1. Introduction. In classical homological algebra, given right $R$-modules $M, N$ and a left $R$-module $L$, the cohomology group $\text{Ext}^n(M,N)$ is obtained by using a right injective resolution of $N$ or a left projective resolution of $M$, and the homology group $\text{Tor}^n(M,L)$ is obtained by using a left projective (flat) resolution of $M$ or $L$. In relative homological algebra [5], if $G$ is a preenveloping class of right $R$-modules, then we can get the relative cohomology group $\overline{\text{Ext}}^n_G(M,N)$ computed by the right $G$-resolution of $N$. Similarly, if $F$ is a precovering class of right $R$-modules, then we can get the relative cohomology group $\mathcal{F}\text{Ext}^n(M,N)$ and the relative homology group $\mathcal{F}\text{Tor}^n(M,L)$ computed by the left $F$-resolution of $M$.

The main goal of the present paper is to extend some important properties of classical (co)homology groups to relative (co)homological groups. We introduce the concepts of an $F$-extension and a $G$-coextension of modules, where $F$ and $G$ denote two classes of right $R$-modules. It is proven that the set of all equivalence classes of $F$-extensions (resp. $G$-coextensions) of $A$ by $C$, denoted by $\mathcal{F}\text{E}(C,A)$ (resp. $\mathcal{E}\text{G}(C,A)$), is an Abelian group. Moreover, we prove that $\text{Ext}^1_{\mathcal{F}}(C,A) \cong \mathcal{E}\text{G}(C,A)$ if $G$ is a monic preenveloping class and $\mathcal{F}\text{Ext}^1(C,A) \cong \mathcal{F}\text{E}(C,A)$ if $F$ is an epic precovering class. As applications, we obtain several properties of relative (co)homology groups. For example, if $F$ is an epic precovering class of right $R$-modules, then we prove that: (1) there is a monomorphism $\mathcal{F}\text{Ext}^1(C,A) \rightarrow \text{Ext}^1(C,A)$ for all right $R$-modules $A$ and $C$; (2) there is an epimorphism $\text{Tor}_1(A,B) \rightarrow \mathcal{F}\text{Tor}_1(A,B)$ for any right $R$-module $A$ and any left $R$-module $B$. In addition, we give a relative version of Wakamatsu’s lemmas.

We next recall some notions and facts needed in the sequel.

Following [3], we say that a right $R$-module homomorphism $\phi : M \rightarrow G$ is a $G$-preenvelope of $M$ if $G \in G$ and the Abelian group homomorphism $\phi^* : \text{Hom}(G,G') \rightarrow \text{Hom}(M,G')$ is surjective for each $G' \in G$. A $G$-preenvelope $\phi : M \rightarrow G$ is said to be a $G$-envelope of $M$ if every endomorphism $g : G \rightarrow G$ such that $g\phi = \phi$ is an isomorphism. Dually we have the definitions of an $F$-precover and an $F$-cover. $G$-envelopes ($F$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.
We say that $G$ is a (resp. monic) preenveloping class of right $R$-modules [8] if every right $R$-module has a (resp. monic) $G$-preenvelope. Dually, $F$ is called a (resp. epic) precovering class of right $R$-modules if every right $R$-module has an (resp. epic) $F$-precover.

Let $G$ be a preenveloping class. Then any right $R$-module $N$ has a right $G$-resolution, i.e., there is a cocomplex $0 \to N \to G^0 \to G^1 \to \ldots$ with each $G^i \in G$ such that $\ldots \to \Hom(G^1, G) \to \Hom(G^0, G) \to \Hom(N, G) \to 0$ is exact for any $G \in G$. Let $G^\cdot$ be the deleted cocomplex corresponding to a right $G$-resolution of $N$, which is unique up to homotopy, then for a right $R$-module $M$, we obtain the $n$th cohomology group of the cocomplex $\Hom(M, G^\cdot)$, denoted by $\Ext^n_G(M, N)$ (see [5], 8.2). Particularly, if $G$ is the class of injective right $R$-modules, then $\Ext^n_G(M, N)$ is just the classical cohomology group $\Ext^n(M, N)$.

Dually, let $F$ be a precovering class, then any right $R$-module $M$ has a left $F$-resolution, i.e., there is a complex $\ldots \to F_1 \to F_0 \to M \to 0$ with each $F_i \in F$ such that $\ldots \to \Hom(F_{i+1}, F_i) \to \Hom(F_i, F_0) \to \Hom(F_i, M) \to 0$ is exact for any $F \in F$. Let $F^\cdot$ be the deleted complex corresponding to a left $F$-resolution of $M$, which is unique up to homotopy. Then for a right $R$-module $N$, we obtain the $n$th homology group of the cocomplex $\Hom(F^\cdot, N)$, denoted by $\Tor^n(F, N)$ (see [5], 8.2). In addition, for a left $R$-module $L$, we get the $n$th homology group of the complex $F_\otimes L$, denoted by $\Tor_n(F, L)$. Particularly, if $F$ is the class of projective right $R$-modules, then $\Tor^n(F, N)$ is just the classical homology group $\Tor^n(M, N)$ and $\Tor_n(F, L)$ is just the classical homology group $\Tor_n(M, L)$.

Throughout this paper, $R$ is an associative ring with identity and all modules are unitary. All classes of modules are closed under isomorphisms and direct summands. $F R$ (resp. $M_R$) denotes a left (resp. right) $R$-module. The character module $\Hom_R(M, \mathbb{Q}/\mathbb{Z})$ of $M$ is denoted by $M^\ast$. The reader is referred to [5, 6, 8, 10, 12, 14] for unexplained concepts and notations.

2. Relative homology groups and relative extensions of modules. Let $A$ and $C$ be two right $R$-modules. Then an exact sequence $0 \to A \to B \to C \to 0$ is called an extension of $A$ by $C$ [10]. We first introduce the concepts of relative (co)extensions as follows.

**Definition 2.1.** Given a class $F$ of right $R$-modules, an exact sequence $0 \to A \to B \to C \to 0$ of right $R$-modules is said to be an $F$-extension of $A$ by $C$ if

$$0 \to \Hom(F, A) \to \Hom(F, B) \to \Hom(F, C) \to 0$$

is exact for any $F \in F$.

Dually, given a class $G$ of right $R$-modules, an exact sequence $0 \to A \to B \to C \to 0$ of right $R$-modules is called a $G$-coextension of $A$ by $C$ if $0 \to \Hom(C, G) \to \Hom(B, G) \to \Hom(A, G) \to 0$ is exact for any $G \in G$.

**Remark 2.1.** (1) Let $F$ (resp. $G$) be the class of projective (resp. injective) right $R$-modules, then an $F$-extension (resp. a $G$-coextension) of $A$ by $C$ is just the usual extension of $A$ by $C$.

(2) Let $F$ (resp. $G$) be the class of pure-projective (resp. pure-injective) right $R$-modules, then an $F$-extension (resp. a $G$-coextension) of $A$ by $C$ is just a pure exact sequence $0 \to A \to B \to C \to 0$.

(3) Let $G$ be the class of cotorsion right $R$-modules (A right $R$-module $M$ is called cotorsion [4] if $\Ext^n(F, M) = 0$ for every flat right $R$-module $F$), then a $G$-coextension of $A$ by $C$ is just an exact sequence $0 \to A \to B \to C \to 0$ with $A \to B$ a strongly pure monomorphism in the sense of [9].

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Two $\mathcal{F}$-extensions ($\mathcal{G}$-coextensions) $\Delta$ and $\Delta'$ of $A$ by $C$ are called equivalent if there is $\sigma : B \to B'$ such that the following diagram is commutative:

$$
\begin{array}{c}
\Delta : 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
\sigma & & \downarrow & & \downarrow & & \downarrow & & \\
\Delta' : 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0.
\end{array}
$$

By the Five lemma, the middle homomorphism $\sigma$ is an isomorphism. So the equivalence of $\mathcal{F}$-extensions ($\mathcal{G}$-coextensions) is a reflexive, symmetric and transitive relation. We write $\mathcal{F}E(C, A)$ (resp. $E_{\mathcal{G}}(C, A)$) to be the set of all equivalence classes of $\mathcal{F}$-extensions (resp. $\mathcal{G}$-coextensions) of $A$ by $C$.

If $\mathcal{F}$ (resp. $\mathcal{G}$) is the class of projective (resp. injective) right $R$-modules, it is well known that $\mathcal{F}E(C, A)$ (resp. $E_{\mathcal{G}}(C, A)$) is an Abelian group using the so called Baer sum (see [10]). We can extend this result to a more general setting as follows.

**Theorem 2.1.** The following are true for right $R$-modules $A$ and $C$:

1. $E_{\mathcal{G}}(C, A)$ is an Abelian group for any class $\mathcal{G}$ of right $R$-modules.
2. $\mathcal{F}E(C, A)$ is an Abelian group for any class $\mathcal{F}$ of right $R$-modules.

**Proof.** (1) Let $\Delta_1 : 0 \to A \xrightarrow{i_1} B_1 \xrightarrow{\pi_1} C \to 0$ and $\Delta_2 : 0 \to A \xrightarrow{i_2} B_2 \xrightarrow{\pi_2} C \to 0$ be two $\mathcal{G}$-coextensions of $A$ by $C$. Then we get the following pushout diagram:

$$
\begin{array}{c}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & A & \longrightarrow & B_1 & \longrightarrow & C & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & B_2 & \longrightarrow & H_{12} & \longrightarrow & C & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C & = & C \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & = & 0
\end{array}
$$

where $H_{12} = (B_1 \oplus B_2)/W, W = \{(i_1(a), -i_2(a)) : a \in A\}, f(b_1) = (b_1, 0)$ for $b_1 \in B_1$, $g(b_2) = (0, b_2)$ for $b_2 \in B_2$. Let $Q = \{(x, y) : \pi_1(x) = \pi_2(y), x \in B_1, y \in B_2\} \subseteq B_1 \oplus B_2$. Then $W \subseteq Q$. Put $Y_{12} = Q/W \subseteq H_{12}$. Then we get the sequence

$$
\Psi_{12} : 0 \to A \xrightarrow{\lambda_{12}} Y_{12} \xrightarrow{\tau_{12}} C \to 0,
$$

where $\lambda_{12}(a) = (i_1(a), 0)$ for $a \in A$ and $\tau_{12}(x, y) = \pi_1(x) = \pi_2(y)$ for $(x, y) \in Q$.

We first claim that $\Psi_{12}$ is exact. In fact, it is clear that $\lambda_{12}$ is monic, $\tau_{12}$ is epic and $\tau_{12}\lambda_{12} = 0$. If $\tau_{12}(x, y) = 0$, then $x = i_1(a_1)$ and $y = i_2(a_2)$ for some $a_1, a_2 \in A$. Thus $(x, y) = (i_1(a_1), i_2(a_2)) = (i_1(a_1 + a_2), 0) = \lambda_{12}(a_1 + a_2)$. So $\Psi_{12}$ is exact.
We now prove that $\Psi_{12}$ is a $G$-coextension of $A$ by $C$. In fact, let $G \in G$ and $\alpha \in \text{Hom}(A,G)$, then there exist $\beta_1 \in \text{Hom}(B_1,G)$ and $\beta_2 \in \text{Hom}(B_2,G)$ such that $\alpha = \beta_1 i_1$ and $\alpha = \beta_2 i_2$ by hypothesis. Thus by the property of a pushout, there exists $\xi \in \text{Hom}(H_{12}, G)$ such that the following diagram is commutative:

\[
\begin{array}{c}
A \xrightarrow{i_1} B_1 \\
\downarrow i_2 \quad \downarrow f \\
B_2 \xrightarrow{g} H_{12} \\
\downarrow \beta_1 \quad \downarrow \beta_2 \\
G
\end{array}
\]

Write $\epsilon : Y_{12} \rightarrow H_{12}$ to be the inclusion. Then

$$\alpha = \beta_1 i_1 = \xi f i_1 = (\xi \epsilon) \lambda_{12}.$$ 

So $\text{Hom}(Y_{12}, G) \rightarrow \text{Hom}(A, G)$ is epic, i.e., $\Psi_{12}$ is a $G$-coextension of $A$ by $C$.


Let $\Delta_3 : 0 \rightarrow A \xrightarrow{i_3} B_3 \xrightarrow{\tau_3} C \rightarrow 0$ be a $G$-coextension of $A$ by $C$. We next prove that $[\Delta_1] + [\Delta_2] + [\Delta_3] = [\Delta_1] + [\Delta_2] + [\Delta_3]$.

Let $([\Delta_1] + [\Delta_2]) + [\Delta_3] = [\Xi]$, where $\Xi : 0 \rightarrow A \xrightarrow{\omega} U/V \xrightarrow{\nu} C \rightarrow 0$ is a $G$-coextension of $A$ by $C$ with $U = \{(x, y, z) : \pi_2((x, y)) = \pi_3(z), (x, y) \in Y_{12}, z \in B_3\} \subseteq Y_{12} \oplus B_3$, $V = \{((\lambda_1(a), -i_3(a)) : a \in A\}$. Let $[\Delta_2] + [\Delta_3] = [\nu_{23}]$ and $[\Delta_1] + ([\Delta_2] + [\Delta_3]) = [\Lambda]$, where $\nu_{23} : 0 \rightarrow A \xrightarrow{\nu_{23}} Y_{23} \xrightarrow{\tau_3} C \rightarrow 0$ is a $G$-coextension of $A$ by $C$ and $\Lambda : 0 \rightarrow A \xrightarrow{\nu} M/N \xrightarrow{\nu_{23}} C \rightarrow 0$ is a $G$-coextension of $A$ by $C$ with $M = \{(x, y, z) : \pi_3((x, y)) = \tau_23((y, z)), x \in B_1, (y, z) \in Y_{23}\} \subseteq B_1 \oplus Y_{23}$, $N = \{(i_1(a), -\lambda_{23}(a)) : a \in A\}$.

Define $\sigma : U/V \rightarrow M/N$ by $\sigma((x, y, z)) = (x, (y, z))$ for $(x, y) \in Y_{12}, z \in B_3$. We claim that $\sigma$ is well defined. In fact, if $(x, y, z) = 0$, then $(x, y, z) = (\lambda_1(a), -i_3(a))$ for some $a \in A$. So $(x, y) = (i_1(a), 0), z = -i_3(a)$. Thus $x = i_1(a) + b, y = -i_2(b)$ for some $b \in B$. Hence $x, y \in i_1(a) + b, y = -i_2(b)$. So $(x, (y, z)) = (i_1(a) + b, (-i_2(b), -\lambda_{23}(a)) = (i_1(a) + b, (-i_2(a) + b, 0) = (i_1(a) + b, -\lambda_{23}(a) + b)) \in N$. Moreover, it is easy to verify that the following diagram is commutative:

$$\Xi : 0 \rightarrow A \xrightarrow{\omega} U/V \xrightarrow{\nu} C \rightarrow 0$$

$$\Lambda : 0 \rightarrow A \xrightarrow{\nu} M/N \xrightarrow{\nu_{23}} C \rightarrow 0.$$

So $([\Delta_1] + [\Delta_2]) + [\Delta_3] = [\Delta_1] + ([\Delta_2] + [\Delta_3])$.

On the other hand, the split exact sequence $\Xi : 0 \rightarrow A \xrightarrow{\delta} A \oplus C \xrightarrow{\psi_1} C \rightarrow 0$ is clearly a $G$-coextension of $A$ by $C$. We claim that $[\Delta_1] + [\Xi] = [\Delta_1]$. In fact, let $[\Delta_1] + [\Xi] = [\psi_1]$, where $\psi_1 : 0 \rightarrow A \xrightarrow{\lambda_1} Q_1/W_1 \xrightarrow{\tau_3} C \rightarrow 0$ is a $G$-coextension of $A$ by $C$ with $W_1 = \{(i_1(a), -(a, 0)) : \}$.
a \in A}, Q_1 = \{(x, (a, \pi_1(x))): x \in B_1, a \in A\}. Define \(\sigma_1: Q_1/W_1 \rightarrow B_1\) by \(\sigma_1((x, (a, \pi_1(x)))) = x + i_1(a)\). It is easy to verify that \(\sigma_1\) is well defined, \(\sigma_1\lambda_1 = i_1\) and \(\pi_1\sigma_1 = \tau_1\). Thus \([\tilde{U}]\) is the zero element in \(E_G(C, A)\).

Finally consider the exact sequence \(\Delta'_1: 0 \rightarrow A \xrightarrow{\psi} B_1 \xrightarrow{\pi} C \rightarrow 0\), which is obviously a \(G\)-coextension of \(A\) by \(C\). We claim that \([\Delta_1] + [\Delta'_1] = [\tilde{U}]\). In fact, let \([\Delta_1] + [\Delta'_1] = [\psi']\), where \(\psi': 0 \rightarrow A \xrightarrow{\psi'} Q'/W' \xrightarrow{\pi'} C \rightarrow 0\) is a \(G\)-coextension of \(A\) by \(C\) with \(W' = \{(i_1(a), i_1(a)): a \in A\}, Q' = \{(x, y): \pi_1(x) = \pi_1(y), x, y \in B_1\} = \{(y + i_1(a), y): a \in A, y \in B_1\}\). Define \(\sigma': Q'/W' \rightarrow A \oplus C\) by \(\sigma'(y + i_1(a), y) = (a, \pi_1(y))\). It is easy to verify that \(\sigma'\) is well defined, \(\sigma'\lambda' = i\) and \(\kappa\sigma' = \tau'\). So \(\Delta'_1\) is the negative element of \(\Delta_1\) in \(E_G(C, A)\).

It follows that \(E_G(C, A)\) is an Abelian group.

(2) can be proved dually.

Theorem 2.1 is proved.

It is well known that, in standard homological algebra, the cohomological group \(\text{Ext}^1(C, A)\) is isomorphic to the group of all equivalence classes of extensions of \(A\) by \(C\). This result can be generalized as follows.

**Theorem 2.2.** The following are true:

(1) If \(G\) is a monic preenveloping class of right \(R\)-modules, then there is an Abelian group isomorphism \(\text{Ext}^3_G(C, A) \cong E_G(C, A)\) for all right \(R\)-modules \(A\) and \(C\).

(2) If \(F\) is an epic precovering class of right \(R\)-modules, then there is an Abelian group isomorphism \(\text{Ext}^3_F(C, A) \cong E_F(C, A)\) for all right \(R\)-modules \(A\) and \(C\).

**Proof.** (1) Let \(0 \rightarrow A \xrightarrow{d^0} G^0 \xrightarrow{d^1} G^1 \xrightarrow{d^2} G_2 \rightarrow \ldots\) be a right \(G\)-resolution of \(A\). Then we get the cocomplex

\[0 \rightarrow \text{Hom}(C, G^0) \xrightarrow{d^1} \text{Hom}(C, G^1) \xrightarrow{d^2} \text{Hom}(C, G^2) \rightarrow \ldots.\]

So \(\text{Ext}^3_G(C, A) = \ker(d^2)/\text{im}(d^1)\).

Let \(\Gamma: 0 \rightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \rightarrow 0\) be a \(G\)-coextension of \(A\) by \(C\), then there exist \(\varepsilon_0: B \rightarrow G^0\) and \(\varepsilon_1: C \rightarrow G^1\) such that the following diagram with exact rows is commutative:

\[
\begin{array}{cccccc}
0 & \rightarrow & A & \xrightarrow{i} & B & \xrightarrow{\pi} & C & \rightarrow & 0 \\
\varepsilon_0 & & \downarrow & & \varepsilon_1 & & \downarrow \\
0 & \rightarrow & A & \xrightarrow{d^0} & G^0 & \xrightarrow{d^1} & G^1 & \rightarrow & G^2.
\end{array}
\]

Note that \(d^2\varepsilon_1\pi = d^2d^1\varepsilon_0 = 0\). Thus \(d^2\varepsilon_1 = 0\), and so \(\varepsilon_1 \in \ker(d^2)\).

Define \(\Theta: E_G(C, A) \rightarrow \text{Ext}^3_G(C, A)\) by \(\Theta([\Gamma]) = [\pi_1]\). We claim that \(\Theta\) is well defined. In fact, if there exist \(\varepsilon'_0: B \rightarrow G^0\) and \(\varepsilon'_1: C \rightarrow G^1\) such that the above diagram also commutes, then \((\varepsilon'_0 - \varepsilon_0)i = 0\), so there exists \(\chi: C \rightarrow G^0\) such that \(\varepsilon'_0 - \varepsilon_0 = \chi\pi\). Thus \((\varepsilon'_1 - \varepsilon_1)\pi = d^1(\varepsilon'_0 - \varepsilon_0) = d^1\chi\pi\). So \(\varepsilon'_1 - \varepsilon_1 = d^1\chi \in \text{im}(d^1)\). Hence \([\pi_1] = [\varepsilon_1]\).

We now prove that \(\Theta\) is a group homomorphism.

Let \(\Upsilon: 0 \rightarrow A \xrightarrow{d} H \xrightarrow{d'} C \rightarrow 0\) be a \(G\)-coextension of \(A\) by \(C\). Then there is the following commutative diagram with exact rows:
Let $[\Gamma] + [\Upsilon] = [\Psi]$, where $\Psi : 0 \to A \xrightarrow{\lambda} Q/W \xrightarrow{\tau} C \to 0$ is a $G$-coextension of $A$ by $C$ with $Q = \{(x, y) : \pi(x) = \rho(y), \ x \in B, \ y \in H\}$ and $W = \{(i(a), -\iota(a)) : a \in A\}$ by Theorem 2.1.

Define $\eta : Q/W \to G^0$ by $\eta((x, y)) = \varepsilon_0(x) + \gamma_0(y)$ for $(x, y) \in Q$. Then $\eta$ is well defined and $\eta \lambda(a) = \eta(i(a), 0)) = \varepsilon_0(i(a)) = d^0(a), (\varepsilon_1 + \gamma_1)(\pi(x)) = d^1\varepsilon_0(x) + d^1\gamma_0(y) = d^1\eta((x, y))$. So we have the following commutative diagram with exact rows:

$$
\begin{array}{c}
0 & \longrightarrow & A & \overset{\lambda}{\longrightarrow} & Q/W & \overset{\tau}{\longrightarrow} & C & \longrightarrow & 0 \\
| & & | & & | & & | & & |
0 & \longrightarrow & A & \overset{d^0}{\longrightarrow} & G^0 & \overset{d^1}{\longrightarrow} & G^1 & \longrightarrow & G^2.
\end{array}
$$

Thus $\Theta([\Gamma] + [\Upsilon]) = \Theta([\Gamma]) + \Theta([\Upsilon])$. We next prove that $\Theta$ is a group isomorphism.

Write $\mu : \text{im}(d^1) \to G^1$ to be the inclusion. Then there exists $\nu : G^0 \to \text{im}(d^1)$ such that $\mu \nu = d^1$.

Let $\beta \in \ker(d^2)$. Then $d^2\beta = 0$. So $\text{im}(\beta) \subseteq \ker(d^2) = \text{im}(d^1)$. Thus there exists $\tilde{\beta} : C \to \text{im}(d^1)$ such that $\beta = \mu \tilde{\beta}$. We obtain the following pullback diagram:

$$
\begin{array}{c}
0 & \longrightarrow & A & \overset{\nu}{\longrightarrow} & \text{im}(d^1) & \longrightarrow & 0 \\
| & & | & & | & & |
0 & \longrightarrow & A & \overset{d^0}{\longrightarrow} & G^0 & \overset{\nu}{\longrightarrow} & \text{im}(d^1) & \longrightarrow & 0.
\end{array}
$$

For any $M \in G$ and any homomorphism $h : A \to M$, there is $j : G^0 \to M$ such that $h = jd^0$. So $(jw)f = jd^0 = h$. Thus the sequence $\text{Hom}(D, M) \to \text{Hom}(A, M) \to 0$ is exact. Hence the exact sequence $\Delta : 0 \to A \xrightarrow{\lambda} D \xrightarrow{\rho} C \to 0$ is a $G$-coextension of $A$ by $C$. Since $\beta = \mu \tilde{\beta}$, we have $\Theta([\Delta]) = \beta$. So $\Theta$ is an epimorphism.

On the other hand, let $\Theta([\Gamma]) = \varepsilon_1 = 0$, then $\varepsilon_1 = d^1\kappa$ for some $\kappa \in \text{Hom}(C, G^0)$. Since $d^2\varepsilon_1 = d^2d^1\kappa = 0$, $\text{im}(\varepsilon_1) \subseteq \ker(d^2) = \text{im}(d^1)$. Thus there exists $\hat{\varepsilon}_1 : C \to \text{im}(d^1)$ such that $\varepsilon_1 = \mu \hat{\varepsilon}_1$. So $\mu \hat{\varepsilon}_1 = d^1\kappa = \varepsilon_1 = \mu \varepsilon_1$. Thus $\mu \kappa = \varepsilon_1$.

Consider the following diagram with exact rows:

$$
\begin{array}{c}
0 & \longrightarrow & A & \overset{i}{\longrightarrow} & B & \overset{\pi}{\longrightarrow} & C & \longrightarrow & 0 \\
| & & | & & \varepsilon_0 & & | & & |
0 & \longrightarrow & A & \overset{d^0}{\longrightarrow} & G^0 & \overset{\nu}{\longrightarrow} & \text{im}(d^1) & \longrightarrow & 0.
\end{array}
$$

Then there exists $\delta : B \to A$ such that $\delta i = 1$ by [6, p. 44] (Lemma 8.4). Therefore $\Gamma$ is a split exact sequence, and so $[\Gamma] = 0$. Hence $\Theta$ is a monomorphism.

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(2) can be proved dually.

Theorem 2.2 is proved.

As an immediate consequence of Theorem 2.2, we have the following corollary.

**Corollary 2.1.** The following are true:

1. If $\mathcal{G}$ is a monic preenveloping class of right $R$-modules, then there is a monomorphism $\text{Ext}^1_{\mathcal{G}}(C, A) \to \text{Ext}^1(C, A)$ for all right $R$-modules $A$ and $C$.
2. If $\mathcal{F}$ is an epic precovering class of right $R$-modules, then there is a monomorphism $\varphi\text{Ext}^1(C, A) \to \text{Ext}^1(C, A)$ for all right $R$-modules $A$ and $C$.

Obviously, a preenveloping class $\mathcal{G}$ of right $R$-modules is monic if and only if $\mathcal{G}$ contains all injective right $R$-modules and a precovering class $\mathcal{F}$ of right $R$-modules is epic if and only if $\mathcal{F}$ contains all projective right $R$-modules. Furthermore, we have the following result.

**Corollary 2.2.** The following are true:

1. Let $\mathcal{G}$ be a monic preenveloping class of right $R$-modules, then $\text{Ext}^1_{\mathcal{G}}(C, A) \cong \text{Ext}^1(C, A)$ for all right $R$-modules $A$ and $C$ and only if $\mathcal{G}$ is the class of injective right $R$-modules.
2. Let $\mathcal{F}$ be an epic precovering class of right $R$-modules, then $\varphi\text{Ext}^1(C, A) \cong \text{Ext}^1(C, A)$ for all right $R$-modules $A$ and $C$ if and only if $\mathcal{F}$ is the class of projective right $R$-modules.

**Proof.** (1) $\Rightarrow$. For any $M \in \mathcal{G}$, there is an exact sequence $0 \to M \to E \to C \to 0$ with $E$ injective. By Theorem 2.2(1), the exact sequence is a $\mathcal{G}$-coextension of $M$ by $C$, and so is split. Thus $M$ is injective.

$\Leftarrow$ is trivial.

(2) can be proved dually.

Corollary 2.2 is proved.

Now we characterize when $\text{Ext}^n_{\mathcal{G}}(-, -)$ and $\varphi\text{Ext}^n(-, -) (n = 1, 2)$ vanish.

**Proposition 2.1.** The following are true:

1. Let $\mathcal{G}$ be a monic preenveloping class of right $R$-modules, then any $\mathcal{G}$-coextension of $A$ by $C$ $0 \to A \to B \to C \to 0$ is split if and only if $\text{Ext}^1_{\mathcal{G}}(C, A) = 0$.
2. Let $\mathcal{F}$ be an epic precovering class of right $R$-modules, then any $\mathcal{F}$-extension of $A$ by $C$ $0 \to A \to B \to C \to 0$ is split if and only if $\varphi\text{Ext}^1(C, A) = 0$.

**Proof.** (1) $\Leftarrow$. Since $\mathcal{G}$ is a monic preenveloping class of right $R$-modules, we have $\text{Ext}^1_{\mathcal{G}}(C, -) \cong \text{Hom}(C, -)$ (see [5, p. 170]) and $\mathcal{G}$ is closed under finite direct sums by [1] (Lemma 1). Thus by [5] (Theorem 8.2.5(1)), the $\mathcal{G}$-coextension of $A$ by $C$ $0 \to A \to B \to C \to 0$ induces the exact sequence

$$0 \to \text{Hom}(C, A) \to \text{Hom}(C, B) \to \text{Hom}(C, C) \to \text{Ext}^1_{\mathcal{G}}(C, A) = 0.$$ 

So $0 \to A \to B \to C \to 0$ is split.

$\Rightarrow$. Since any $\mathcal{G}$-coextension of $A$ by $C$ $0 \to A \to B \to C \to 0$ is equivalent to the exact sequence $0 \to A \to A \oplus C \to C \to 0$, $E\mathcal{G}(C, A) = 0$. So $\text{Ext}^1_{\mathcal{G}}(C, A) \cong E\mathcal{G}(C, A) = 0$ by Theorem 2.2(1).

(2) $\Leftarrow$. Since $\mathcal{F}$ is an epic precovering class of right $R$-modules, we have $\varphi\text{Ext}^1(-, A) \cong \text{Hom}(-, A)$ (see [5, p. 170]) and $\mathcal{F}$ is closed under direct sums by [7] (Proposition 1). So by [5] (Theorem 8.2.3(2)), the $\mathcal{F}$-extension of $A$ by $C$ $0 \to A \to B \to C \to 0$ induces the exact sequence

$$0 \to \text{Hom}(C, A) \to \text{Hom}(B, A) \to \text{Hom}(A, A) \to \varphi\text{Ext}^1(C, A) = 0.$$ 

Thus $0 \to A \to B \to C \to 0$ is split.
⇒. Since any $\mathcal{F}$-extension of $A$ by $C$ is equivalent to the exact sequence $0 \to A \to A \oplus C \to C \to 0$, we have $\mathcal{F}E(C, A) = 0$. Thus $\mathcal{F}\text{Ext}^1(C, A) \cong \mathcal{F}E(C, A) = 0$ by Theorem 2.2(2).

Proposition 2.1 is proved.

**Corollary 2.3.** The following are true:

1. Let $\mathcal{G}$ be a monic preenveloping class of right $R$-modules, then a right $R$-module $A$ belongs to $\mathcal{G}$ if and only if $\text{Ext}^1_{\mathcal{G}}(C, A) = 0$ for any right $R$-module $C$.

2. Let $\mathcal{F}$ be an epic precovering class of right $R$-modules, then a right $R$-module $C$ belongs to $\mathcal{F}$ if and only if $\mathcal{F}\text{Ext}^1(C, A) = 0$ for any right $R$-module $A$.

**Proof.** It is easy by Proposition 2.1.

**Proposition 2.2.** The following are true:

1. Let $\mathcal{G}$ be a monic preenveloping class of right $R$-modules, then $\text{Ext}^2_{\mathcal{G}}(N, M) = 0$ for all right $R$-modules $M$ and $N$ if and only if $C \in \mathcal{G}$ for any $\mathcal{G}$-coextension $0 \to A \to B \to C \to 0$ with $B \in \mathcal{G}$.

2. Let $\mathcal{F}$ be an epic precovering class of right $R$-modules, then $\mathcal{F}\text{Ext}^2(N, M) = 0$ for all right $R$-modules $M$ and $N$ if and only if $A \in \mathcal{F}$ for any $\mathcal{F}$-extension $0 \to A \to B \to C \to 0$ with $B \in \mathcal{F}$.

**Proof.** (1) $\Rightarrow$. By [5] (Theorem 8.2.5(1)), for any right $R$-module $N$, any $\mathcal{G}$-coextension $0 \to A \to B \to C \to 0$ with $B \in \mathcal{G}$ induces the exact sequence

$$0 = \text{Ext}^1_{\mathcal{G}}(N, B) \to \text{Ext}^1_{\mathcal{G}}(N, C) \to \text{Ext}^2_{\mathcal{G}}(N, A) = 0.$$ 

So $\text{Ext}^1_{\mathcal{G}}(N, C) = 0$. Thus $C \in \mathcal{G}$ by Corollary 2.3(1).

$\Leftarrow$. For any right $R$-module $M$, by hypothesis, there exists a $\mathcal{G}$-coextension $0 \to M \to B \to C \to 0$ with $B \in \mathcal{G}$. So $C \in \mathcal{G}$. Thus by [5] (Theorem 8.2.5(1)), for any right $R$-module $N$, we get the induced exact sequence

$$0 = \text{Ext}^1_{\mathcal{G}}(N, C) \to \text{Ext}^2_{\mathcal{G}}(N, M) \to \text{Ext}^2_{\mathcal{G}}(N, B) = 0.$$ 

So $\text{Ext}^2_{\mathcal{G}}(N, M) = 0$.

(2) can be proved dually.

Proposition 2.2 is proved.

The following result may be viewed as a relative version of Wakamatsu’s lemmas.

**Theorem 2.3.** The following are true:

1. Suppose that $\mathcal{G}$ is a monic preenveloping class of right $R$-modules and $\mathcal{C}$ is a class of right $R$-modules closed under $\mathcal{G}$-coextensions. If $\alpha : N \to M$ is a $\mathcal{C}$-envelope of $N$, then $\text{Ext}^1_{\mathcal{G}}(\text{coker}(\alpha), C) = 0$ for any $C \in \mathcal{C}$.

2. Suppose that $\mathcal{F}$ is an epic precovering class of right $R$-modules and $\mathcal{D}$ is a class of right $R$-modules closed under $\mathcal{F}$-extensions. If $\alpha : N \to M$ is a $\mathcal{D}$-cover of $M$, then $\mathcal{F}\text{Ext}^1(D, \text{ker}(\alpha)) = 0$ for any $D \in \mathcal{D}$.

**Proof.** (1) By Proposition 2.1(1), it is enough to show that any $\mathcal{G}$-coextension $0 \to C \to B \xrightarrow{\rho} \text{coker}(\alpha) \to 0$ with $C \in \mathcal{C}$ is split.

Let $\lambda : \text{im}(\alpha) \to M$ be the inclusion and $\pi : M \to \text{coker}(\alpha)$ the canonical map. Then there exists $\gamma : N \to \text{im}(\alpha)$ such that $\lambda \gamma = \alpha$.

Consider the following pullback diagram:
Since $0 \to C \to B \xrightarrow{\rho} \text{coker}(\alpha) \to 0$ is a $G$-coextension, it is easy to see that $0 \to C \to X \to M \to 0$ is also a $G$-coextension. Thus $X \in C$ since $C$ is closed under $G$-coextensions. Because $\alpha: N \to M$ is a $C$-envelope, there exists $g: M \to X$ such that $i\gamma = ga$. Thus $\alpha = \lambda\gamma = \beta\gamma = \beta ga$. Hence $\beta g$ is an isomorphism.

Define $\varphi: \text{coker}(\alpha) \to B$ by $\varphi(\tau) = \theta g(\beta g)^{-1}(x)$ for $x \in M$. Since $\theta g(\beta g)^{-1} = \theta ga = \theta\lambda = 0$, $\varphi$ is well defined. Note that

$$\rho\varphi(\tau) = \rho\theta g(\beta g)^{-1}(x) = \pi\beta g(\beta g)^{-1}(x) = \pi(x) = \tau$$

for $x \in M$. Thus $\rho \varphi = 1$. Hence $0 \to C \to B \xrightarrow{\rho} \text{coker}(\alpha) \to 0$ is split, and so $\text{Ext}^1_B(\text{coker}(\alpha), C) = 0$.

(2) can be proved dually.

Theorem 2.3 is proved.

**Remark 2.2.** (1) Let $\mathcal{F}$ (resp. $\mathcal{G}$) in Theorem 2.3 be the class of projective (resp. injective) right $R$-modules, then Theorem 2.3 is just the usual Wakamatsu’s lemmas (see [5], Corollary 7.2.3 and Proposition 7.2.4 or [14], Section 2.1).

(2) Following [13], an exact sequence $0 \to A \to B \to C \to 0$ of left $R$-modules is called $RD$-exact if the sequence $\text{Hom}(R/\pi A, B) \to \text{Hom}(R/\pi A, C) \to 0$ is exact for every $\pi \in R$. A left $R$-module $G$ is called $RD$-injective if for every $RD$-exact sequence $0 \to A \to B \to C \to 0$ of left $R$-modules, the sequence $0 \to \text{Hom}(C, G) \to \text{Hom}(B, G) \to \text{Hom}(A, G) \to 0$ is exact. According to [2], a right $R$-module $F$ is called $RD$-flat if for every $RD$-exact sequence $0 \to A \to B \to C \to 0$ of left $R$-modules, the sequence $0 \to F \otimes A \to F \otimes B \to F \otimes C \to 0$ is exact.

Let $\mathcal{F}$ be the class of pure-projective right $R$-modules. It is well known that $\mathcal{F}$ is an epic precovering class of right $R$-modules (see [5], Example 8.3.2). Let $0 \to X \to Y \to Z \to 0$ be a pure exact sequence of right $R$-modules with $X$ and $Z$ $RD$-flat, then we get the split exact sequence $0 \to Z^+ \to Y^+ \to X^+ \to 0$. Since $X^+$ and $Z^+$ are $RD$-injective by [2] (Proposition 1.1), we have $Y^+$ is $RD$-injective. Hence $Y$ is $RD$-flat. So the class of $RD$-flat right $R$-modules is closed under $\mathcal{F}$-extensions by Remark 2.1(2). Note that any right $R$-module $M$ has an $RD$-flat cover $\alpha: N \to M$ by [11] (Theorem 2.6(2)). Thus $\mathcal{F}\text{Ext}^1(D, \ker(\alpha)) = 0$ for any $RD$-flat right $R$-module $D$ by Theorem 2.3(2).
We next give some isomorphism formulas about relative (co)homological groups.

**Theorem 2.4.** The following are true:

1. Let \( G \) be a preenveloping class of right \( R \)-modules, \( A_S \) a projective right \( S \)-module, \( S B_R \) an \((S, R)\)-bimodule, \( C_R \) a right \( R \)-module and \( n \geq 0 \). Then

\[
\text{Ext}_r^n(G, \text{Hom}_S(A, \text{Ext}_r^n(B, C))) = \text{Hom}_S(A, \text{Ext}_r^n(B, C)).
\]

2. Let \( F \) be a precovering class of right \( R \)-modules, \( A_R \) a right \( R \)-module, \( R B_S \) an \((R, S)\)-bimodule, \( E_S \) an injective right \( S \)-module and \( n \geq 0 \). Then

\[
\text{Ext}_r^n(F, \text{Hom}_S(B, E)) \cong \text{Hom}_S(\text{Ext}_r^n(A, B), E).
\]

**Proof.** (1) Let \( G : 0 \rightarrow G^0 \rightarrow G^1 \rightarrow \ldots \) be a deleted right \( G \)-resolution of \( C \). Then we obtain the complex \( \text{Hom}_R(A \otimes S B, G') \):

\[
0 \rightarrow \text{Hom}_R(A \otimes S B, G^0) \rightarrow \text{Hom}_R(A \otimes S B, G^1) \rightarrow \ldots,
\]

which is isomorphic to the complex \( \text{Hom}_S(A, \text{Hom}_R(B, G')) \):

\[
0 \rightarrow \text{Hom}_S(A, \text{Hom}_R(B, G^0)) \rightarrow \text{Hom}_S(A, \text{Hom}_R(B, G^1)) \rightarrow \ldots.
\]

Note that \( \text{Hom}_S(A, -) \) is an exact functor. So by [12, p. 170] (Exercise 6.4), we have \( \text{Ext}_r^n(A \otimes S B, C) = H^n(\text{Hom}_R(A \otimes S B, G')) \cong \text{Hom}_S(A, H^n(\text{Hom}_R(B, G'))) \cong \text{Hom}_S(A, \text{Ext}_r^n(B, C')). \)

(2) Let \( F : \ldots \rightarrow F_1 \rightarrow F_0 \rightarrow 0 \) be a deleted left \( F \)-resolution of \( A \). Then we obtain the complex \( \text{Hom}_S(F, \text{Hom}_R(B, E)) \):

\[
0 \rightarrow \text{Hom}_S(F_0 \otimes R B, E) \rightarrow \text{Hom}_S(F_1 \otimes R B, E) \rightarrow \ldots,
\]

which is isomorphic to the complex \( \text{Hom}_R(F, \text{Hom}_S(B, E)) \):

\[
0 \rightarrow \text{Hom}_R(F_0, \text{Hom}_S(B, E)) \rightarrow \text{Hom}_R(F_1, \text{Hom}_S(B, E)) \rightarrow \ldots.
\]

Note that \( \text{Hom}_S(-, E) \) is an exact functor. So by [12, p. 170] (Exercise 6.4), we have \( \text{Hom}_S(\text{Ext}_r^n(A, B), E) = \text{Hom}_S(H_n(F, \otimes R B), E) \cong H^n(\text{Hom}_S(F, \otimes R B), E) \cong H^n(\text{Hom}_R(F, \text{Hom}_S(B, E))) = \text{Ext}_r^n(A, \text{Hom}_S(B, E)). \)

Theorem 2.4 is proved.

**Corollary 2.4.** Let \( F \) be a precovering class of right \( R \)-modules, \( A_R \) a right \( R \)-module, \( R B \) a left \( R \)-module and \( n \geq 0 \). Then \( \text{Ext}_r^n(A, B^+) \cong \text{Ext}_r^n(A, B^+). \)

**Proof.** Let \( S = \mathbb{Z} \) and \( E = \mathbb{Q}/\mathbb{Z} \) in Theorem 2.4(2). Then we get the isomorphism \( \text{Ext}_r^n(A, B^+) \cong \text{Ext}_r^n(A, B^+). \)

Finally we discuss the relationship between \( \text{Tor}_r(A, B) \) and \( \text{Tor}_r(A, B) \).

Suppose that \( F \) is an epic precovering class of right \( R \)-modules. Let

\[
\ldots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow A \rightarrow 0
\]

be a left \( F \)-resolution of a right \( R \)-module \( A \) and let

\[
\ldots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0
\]
be a left projective resolution of $A$. Then there exist $f_i : P_i \rightarrow F_i$ such that the following diagram is commutative:

$$
\begin{array}{ccccccc}
\ldots & \rightarrow & P_2 & \rightarrow & P_1 & \rightarrow & P_0 & \rightarrow & A & \rightarrow & 0 \\
& f_2 & & f_1 & & f_0 & & & & \\
\ldots & \rightarrow & F_2 & \rightarrow & F_1 & \rightarrow & F_0 & \rightarrow & A & \rightarrow & 0.
\end{array}
$$

Applying $- \otimes B$ to the above diagram, we have the following commutative diagram of complexes:

$$
\begin{array}{ccccccc}
\ldots & \rightarrow & P_2 \otimes B & \rightarrow & P_1 \otimes B & \rightarrow & P_0 \otimes B & \rightarrow & 0 \\
& f_2 \otimes 1 & & f_1 \otimes 1 & & f_0 \otimes 1 & & & & \\
\ldots & \rightarrow & F_2 \otimes B & \rightarrow & F_1 \otimes B & \rightarrow & F_0 \otimes B & \rightarrow & 0.
\end{array}
$$

Then it is easy to check that there exist group homomorphisms:

$$
\eta_n : \text{Tor}_n(A, B) \rightarrow \mathcal{F}\text{Tor}_n(A, B), \quad n \geq 0.
$$

**Theorem 2.5.** If $\mathcal{F}$ is an epic precovering class of right $R$-modules, then $\eta_1 : \text{Tor}_1(A, B) \rightarrow \mathcal{F}\text{Tor}_1(A, B)$ is an epimorphism for any right $R$-module $A$ and any left $R$-module $B$.

**Proof.** Consider the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{F}\text{Ext}^1(A, B^+) & \rightarrow & \text{Ext}^1(A, B^+) \\
\gamma & \downarrow & \beta \\
\mathcal{F}\text{Tor}_1(A, B)^+ & \rightarrow & \text{Tor}_1(A, B)^+.
\end{array}
$$

Note that $\alpha$ and $\beta$ are isomorphisms by Corollary 2.4 and $\gamma$ is a monomorphism by Corollary 2.1(2). So $\eta_1^+ : \mathcal{F}\text{Tor}_1(A, B)^+ \rightarrow \text{Tor}_1(A, B)^+$ is a monomorphism. Thus $\eta_1 : \text{Tor}_1(A, B) \rightarrow \mathcal{F}\text{Tor}_1(A, B)$ is an epimorphism.

Theorem 2.5 is proved.

**Remark 2.3.** Let $\mathcal{F}$ be an epic precovering class of right $R$-modules. Although $\eta_1 : \text{Tor}_1(A, B) \rightarrow \mathcal{F}\text{Tor}_1(A, B)$ is an epimorphism by Theorem 2.5, this is not an isomorphism in general. For example, if $\mathcal{F}$ is the class of pure-projective $\mathbb{Z}$-modules, then $\mathcal{F}\text{Tor}_1(\mathbb{Z}_2, \mathbb{Z}_2) = 0$, but $\text{Tor}_1(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$.

Note that $\text{Tor}_0(A, B) \cong \mathcal{F}\text{Tor}_0(A, B) \cong A \otimes B$ for any right $R$-module $A$ and any left $R$-module $B$. It is natural to ask when $\text{Tor}_n(A, B) \rightarrow \mathcal{F}\text{Tor}_n(A, B)$ is an isomorphism. We give the following answer which is easy to verify.

**Proposition 2.3.** Let $\mathcal{F}$ be an epic precovering class of right $R$-modules. Then the following are equivalent:

1. $\text{Tor}_n(A, B) \cong \mathcal{F}\text{Tor}_n(A, B)$ for any right $R$-module $A$, left $R$-module $B$ and $n \geq 1$.
2. $\text{Tor}_1(A, B) \cong \mathcal{F}\text{Tor}_1(A, B)$ for any right $R$-module $A$ and left $R$-module $B$.
3. Every $M \in \mathcal{F}$ is flat.

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