, \) the R–L fractional integrals of order \( \mu \) [10] are defined as follows.

The left-sided R–L fractional integral operator of order \( \mu \) is defined as
\[
aI^\mu_a f(x) = I^\mu_a f(x) = \frac{1}{\Gamma(\mu)} \int_a^x \frac{f(t)}{(x-t)^{1-\mu}} dt, \quad x > a, \tag{1.4}
\]
where the right-sided R–L fractional integral operator of order \( \mu \) is defined as
\[
xI^\mu_b f(x) := I^\mu_b f(x) = \frac{1}{\Gamma(\mu)} \int_x^b \frac{f(t)}{(x-t)^{1-\mu}} dt, \quad x < b. \tag{1.5}
\]

Further, if \( \mu, \beta \in \mathbb{C}, \text{Re}(\mu, \beta) > 0, \) then [6, 10]
\[
I^\mu_a [(t-a)^{\beta-1}](x) = \frac{\Gamma(\beta)}{\Gamma(\mu + \beta)} (x-a)^{\mu + \beta - 1}. \tag{1.6}
\]  

For \( \mu \in \mathbb{C}, \text{Re}(\mu) > 0; n = \lfloor \text{Re}(\mu) \rfloor + 1, \) the R–L fractional derivative is
\[
(D^\alpha_{a+} f)(x) = \left( \frac{d}{dx} \right)^n (I^{n-\alpha}_{a+} f)(x). \tag{1.7}
\]

Then for \( \alpha, \beta, \gamma, \lambda, \rho, \in \mathbb{C}, \text{Re}(\alpha, \beta, \gamma, \lambda, \rho, (\beta - m)) > 0, \) and \( \delta, \mu, p, m \in \mathbb{N}, \) we have shown that [7]
\[
\left( \frac{d}{dz} \right)^m \left[ z^{\beta-1} E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(\omega (cz)^\alpha; s,r) \right] = z^{\beta-m-1} E_{\alpha,\beta-m,\lambda,\mu,\rho,p}^{\gamma,\delta}(\omega (cz)^\alpha; s,r). \tag{1.8}
\]

The fractional integral operator investigated by Erdélyi–Kober is defined and represented as
\[
I^\eta_{x} f(x) = \frac{x^{-\eta+1}}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} t^{-\nu-1} f(t) dt, \quad \text{Re}(\nu) > 0, \quad \eta > 0, \tag{1.9}
\]
which is a generalization of the R–L fractional integral operator (1.5).

Hilfer [2, 3] generalized the R–L fractional derivative operator \( D^\mu_{a+} \) in (1.6) by introducing a right-sided fractional derivative operator \( D^{\mu,\nu}_{a+} \) of order \( 0 < \mu < 1 \) and type \( 0 \leq \nu \leq 1 \) with respect to \( x \) as follows:
\[
(D^{\mu,\nu}_{a+} f)(x) = \left( I^\nu_{a+} \frac{d}{dx} (I^{\nu(1-\mu)}_{a+} f)(x) \right). \tag{1.10}
\]

The difference between the fractional derivatives of various types becomes apparent from the following formula involving the Laplace transformation [2, 3]:
\[
\mathcal{L}[D^{\mu,\nu}_{a+} f(x)](s) = s^\mu \mathcal{L}[f(x)](s) - s^{\nu(1-mu)} (f^{(1-\nu)(1-\mu)}_0 f)(0+), \tag{1.11}
\]
where \( 0 < \mu < 1, \) and the initial-value term: \( (f^{(1-\nu)(1-\mu)}_0 f)(0+) \) involves the R–L fractional integral operator of order \( (1 - \nu)(1 - \mu) \) evaluated in the limit as \( t \to 0+. \) Here, as usual
provided that the defining integral exists.

Prajapati, Dave and Nathwani [7] has shown that the Mellin–Barnes integral for the function defined by (1.1) is given by

\[
E_{\alpha, \beta, \lambda, \rho, p}^{c, d} \left( z; s, r \right) = \frac{[\Gamma(\lambda)]^r [\Gamma(\rho)]^p z^{\rho - 1}}{2\pi i [\Gamma(\gamma)]} \times \frac{\Gamma(p \xi) \Gamma(1 + p \xi) [\Gamma(\gamma + \delta \xi)]^s (-z)^r}{\Gamma(\beta + \alpha \rho - \alpha + \alpha \rho \xi) [\Gamma(\lambda + \mu \xi)]^r \Gamma(\rho + p \xi)} d\xi. \tag{1.13}
\]

Wright generalized hypergeometric function [1] is defined as

\[
p \psi \left[ \begin{array}{c}
(a_1, A_1), \ldots, (a_p, A_p); \\
(b_1, B_1), \ldots, (b_q, B_q);
\end{array} \right] = \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + r A_j) z^r}{\prod_{j=1}^{q} \Gamma(b_j + r B_j) r!}
\]

\[
= H_{p,q}^{m,n} \left[ \begin{array}{c}
-(a_1, A_1), \ldots, (a_p, A_p); \\
(b_1, B_1), \ldots, (b_q, B_q);
\end{array} \right], \tag{1.14}
\]

where \( H_{p,q}^{m,n} \) denotes the Fox H-function and \( a_i, b_j \in \mathbb{C}, A_i, B_j \in \mathbb{R}, i = 1, 2, \ldots, p; j = 1, 2, \ldots, q, 1 + \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > 0. \)

2. Main results. We prove in this section the following results.

Theorem 2.1. Let \( a \in \mathbb{R}_+ = [0, \infty), \alpha, \beta, \gamma, \lambda, \rho, \eta \in \mathbb{C}, \text{Re}(\alpha, \beta, \gamma, \lambda, \rho, \eta) > 0; \delta, \mu, p > 0 \)

for \( x > a, \) then

\[
\left( I_{a+} (t-a)^{\beta - 1} E_{\alpha, \beta, \lambda, \rho, p}^{c, d} \left( \omega(c(t-a))^\alpha; s, r \right) \right) (x) = (x-a)^{(\alpha + \beta - 1)} E_{\alpha, \beta, \lambda, \rho, p}^{c, d} \left( \omega(c(x-a))^\alpha; s, r \right) \tag{2.1}
\]

and

\[
\left( D_{a+} (t-a)^{\beta - 1} E_{\alpha, \beta, \lambda, \rho, p}^{c, d} \left( \omega(c(t-a))^\alpha; s, r \right) \right) (x) = (x-a)^{(\beta - \eta - 1)} E_{\alpha, \beta, \lambda, \rho, p}^{c, d} \left( \omega(c(x-a))^\alpha; s, r \right). \tag{2.2}
\]

Proof. Applying (1.1) to the left-hand side of (1.13) and then using (1.6), it yields

\[
\left( I_{a+} (t-a)^{\beta - 1} E_{\alpha, \beta, \lambda, \rho, p}^{c, d} \left( \omega(c(t-a))^\alpha; s, r \right) \right) (x) = (x-a)^{(\beta + \eta - 1)} \sum_{n=0}^{\infty} \frac{[\gamma]_n^\alpha \omega(c(x-a))^\alpha}{\Gamma(\alpha(n + \beta + \eta - 1)) \Gamma(\gamma)^n} \frac{[\Gamma(\lambda)^n]}{(\rho)^n}. \tag{1.14}
\]
Here using (1.1) once again leads us to (2.1).

Now, using (1.7) to the left-hand side of (2.2) and then applying (2.1), we get

\[ \left( D^\eta_{\alpha} (t - a)^{\beta - 1} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p} (\omega(c(t - a))^\alpha; s, r) \right) (x) = \]

\[ = \left( \frac{d}{dx} \right)^n [(x - a)^{\beta - \eta - 1} E^{\gamma,\delta}_{\alpha,\beta - \eta,\lambda,\mu,\rho,p} (\omega(x - a)^\alpha; s, r)]. \]

Further use of (1.7), gives the proof of (2.2).

**Theorem 2.2.** Let \( \alpha, \gamma, \lambda, \rho, \eta \in \mathbb{C}; \text{Re}(\alpha), \text{Re}(\gamma), \text{Re}(\lambda), \text{Re}(\rho), \text{Re}(\eta) > 0; \delta, \mu, p > 0, \) then

\[ 0 I_\eta^\nu \left[ \nu E^{1,\delta}_{\alpha,\beta,\lambda,\mu,\rho} (\nu cx)^\alpha; s, r \right] = \nu x^\eta E^{1,\delta}_{\alpha,\beta + 1,\lambda,\mu,\rho} (\nu(c x)^\alpha; s, r). \]  

**Proof.** Applying (1.4) to the left-hand side of (2.3) and then using (1.1), we get

\[ 0 I_\eta^\nu \left[ \nu E^{1,\delta}_{\alpha,\beta,\lambda,\mu,\rho} (\nu cx)^\alpha; s, r \right] = \]

\[ = \frac{1}{\Gamma(\eta)} \sum_{n=0}^\infty \frac{[(\gamma)_{\delta n}]^n \omega^{(p + \rho - 1) \nu (p + \rho)}}{\Gamma(\alpha(p + \rho) - 1) + \beta} (\nu c^\alpha x) \int_0^x \nu^{(p + \rho - 1)} (x - t)^{\eta - 1} dt. \]

After some simplification and further use of (1.1), gives the proof of (2.3).

**Theorem 2.3.** Let \( \alpha, \beta, \gamma, \lambda, \rho, \nu, \omega \in \mathbb{C}; \text{Re}(\alpha, \beta, \gamma, \lambda, \rho, \nu) > 0; \delta, \mu, p > 0, \) then

\[ (E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,\omega} (t - a)^{\nu - 1}) (x) = (x - a)^{\beta + \nu - 1} \Gamma(\nu) E^{\gamma,\delta}_{\alpha,\beta + \nu, \lambda,\mu,\rho} (\omega(x - a)^\alpha). \]  

**Proof.** Putting \( f(t) = (t - a)^{\nu - 1} \) in (1.2), we get

\[ (E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,\omega} (t - a)^{\nu - 1}) (x) = \int_a^x (x - t)^{\beta - 1} E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,\omega} (\omega(x - t)^\alpha) (t - a)^{\nu - 1} dt, \]

and using (1.1), this reduced to

\[ = \sum_{n=0}^\infty \frac{[(\gamma)_{\delta n}]^n \omega^{(p + \rho - 1)}}{\Gamma(\alpha(p + \rho - 1) + \beta) (\nu c^\alpha x) \int_0^x (x - t)^{\alpha(p + \rho - 1) + \beta - 1} (t - a)^{\nu - 1} dt} \]

and simplifying the above equation, it becomes

\[ = \sum_{n=0}^\infty \frac{[(\gamma)_{\delta n}]^n (\nu c^\alpha x)^{\nu - 1}}{\Gamma(\alpha(p + \rho - 1) + \beta) (\nu c^\alpha x)^{\nu - 1} B(\alpha(p + \rho) + \beta - 1, \nu)} \]

and further simplification of the above equation gives the proof of (2.4).
We show that the operator defined (1.2) is in fact bounded; whose proof is given below in the form of the following theorem.

**Theorem 2.4.** Let the function \( \phi \) be in the space \( L(a,b) \) of Lebesgue measurable functions on a finite interval \([a,b]\) of the real line \( \mathbb{R} \) given by

\[
L(a,b) = \left \{ f : \|f\|_1 = \int_{a}^{b} |f(t)| \, dt < \infty \right \}.
\]

Then the integral operator \( \mathcal{E}^{(\alpha,\beta,\lambda,\mu,\rho,p)}_{a+;\, \omega;\, \alpha+;\, \delta} \) is bounded on \( L(a,b) \) and

\[
\left\| \mathcal{E}^{(\alpha,\beta,\lambda,\mu,\rho,p)}_{a+;\, \omega;\, \alpha+;\, \delta} \phi \right\|_1 \leq M \|\phi\|_1,
\]

where the constant \( M, 0 < M < \infty \), given by

\[
M = (b-a)^{\text{Re}(\beta)} \sum_{k=0}^{\infty} \frac{|(\gamma)\delta k|^s}{\Gamma(\alpha(pk+\rho-1)+\beta)} (\text{Re}(\alpha pk + \rho - 1 + \beta)) \times \frac{\omega((b-a)c)^{\text{Re}(pk+\rho-1)}}{(|\lambda\mu k|^r|\rho pk|}.
\]

**Proof.** Using (1.2) and (1.3) and interchanging the order of integration by applying the Dirichlet formula [9], we have

\[
\left\| \mathcal{E}^{(\alpha,\beta,\lambda,\mu,\rho,p)}_{a+;\, \omega;\, \alpha+;\, \delta} \phi \right\|_1 =
\]

\[
= \int_{a}^{b} \left| \int_{a}^{x} (x-t)^{\beta-1} E^{(\alpha,\beta,\lambda,\mu,\rho,p)}_{a+;\, \omega;\, \alpha+;\, \delta} \left( \omega((x-t)^{\alpha}; s, r) \right) \phi(t) \, dt \right| \, dx \leq
\]

\[
\leq \int_{a}^{b} \left\| F^{(\alpha,\beta,\lambda,\mu,\rho,p)}_{a+;\, \omega;\, \alpha+;\, \delta} \left( \omega((x-t)^{\alpha}; s, r) \right) \right\|_1 \, dx \, |\phi(t)| \, dt.
\]

On substituting \( x-t = u \), using (1.1) and simplification of the above equation yields

\[
= \int_{a}^{b} \int_{0}^{b-t} u^{\text{Re}(\beta)-1} \left| E^{(\alpha,\beta,\lambda,\mu,\rho,p)}_{a+;\, \omega;\, \alpha+;\, \delta} \left( \omega(u^{\alpha}; s, r) \right) \right| \, du \, |\phi(t)| \, dt \leq
\]

\[
\leq \int_{a}^{b} \sum_{k=0}^{\infty} \frac{|(\gamma)\delta k|^s}{\Gamma(\alpha(pk+\rho-1)+\beta)} \left( |\lambda\mu k|^r|\rho pk| \right) \times \int_{0}^{u} u^{\text{Re}(\alpha(pk+\rho-1)+\beta)} \, |\phi(t)| \, dt =
\]

\[
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\]
\[
\int_{a}^{b} \frac{|(\gamma)_{\delta} k|^s (\omega c^\alpha)_{p+1} |b - a|^\text{Re}(\alpha(pk + p - 1) + \beta) |\phi(t)|}{|\Gamma(\alpha(pk + p - 1) + \beta)| |(\lambda)_{\mu} k|^r |(\rho)_{pk}|} \, dt = \\
= (b - a)^\text{Re}(\beta) \sum_{k=0}^{\infty} \frac{|(\gamma)_{\delta} k|^s}{|\Gamma(\alpha(pk + p - 1) + \beta)| |(\lambda)_{\mu} k|^r |(\rho)_{pk}|} 	imes \\
\times \frac{|\omega (c(b - a))^\text{Re}(\alpha)|^s}{|\text{Re}(\alpha(pk + p - 1) + \beta)|} \int_{a}^{b} |\phi(t)| \, dt = \\
= (b - a)^\text{Re}(\beta) \sum_{k=0}^{\infty} \frac{|(\gamma)_{\delta} k|^s |\omega (b - a)^{\alpha}|^s |\text{Re}(pk + p - 1)|}{|\Gamma(\alpha(pk + p - 1) + \beta)| |\text{Re}(\alpha(pk + p - 1) + \beta)| |(\lambda)_{\mu} k|^r |(\rho)_{pk}|} = \\
= \mathcal{M} \|\phi\|_1,
\]

where \(\mathcal{M}\) is finite and given by (2.6). This completes proof of the boundedness property of the integral operator \(E_{\alpha, \beta, \lambda, \mu, p, \omega, \alpha+}\) as asserted by Theorem 2.4.

The following theorem incorporates the fractional differential equation for (1.2).

**Theorem 2.5.** If \(0 < \eta < 1, 0 \leq \nu \leq 1, \omega, \xi \in \mathbb{C}, R(\alpha) = R(\delta) - 1 > 0\) and \(\min\{\text{Re}(\beta, \gamma, \lambda, \mu, \rho)\} > 0\), then

\[
(D_{0+}^{\alpha, \delta} y)(x) = \xi \left(E_{\alpha, \beta, \lambda, \mu, p, \omega; \alpha+}^{\gamma, \delta}(x) + f(x)ight) \tag{2.7}
\]

with the initial condition

\[
\left(I_{0+}^{(1-\nu)(1-\eta)} y\right)(0+) = C,
\]

has solution in the space \(L(0, \infty)\) given by

\[
y(x) = C x^{\frac{\eta - \nu (1 - \eta) - 1}{\Gamma(\eta - \nu + \eta \nu)}} + \xi x^{\eta + \beta} E_{\alpha, \beta+\eta+1, \lambda, \mu, p} (\omega(\alpha x^\alpha)) + \\
+ \frac{1}{\Gamma(\eta)} \int_{0}^{x} (x - t)^{\eta-1} f(t) \, dt, \tag{2.8}
\]

where \(C\) is arbitrary constant.

**Proof.** Applying the Laplace transform of each side of (2.7), and using the formulas (1.2) and (1.11), we find by means of the Laplace convolution theorem that

\[
s^\eta Y(s) - C s^\nu (1-\eta) = \xi L\left[x^{\beta-1} E_{\alpha, \beta, \lambda, \mu, p} (\omega x^\alpha)\right](s) L(1)(s) + F(s) = \\
= \xi s^{-\beta-1} \sum_{n=0}^{\infty} \frac{|(\gamma)_{\delta} n|^s (\omega(\alpha s)^\alpha)^n + \rho - 1}{|(\lambda)_{\mu} n|^r |(\rho)_{pn}|} + F(s)
\]

which readily yields

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Now, by taking the inverse Laplace transform of each side of equation (2.9), we get
\[
Y(s) = C \ s^{\nu(1-\eta) - \eta} + \xi \ s^{-\beta-\eta-1} \sum_{n=0}^{\infty} \frac{[(\gamma)_{n}]^\eta \ (\omega (a s)_{\eta})^{n+\rho-1}}{[(\lambda)_{\mu}]^r (\rho)^n} + F(s) \ s^{-\eta}.
\] (2.9)

Now, by taking the inverse Laplace transform of each side of equation (2.9), we get
\[
y(x) = C \ \mathcal{L}^{-1}(s^{\nu(1-\eta) - \eta})(x) + \\
+ \xi \sum_{n=0}^{\infty} \frac{[(\gamma)_{n}]^\eta \ (\omega (a)^{n+\rho-1})}{[(\lambda)_{\mu}]^r (\rho)^n} \mathcal{L}^{-1}(s^{-\alpha(\mu+\rho-1)-\beta-\eta-1})(x) + \mathcal{L}^{-1}(s^{-\eta} F(s)) = \\
= C \ \frac{x^{\eta+\rho-1}}{\Gamma(\eta+\rho+1)} + \xi \ x^{\eta+\beta} \ E_{\alpha,\beta+\eta+1,\lambda,\mu,\rho}(\omega (a x)^\eta) + \\
+ \frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} f(t) \ dt
\]

which completes the proof of Theorem 2.5 under the various already stated parametric constraints.

**Theorem 2.6.** Let \(0 < \eta < 1, 0 \leq \nu \leq 1, \omega, \xi \in \mathbb{C}, R(\alpha) = R(\delta) - 1 > 0\) and \(\min\{R(\beta, \gamma, \lambda, \mu, \rho)\} > 0\), then
\[
\left( D_{0^+}^{\nu, \eta} y \right)(x) = \xi \left( E_{\alpha,\beta+\eta+1,\lambda,\mu,\rho}\omega (a x)^\eta; s, r \right)
\]
with the initial condition
\[
\left( I_{0^+}^{(1-\nu)(1-\eta)} y \right)(0^+) = C
\]
has solution in the space \(L(0, \infty)\) given by
\[
y(x) = C \ \frac{x^{\eta+\rho-1}}{\Gamma(\eta+\rho+1)} + \xi \ x^{\eta+\beta} \ E_{\alpha,\beta+\eta+1,\lambda,\mu,\rho}(\omega (a x)^\eta; s, r),
\]
where \(C\) is arbitrary constant.

**Proof.** Now, substituting
\[
f(t) = t^{\beta} \ E_{\alpha,\beta+\eta+1,\lambda,\mu,\rho}(\omega (a t)^\eta; s, r)
\]
in above Theorem 2.5, we get
\[
y(x) = C \ \frac{x^{\eta+\rho-1}}{\Gamma(\eta+\rho+1)} + \xi \ x^{\eta+\beta} \ E_{\alpha,\beta+\eta+1,\lambda,\mu,\rho}(\omega (a x)^\eta; s, r) + \\
+ \frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} t^{\beta} \ E_{\alpha,\beta+\eta+1,\lambda,\mu,\rho}(\omega (a x)^\eta; s, r) \ dt.
\] (2.12)
Here

\[
\int_0^x (x-t)^{\eta-1} t^\beta E_{\alpha, \beta+1, \lambda, \mu, \rho, p}^\gamma (\omega(ax)^n; s, r) \, dt =
\]

\[
= \int_0^x (x-t)^{\eta-1} t^\beta \sum_{n=0}^{\infty} \frac{[(\gamma)_n]^{s}(\omega (at)^n)^{pn+\rho-1}}{\Gamma(\alpha(pm + \rho - 1) + \beta) \Gamma(\lambda_m)^r (p)_{pn}} \, dt =
\]

\[
= \sum_{n=0}^{\infty} \frac{[(\gamma)_n]^{s}(\omega (a)^n)^{pn+\rho-1}}{\Gamma(\alpha(pm + \rho - 1) + \beta) \Gamma(\lambda_m)^r (p)_{pn}} \times \int_0^x (x-t)^{\eta-1} t^{\alpha(pm+\rho-1)+\beta} \, dt.
\]

(2.13)

Take \( t = xu \), then \( dt = xdu \) and as \( t \to 0, u \to 0 \) and as \( t \to x, u \to 1 \)

\[
= \sum_{n=0}^{\infty} \frac{[(\gamma)_n]^{s}(\omega (a)^n)^{pm+\rho-1} x^{\alpha(pm+\rho-1)+\eta+\beta}}{\Gamma(\alpha(pm + \rho - 1) + \beta) \Gamma(\lambda_m)^r (p)_{pn}} \int_0^1 (1-u)^{\eta-1} u^{\alpha(pm+\rho-1)+\beta} \, dt =
\]

\[
= \sum_{n=0}^{\infty} \frac{[(\gamma)_n]^{s}(\omega (a)^n)^{pm+\rho-1} x^{\alpha(pm+\rho-1)+\eta+\beta}}{\Gamma(\alpha(pm + \rho - 1) + \beta) \Gamma(\lambda_m)^r (p)_{pn}} \Gamma(\eta) \Gamma(\alpha(pm + \rho - 1) + \beta + \eta + 1) =
\]

\[
= \frac{\Gamma(\eta) \sum_{n=0}^{\infty} [(\gamma)_n]^{s}(\omega (a)^n)^{pm+\rho-1} x^{\alpha(pm+\rho-1)+\eta+\beta}}{\Gamma(\alpha(pm + \rho - 1) + \beta + \eta + 1) \Gamma(\lambda_m)^r (p)_{pn}} =
\]

\[
x^{(\eta+\beta)} \Gamma(\eta) E_{\alpha, \beta+\eta+1, \lambda, \mu, \rho, p}^\gamma (\omega(ax)^n; s, r)
\]

using this in (2.12) we get (2.11).

Which completes the proof of Theorem 2.6.

**Theorem 2.7** (Mellin transform of the operator \( \{E_{\alpha, \beta, \lambda, \mu, \rho, p; \omega; 0, f}(x)\} \)). Let \( \alpha, \beta, \gamma, \lambda, \rho, \omega \in \mathbb{C}, \) \( \text{Re}(\alpha, \beta, \gamma, \lambda, \rho) > 0, \delta, \mu > 0, p \in \mathbb{N}, \text{Re}(1-S - \alpha \rho + \alpha - \beta) > 0, \) then

\[
M \left\{ \left( E_{\alpha, \beta, \lambda, \mu, \rho, p; \omega; 0, f}(x) \right); S \right\} = \frac{[\Gamma(\lambda)]^{s} \Gamma(p)}{2\pi i [\Gamma(\gamma)]^{s} \Gamma(1-S)} H_{s+1,r+3}^{r+3,s+1} \times
\]

\[
\left[ -wt^\alpha \begin{pmatrix} [(1-\gamma, \delta)]^{s}, (0, p) \\ (0, 1), (1-S - \alpha \rho + \alpha - \beta, \alpha p), [(1-\lambda, \mu)]^{s}, (1-\rho, p) \end{pmatrix} \right] M \{ t^\beta f(t); S \}.
\]

(2.14)

**Proof.** By the definition of the Mellin transform, we have

\[
M \left\{ \left( E_{\alpha, \beta, \lambda, \mu, \rho, p; \omega; 0, f}(x) \right); S \right\} =
\]
\[
\int_0^\infty x^{S-1} \int_0^x (x-t)^{\beta-1} E_{\alpha,\beta,\lambda,\mu,\rho,p}(\omega(x-t)^{\alpha}; s, r) f(t) \, dt \, dx.
\]

Interchanging the order of integration, which is permissible under the given conditions, we find that

\[
M \{ (E_{\alpha,\beta,\lambda,\mu,\rho,p,\omega}^{\gamma,\delta}; 0 + f) (x); S \} = \int_0^\infty f(t) \int_t^\infty x^{S-1} (x-t)^{\beta-1} E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(\omega(x-t)^{\alpha}; s, r) \, dx \, dt.
\]

If we set \( x = t + u \) the above integral takes the form

\[
M \{ (E_{\alpha,\beta,\lambda,\mu,\rho,p,\omega}^{\gamma,\delta}; 0 + f) (x); S \} = \int_0^\infty f(t) \int_t^\infty (t+u)^{S-1} u^{\beta-1} E_{\alpha,\beta,\lambda,\mu,\rho,p}^{\gamma,\delta}(\omega u^{\alpha}) \, du \, dt.
\]

To evaluate the \( u \)-integral, we express the Mittag-Leffler function in terms of its Mellin–Barnes contour integral by means of the formula (1.14), then the above expression transforms into the form

\[
M \{ (E_{\alpha,\beta,\lambda,\mu,\rho,p,\omega}^{\gamma,\delta}; 0 + f) (x); S \} = \int_0^\infty f(t) \frac{\Gamma(\lambda)}{2\pi i \Gamma(\gamma) \Gamma(\rho) \Gamma(\mu)} \times \int_{-i\infty}^{i\infty} \frac{\Gamma(-p\xi) \Gamma(1+p\xi) \Gamma(\gamma+\delta \xi)}{\Gamma(\beta+\alpha \rho - \alpha + \alpha p \xi) \Gamma(\lambda+\mu \xi)} \times \int_0^\infty (t+u)^{s-1} u^{(p\xi+\rho-1)+\beta-1} \, du \, d\xi \, dt.
\]

If we evaluate the \( u \)-integral with the help of the formula

\[
\int_0^\infty x^{s-1}(x+a)^{-\rho} \, dx = \frac{\Gamma(s) \Gamma(\rho - s)}{\Gamma(\rho)}, \quad \text{Re}(\rho) > \text{Re}(s) > 0,
\]

then after some simplification, it is seen that the right-hand side of above equation simplifies to

\[
\frac{\Gamma(\lambda)}{2\pi i \Gamma(\gamma) \Gamma(1-s)} \int_{-i\infty}^{i\infty} \frac{\Gamma(-p\xi) \Gamma(1+p\xi) \Gamma(\gamma+\delta \xi) \Gamma(1-s-\alpha(p\xi+\rho-1)-\beta)}{\Gamma(\lambda+\mu \xi) \Gamma(\rho+p\xi)} \times (-\omega t^\alpha)^{p\xi+\rho-1} \, d\xi \int_0^\infty t^{\beta+s-1} f(t) \, dt.
\]

By using the definition of \( H \)-function yields the desired result.

For \( s = 1, r = 0, \rho = 1, p = 1, \delta = q \) the Theorem 2.7 reduces to the following corollary.
Corollary 2.1.

\[
M \left\{ \left. \mathcal{E}_{\alpha,\beta,\mu,\lambda,\rho,\omega}^{\gamma,\delta,\omega,\rho,\omega} f \right| (x) ; S \right\} = \frac{1}{\Gamma(\gamma)\Gamma(1 - S)} \times \nabla \beta f(t) ; S \right\},
\]

where \( \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \text{Re}(\gamma) > 0; q \in (0, 1) \cup \mathbb{N}, \text{Re}(1 - S - \beta) > 0 \) and \( H_{1,2}^{2,1} (\gamma) \) is the H-function defined by (1.15).

Theorem 2.8 (Laplace transform of the operator \( \mathcal{E}_{\alpha,\beta,\mu,\lambda,\rho,\omega}^{\gamma,\delta,\omega,\rho,\omega} f \) (x)).

\[
\mathcal{L} \left\{ \left. \mathcal{E}_{\alpha,\beta,\mu,\lambda,\rho,\omega}^{\gamma,\delta,\omega,\rho,\omega} f \right| (x) ; P \right\} = \frac{[\Gamma(\lambda)]^r}{\Gamma(\gamma)^s} \int_{\psi_{t+1}} \left[ \frac{[\gamma, q]_s, (1, 1); \omega^p / P \rho^\alpha}{\beta+\alpha - \alpha} \right] F(P),
\]

where \( \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \text{Re}(\gamma) > 0; \text{Re}(p) > |\omega|^{1/\text{Re}(\alpha)} \) and \( F(P) \) is the Laplace transform of \( f(t) \), defined by

\[
\mathcal{L} \{ f(t) ; p \} = F(P) = \int_0^\infty e^{-P t} f(t) \, dt,
\]

where \( \text{Re}(p) > 0 \) and the integral is convergent.

Proof. By virtue of the definition of Laplace transform, it follows that

\[
\mathcal{L} \left\{ \left. \mathcal{E}_{\alpha,\beta,\mu,\lambda,\rho,\omega}^{\gamma,\delta,\omega,\rho,\omega} f \right| (x) ; P \right\} = \int_0^\infty e^{-P t} \int_0^x (x - t)^{\alpha - 1} E_{\alpha,\beta,\mu,\lambda,\rho,\omega}^{\gamma,\delta,\omega,\rho,\omega} [\omega(x - t)^\alpha] f(t) \, dt \, dx.
\]

Interchanging the order of integration, which is permissible under the conditions given in the theorem, we find that

\[
\int_0^\infty f(t) dt \int_0^\infty e^{-P t} (x - t)^{\alpha - 1} E_{\alpha,\beta,\mu,\lambda,\rho,\omega}^{\gamma,\delta,\omega,\rho,\omega} [\omega(x - t)^\alpha] dx.
\]

If we set \( x = t + u \) we obtain

\[
\int_0^\infty e^{P t} f(t) dt \int_0^\infty e^{-P u} u^{\beta - 1} E_{\alpha,\beta,\mu,\lambda,\rho,\omega}^{\gamma,\delta,\omega,\rho,\omega} [\omega u^\alpha] du.
\]

On making use of the series definition (1.1), the above expression becomes

\[
= \sum_{k=0}^\infty \frac{[\gamma, q]_s, \omega^{(pk + p - 1)}}{\Gamma(\alpha k + \beta)} [\Gamma(\lambda)]^{r}(\rho)_{pk} \int_0^\infty e^{P t} f(t) dt \int_0^\infty e^{-P u} u^{\beta + \alpha (pk + p - 1) - 1} du =
\]

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After some simplification and using (1.1), we can write

$$\sum_{k=0}^{\infty} \frac{([\gamma]_{\delta k})^s \omega^{pk+\rho-1}}{P^{\beta+\alpha(pk+\rho-1)[[(\lambda)_{\mu k}]^r] (\rho)_{pk}}} \int_0^\infty e^{-pt} f(t) \, dt = \frac{\Gamma(\gamma)^r \Gamma(\rho) \omega^{\rho-1}}{\Gamma(\gamma)^s P^{\beta+\alpha pk+\beta}} \left[ \left[ (\gamma, q) \right]^s, (1, 1); \omega P^{\rho \alpha} \right] F(p)$$

and $F(P)$ is the Laplace transforms of $f(t)$.

For $s = 1, r = 0, \rho = 1, p = 1, \delta = q$ the Theorem 2.8 reduces to the following corollary.

**Corollary 2.2.**

$$L \left\{ \left( e^{\gamma q} \right)(x); P \right\} = \frac{1}{\Gamma(\gamma)} P^{-\beta} \psi_0 \left[ \left( \frac{\gamma}{\omega} \right) P^{\alpha} \right] F(p),$$

where $\text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \text{Re}(\gamma) > 0; \text{Re}(P) > |\omega|^{1/\text{Re}(\alpha)}$ and $F(P)$ is the Laplace transform of $f(t)$, defined by

$$L \{ f(t); P \} = F(P) = \int_0^\infty e^{-Pt} f(t) \, dt,$$

where $\text{Re}(P) > 0$ and the integral is convergent.

**3. Properties.** In this section certain properties of the functions $E_t(c, \nu, \gamma, \delta, \lambda, \mu, \rho, p)$ and $E_t(c, -\eta, \gamma, \delta, \lambda, \mu, \rho, p)$ will be obtained. We begin with the function

$$f(t) = \sum_{n=0}^{\infty} \frac{([\gamma]_{\delta n})^s (ct)^{\nu-1}}{\Gamma(\rho) (\nu)^2 (\lambda)^2}$$

where $\gamma \in \mathbb{C}, \delta > 0, c$ — arbitrary constant.

Now, using (1.4), the fractional integral operator of order $\nu$ is given as

$$I^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t - \xi)^{\nu-1} \sum_{n=0}^{\infty} \frac{([\gamma]_{\delta n})^s (ct)^{\nu-1}}{\Gamma(\rho) (\nu)^2 (\lambda)^2} \, d\xi = \frac{1}{\Gamma(\nu)} \sum_{n=0}^{\infty} \frac{([\gamma]_{\delta n})^s e^{(\nu-1)ct}}{\Gamma(\rho) (\nu)^2 (\lambda)^2} \int_0^t \xi^{\nu-1} (t - \xi)^{\nu-1} \, d\xi.$$

After some simplification and using (1.1), we can write

$$I^\nu f(t) = t^\nu \sum_{n=0}^{\infty} \frac{([\gamma]_{\delta n})^s (ct)^{\nu-1}}{\Gamma(1(pn+\rho-1)+\nu+1) (\nu)^2 (\lambda)^2} = t^\nu E_{1, \nu+1, \lambda, \mu, \rho, p}^\nu (ct) \text{, (3.1)}$$

We denote the function (3.2) as $E_t(c, \nu, \gamma, \delta, \lambda, \mu, \rho, p)$, i.e.,
Now, using (1.7), the fractional differential operator of order $\eta$ is given as

$$D^\eta f(t) = D^k \left[ I^{k-\eta} \sum_{n=0}^{\infty} \frac{[\gamma]_n (ct)^{(pn+\rho-1)}}{\Gamma((\rho)_{pn})^2} \right].$$

Applying (3.1), after some simplification and using (1.1) it yields

$$D^\eta f(t) = t^{-\eta} \sum_{n=0}^{\infty} \frac{[\gamma]_n (ct)^{(pn+\rho-1)}}{\Gamma((\rho)_{pn})^2} = t^{-\eta} E_{1,1,\gamma,\delta,\lambda,\mu,\rho,p} (ct, s, r).$$

We denote the function (3.4) as $E_t(c, -\eta, \gamma, \delta, \lambda, \mu, \rho, p)$, i.e.,

$$E_t(c, -\eta, \gamma, \delta, \lambda, \mu, \rho, p) = t^{-\eta} E_{1,1,\gamma,\delta,\lambda,\mu,\rho,p} (ct, s, r).$$

**Theorem 3.1.** Let $\gamma \in \mathbb{C}, \text{Re}(\gamma) > 0, \delta > 0, c$ is arbitrary constant and fractional integral and differential operator is of order $\sigma$, then

$$I^\sigma E_t(c, \nu, \gamma, \delta, \lambda, \mu, \rho, p) = E_t(c, \sigma + \nu, \gamma, \delta, \lambda, \mu, \rho, p)$$

and

$$D^\sigma (E_t(c, \nu, \gamma, \delta, \lambda, \mu, \rho, p)) = E_t(c, \nu - \sigma, \gamma, \delta, \lambda, \mu, \rho, p).$$

**Proof.** From (1.4), we get

$$I^\sigma E_t(c, \nu, \gamma, \delta, \lambda, \mu, \rho, p) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-\xi)^{\sigma-1} E_{\xi}(c, \nu, \gamma, \delta, \lambda, \mu, \rho, p) \, d\xi.$$  

Using (3.3), above equation becomes,

$$I^\sigma E_t(c, \nu, \gamma, \delta, \lambda, \mu, \rho, p) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-\xi)^{\sigma-1} \xi^\nu \times$$

$$\times \sum_{n=0}^{\infty} \frac{[\gamma]_n (c\xi)^{(pn+\rho-1)}}{\Gamma((\rho)_{pn})^2} \frac{(\eta)_n (ct)^{(pn+\rho-1)}}{(1+\eta)(1+\nu+1)} \, d\xi.$$  

Now, substituting $\xi = xt$, after some simplification and once again use of (3.3) gives (3.6).

From (1.7) and using (3.6), we get

$$D^\sigma (E_t(c, \nu, \gamma, \delta, \lambda, \mu, \rho, p)) = D^k \{ t^{k-\sigma+\nu} E_{1,k-\eta,\nu+1,\gamma,\delta,\lambda,\mu,\rho,p} (ct, s, r) \}.$$  

Using (1.1) and (3.3), we get (3.7).
Theorem 3.2. Let \( \eta \in \mathbb{C}, \text{Re}(\eta) > 0, \delta > 0, c \) is arbitrary constant and fractional integral and differential operator is of order \( \sigma \), then

\[
I^\sigma E_t(c, -\eta, \gamma, \delta, \lambda, \mu, \rho, p) = E_t(c, \sigma - \eta, \gamma, \delta, \lambda, \mu, \rho, p),
\]

\[
D^\sigma \left( E_t(c, -\eta, \gamma, \delta, \lambda, \mu, \rho, p) \right) = E_t(c, -\sigma - \eta, \gamma, \delta, \lambda, \mu, \rho, p).
\]

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