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THE BIDUAL OF r-ALGEBRAS

БІДУАЛ r-АЛГЕБР

We prove that the order continuous bidual of an Archimedean r-algebra is a Dedekind complete r-algebra with respect to the Arens multiplications.

Доведено, що порядковий неперервний бідуал архімедової r-алгебри є повною r-алгеброю Дедекінда відносно множень Аренса.

1. Introduction. In [11] we studied a new class of lattice ordered algebras; so-called r-algebras and presented its relation with the certain lattice ordered algebras; *f-algebras* [5] (a lattice ordered algebra A with the property that $a \wedge b = 0$ implies $ac \wedge b = ca \wedge b = 0$ for all $c \in A^+$), *almost f-algebras* [6] (a lattice ordered algebra A for which $a \wedge b = 0$ in A implies $ab = 0$), *d-algebras* [9] (a lattice ordered algebra A such that $a \wedge b = 0$ in A implies $ac \wedge bc = ca \wedge cb = 0$ for all $c \in A^+$), *pseudo f-algebras* [7] (a lattice ordered algebra A having the property that $ab = 0$ if $a \wedge b$ is a nilpotent element of A) and *generalized almost f-algebras* [8] (a lattice ordered algebra A such that ab is an annihilator of A if $a \wedge b = 0$). A lattice ordered algebra A in which $a \wedge b = 0$ in A implies $ab \wedge ba = 0$ is called an *r-algebra*. This is a wider class than both the classes of almost f-algebras and d-algebras but in general independent of generalized almost f-algebras. Hence an r-algebra is a generalization of a d-algebra in much the same way as an almost f-algebra is a generalization of an f-algebra. Observe also that the Archimedean r-algebra A is not commutative (for details, see [11]).

In this paper we concentrate on the Arens multiplications [2, 3] in the algebraic bidual of r-algebras and prove that the order continuous bidual of an Archimedean r-algebra is again a Dedekind complete (and hence Archimedean) r-algebra. This is the extension of a result of Bernau and Huijsmans in [4] in which they prove that the order continuous bidual of an almost f-algebra (respectively d-algebra) is again an almost f-algebra (respectively d-algebra).

We now assume that A is a lattice ordered algebra, which is not necessarily commutative or unital. The following two multiplications in A'' can be introduced, which are referred to as the *first* and *second Arens multiplications* [2, 3]. They are accomplished in three steps: for $a, b \in A$, $f \in A'$ and $F, G \in A''$, define $a \cdot f, f \cdot F \colon A \mapsto \mathbb{R}$ and $F \cdot G \colon A' \mapsto \mathbb{R}$ (fa, Ff and FG for the second multiplication) by

$$(a \cdot f)(b) = f(ba),$$

$$(f \cdot F)(a) = F(a \cdot f),$$

$$(F \cdot G)(f) = G(f \cdot F)$$

and

$$(fa)(b) = f(ab),$$

$$(Ff)(a) = F(fa),$$

$$(FG)(f) = F(Gf).$$

We shall concentrate on the first Arens multiplication; similar results hold for the second.

For the elementary theory of ℓ-spaces and terminology not explained here we refer to [1, 10, 12].

2. The order continuous bidual of r-algebras. In this section we consider the order continuous bidual of the class of Archimedean r-algebras and prove that the order continuous bidual of an r-algebra A is again an r-algebra with respect to the Arens multiplication. We first recall some relevant notions. The *canonical mapping* $a \mapsto \widehat{a}$ of a vector lattice A into its order bidual A'' is defined by $\widehat{a}(f) = f(a)$ for all $f \in A'$. For each $a \in A$, $\widehat{a}$ defines an order continuous algebraic lattice homomorphism on A' and the canonical image $\widehat{A}$ of A is a subalgebra of $(A')'_c$. Moreover the band $S_{\widehat{A}} = \{F \in (A')'_c \colon |F| \leq \widehat{x}$ for some $x \in A^+\}$ generated by $\widehat{A}$ is order dense in $(A')'_c$; that is, for each $F \in (A')'_c$, there exists an upwards directed net $\{G_\lambda \colon \lambda \in \Lambda\}$ in $S_{\widehat{A}}$ such that $0 < G_\lambda \uparrow F$.

Lemma 2.1. *Let A be an r-algebra and $0 \leq G, H \in (A')'_c$. If $G \wedge H = 0$ and $G, H \leq \widehat{x}$ for some $x \in A^+$, then $G \cdot H \wedge H \cdot G = 0$.*

Proof. Let $0 \leq f \in A'$ and $x \in A^+$. Then define the positive linear functional fx in A' by $(fx)(y) = f(xy)$ for all $y \in A$. It follows that $f \cdot \widehat{x} = fx$ since

$$(f \cdot \widehat{x})(y) = \widehat{x}(y \cdot f) = (y \cdot f)(x) = f(xy) = (fx)(y)$$

for all $y \in A$. Hence $0 \leq x \cdot f + fx \in A'$, and so, by Corollary 1.2 of [4], there exist $g, h \in A'$ with $g \wedge h = 0$, and $G(g) = 0 = H(h)$ such that $x \cdot f + fx = g + h$. Hence

$$\inf\{g(y) + h(z) \colon x = y + z,\ y, z \in A^+\} = (g \wedge h)(x) = 0,$$

which implies that, for $\epsilon > 0$, there exist $y, z \in A^+$ such that $x = y + z$ and $g(y) < \epsilon$ and $h(z) < \epsilon$.

We now define the linear functionals G_1 and H_1 on A' by

$$G_1 = G \wedge (\widehat{y - y \wedge z}) \quad \text{and} \quad H_1 = H \wedge (\widehat{z - y \wedge z}).$$

Clearly, $0 \leq G_1,\ H_1 \in (A')'_c$ and the following inequalities hold:

$$0 \leq H - H_1 = (H - (\widehat{z - y \wedge z}))^+ \leq (\widehat{x} - (\widehat{z - y \wedge z}))^+$$

$$= (\widehat{y + z - (z - y \wedge z)})^+ = (\widehat{y + y \wedge z})^+ \leq 2\widehat{y}, \tag{1}$$

and similarly

$$0 \leq G - G_1 \leq 2\widehat{z}. \tag{2}$$

Since $(y - y \wedge z) \wedge (z - y \wedge z) = (y \wedge z) - (y \wedge z) = 0$ in A and A is an r-algebra,

$$(y - y \wedge z)(z - y \wedge z) \wedge (z - y \wedge z)(y - y \wedge z) = 0,$$

and so $0 \leq G_1 \cdot H_1 \wedge H_1 \cdot G_1 \leq (\widehat{y - y \wedge z}) \cdot (\widehat{z - y \wedge z}) \wedge (\widehat{z - y \wedge z}) \cdot (\widehat{y - y \wedge z}) = 0$; i.e.,

$$G_1 \cdot H_1 \wedge H_1 \cdot G_1 = 0. \tag{3}$$

We next consider the elements $0 \leq G \cdot (H - H_1)$, $(G - G_1) \cdot H_1$, $H \cdot (G - G_1)$ and $(H - H_1) \cdot G_1$ of $(A')'_c$. Then, by (1),

$$(G\cdot(H-H_1))(f)\leq(\widehat{x}\cdot(H-H_1))(f)=(H-H_1)(f\cdot\widehat{x})=$$
$$=(H-H_1)(fx)\leq(H-H_1)(fx+x\cdot f)=$$
$$=(H-H_1)(g+h)=(H-H_1)(g)+(H-H_1)(h)\leq$$
$$\leq(H-H_1)(g)+H(h)\leq 2\widehat{y}(g)+0=2g(y) \tag{4}$$

and, by (2),

$$((G-G_1)\cdot H_1)(f)\leq((G-G_1)\cdot\widehat{x})(f)=\widehat{x}(f\cdot(G-G_1))=$$
$$=(f\cdot(G-G_1))(x)=(G-G_1)(x\cdot f)\leq$$
$$\leq(G-G_1)(x\cdot f+fx)=(G-G_1)(g+h)=$$
$$=(G-G_1)(g)+(G-G_1)(h)\leq$$
$$\leq G(g)+(G-G_1)(h)\leq 2h(z). \tag{5}$$

It follows by symmetry that

$$(H\cdot(G-G_1))(f)\leq 2h(z)\quad\text{and}\quad((H-H_1)\cdot G_1)(f)\leq 2g(y). \tag{6}$$

Using the fact that $(a+b)\wedge c\leq a\wedge c+b\wedge c\leq a+b\wedge c$ in ℓ-spaces and (3), we find

$$G\cdot H\wedge H\cdot G=[\big(G\cdot(H-H_1)+(G-G_1)\cdot H_1+G_1\cdot H_1\big)]\wedge$$
$$\wedge[H\cdot(G-G_1)+(H-H_1)\cdot G_1+H_1\cdot G_1]\leq$$
$$\leq G\cdot(H-H_1)+(G-G_1)\cdot H_1+$$
$$+G_1\cdot H_1\wedge\big[H\cdot(G-G_1)+(H-H_1)\cdot G_1+H_1\cdot G_1\big]\leq$$
$$\leq G\cdot(H-H_1)+(G-G_1)\cdot H_1+$$
$$+G_1\cdot H_1\wedge(H\cdot(G-G_1)+(H-H_1)\cdot G_1)+$$
$$+G_1\cdot H_1\wedge H_1\cdot G_1\leq$$
$$\leq G\cdot(H-H_1)+(G-G_1)\cdot H_1+$$
$$+H\cdot(G-G_1)+(H-H_1)\cdot G_1.$$

Hence, by (4), (5) and (6),

$$0\leq(G\cdot H\wedge H\cdot G)(f)\leq((G\cdot(H-H_1))(f)+((G-G_1)\cdot H_1))(f)+$$
$$+(H\cdot(G-G_1))(f)+((H-H_1)\cdot G_1))(f)\leq$$

$$\leq 2g(y) + 2h(z) + 2h(z) + 2g(y) \leq 8\epsilon.$$

Since this holds for an arbitrary $\epsilon > 0$, we have $(G \cdot H \wedge H \cdot G)(f) = 0$ for all $0 \leq f \in A'$. It now follows that for all $f \in A'$

$$(G \cdot H \wedge H \cdot G)(f) = (G \cdot H \wedge H \cdot G)(f^+) - (G \cdot H \wedge H \cdot G)(f^-) = 0,$$

and so $G \cdot H \wedge H \cdot G = 0$.

We now in a position to express the main result of this work.

Theorem 2.1. *The order continuous bidual of an r-algebra is a Dedekind complete* (*and hence Archimedean*) *r-algebra.*

Proof. Let A be an r-algebra. We only need to prove $G \cdot H \wedge H \cdot G = 0$ whenever $0 \leq G, H \in (A')'_c$ with $G \wedge H = 0$. To this end, consider the band $S_{\widehat{A}}$ generated by the canonical image $\widehat{A}$ of A in $(A')'_c$. Since $S_{\widehat{A}}$ is order dense in $(A')'_c$, there exist $G_\alpha, H_\beta \in S_{\widehat{A}}$ such that $0 \leq G_\alpha \uparrow G$ and $0 \leq H_\beta \uparrow H$ with $0 \leq G_\alpha \leq \widehat{x}_\alpha$ and $0 \leq H_\beta \leq \widehat{y}_\beta$ for some $x_\alpha, y_\beta \in A^+$. It follows from $G \wedge H = 0$ that $G_\alpha \wedge H_\beta = 0$ for all α, β. Furthermore, $0 \leq G_\alpha, H_\beta \leq \widehat{x_\alpha + y_\beta}$. Hence, by Lemma 2.1, we see that

$$G_\alpha \cdot H_\beta \wedge H_\beta \cdot G_\alpha = 0 \tag{7}$$

for all α and β. Now let $0 \leq f \in A'$. It follows from $0 \leq G_\alpha \uparrow G$ that $0 \leq \leq G_\alpha(x \cdot f) \uparrow G(x \cdot f)$; i.e., $0 \leq (f \cdot G_\alpha)(x) \uparrow (f \cdot G)(x)$ for all $x \in A^+$. This shows that $0 \leq f \cdot G_\alpha \uparrow f \cdot G$. Hence, by the order continuity of H_β for each β, $0 \leq H_\beta(f \cdot G_\alpha) \uparrow H_\beta(f \cdot G)$; i.e., $0 \leq (G_\alpha \cdot H_\beta)(f) \uparrow (G \cdot H_\beta)(f)$, which implies that, for each $_\beta$,

$$0 \leq G_\alpha \cdot H_\beta \uparrow G \cdot H_\beta. \tag{8}$$

Similarly, since $0 \leq H_\beta \uparrow H$, $0 \leq H_\beta(f \cdot G) \uparrow H(f \cdot G)$; i.e., $0 \leq \leq (G \cdot H_\beta)(f) \uparrow (G \cdot H)(f)$ for all $0 \leq f \in A'$, and so

$$0 \leq G \cdot H_\beta \uparrow G \cdot H. \tag{9}$$

In the same way, using the order continuity of G_α for each α, we obtain

$$0 \leq H_\beta \cdot G_\alpha \uparrow H \cdot G_\alpha, \tag{10}$$

leading to

$$0 \leq H \cdot G_\alpha \uparrow H \cdot G. \tag{11}$$

It follows from (8) and (10) that $0 \leq G_\alpha \cdot H_\beta \wedge H_\beta \cdot G_\alpha \uparrow G \cdot H_\beta \wedge H \cdot G_\alpha$, and so, by (7), $G \cdot H_\beta \wedge H \cdot G_\alpha = 0$ for all α and β. On the other hand, $0 \leq G \cdot H_\beta \wedge H \cdot G_\alpha \uparrow G \cdot H \wedge H \cdot G$ by (9) and (11), and so $G \cdot H \wedge H \cdot G = 0$, as required.

As remarked earlier, the order bidual A'' of an almost f-algebra (respectively f-algebra) A is an almost f-algebra (respectively f-algebra) which may not be true for the order biduals of either d-algebras [4] or r-algebras. However we have the following consequence.

***Corollary* 2.1.** *The order bidual of a commutative r-algebra is a Dedekind complete r-algebra.*

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