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A. K. Asboei
(Farhangian Univ., Shariati Mazandaran, Iran and Buinzahra Branch, Islamic Azad Univ., Buinzahra, Iran),
S. S. S. Amiri (Babol Branch, Islamic Azad Univ., Babol, Iran),
A. Iranmanesh (Tarbiat Modares Univ., Tehran, Iran)

A NEW CHARACTERIZATION OF $\mathrm{PSL}(2,q)$ FOR SOME q*

НОВА ХАРАКТЕРИСТИКА $\mathrm{PSL}(2,q)$ ДЛЯ ДЕЯКОГО q

Let G be a finite group and let $\pi_e(G)$ be the set of element orders of G. Let $k \in \pi_e(G)$ and let m_k be the number of elements of order k in G. We set $\mathrm{nse}\,(G) := \{m_k | k \in \pi_e(G)\}$. It is proved that $\mathrm{PSL}(2,q)$ are uniquely determined by $\mathrm{nse}\,(\mathrm{PSL}(2,q))$, where $q \in \{5,7,8,9,11,13\}$. As the main result of the paper, we prove that if G is a group such that $\mathrm{nse}\,(G) = \mathrm{nse}\,(\mathrm{PSL}(2,q))$, where $q \in \{16,17,19,23\}$, then $G \cong \mathrm{PSL}(2,q)$.

Нехай G — скінченна група, а $\pi_e(G)$ — множина порядків елементів з G. Нехай також $k \in \pi_e(G)$, а m_k — число елементів порядку k в G. Покладемо $\mathrm{nse}\,(G) := \{m_k | k \in \pi_e(G)\}$. Доведено, що $\mathrm{PSL}(2,q)$ однозначно визначаються $\mathrm{nse}\,(\mathrm{PSL}(2,q))$, де $q \in \{5,7,8,9,11,13\}$. Основним результатом роботи є доведення того факту, що якщо G є групою, для якої $\mathrm{nse}\,(G) = \mathrm{nse}\,(\mathrm{PSL}(2,q))$, де $q \in \{16,17,19,23\}$, то $G \cong \mathrm{PSL}(2,q)$.

1. Introduction. If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a finite group. Denote by $\pi(G)$ the set of primes p such that G contains an element of order p. Also the set of element orders of G is denoted by $\pi_e(G)$. A finite group G is called a simple K_n-group, if G is a simple group with $|\pi(G)| = n$. Set $m_i = m_i(G) = |\{g \in G|$ the order of g is $i\}|$. In fact, m_i is the number of elements of order i in G, and nse$(G) := \{m_i | i \in \pi_e(G)\}$, the set of sizes of elements with the same order.

In [11], it is proved that all simple K_4-groups can be uniquely determined by $\mathrm{nse}\,(G)$ and $|G|$. However, in [13], it is proved that the groups: A_4, A_5, and A_6, and in [9] the groups $\mathrm{PSL}(2,q)$ for $q \in \{7,8,11,13\}$ are uniquely determined by only $\mathrm{nse}\,(G)$. Similar characterizations have been found for the following groups: A_7 and A_8 [2], the sporadic simple groups [3], $\mathrm{PSL}(2,p)$ [1], the alternating groups [5], and the symmetric groups S_r, where r is prime [4]. In [9], the authors gave the following problem:

Let G be a group such that $\mathrm{nse}\,(G) = \mathrm{nse}\,(\mathrm{PSL}(2,q))$, where q is a prime power. Is G isomorphic to $\mathrm{PSL}(2,q)$?

In this paper, we give a positive answer to this problem and show that the group $\mathrm{PSL}(2,q)$ is characterizable by only $\mathrm{nse}\,(G)$ for $q \in \{16,17,19,23\}$. In fact, the main theorem of our paper is as follows:

Main theorem. *Let G be a group such that* $\mathrm{nse}\,(G) = \mathrm{nse}\,(\mathrm{PSL}(2,q))$, *where* $q \in \{16,17,19,23\}$. *Then* $G \cong \mathrm{PSL}(2,q)$.

In this paper, we use a new technique for the proof of our main result. Also we apply the technique of used in [9].

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We note that there are finite groups which are not characterizable by $\mathrm{nse}\,(G)$ and $|G|$. In 1987, Thompson gave an example as follows:

Let $G_1 = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_7$ and $G_2 = L_3(4) \rtimes C_2$ be the maximal subgroups of M_{23}. Then $\mathrm{nse}\,(G_1) = \mathrm{nse}\,(G_2)$ and $|G_1| = |G_2|$, but $G_1 \ncong G_2$. Throughout this paper, we denote by $\phi(n)$ the Euler totient function. If G is a finite group, then we denote by P_q a Sylow q-subgroup of G and $n_q(G)$ is the number of Sylow q-subgroup of G, that is, $n_q(G) = |\mathrm{Syl}_q(G)|$. All further unexplained notations are standard, and the reader is referred to [6] if necessary.

2. Preliminary results. In this section, we bring some preliminary lemmas to be used in the proof of main theorem.

Lemma 2.1 [7] (9.3.1). *Let G be a finite solvable group and $|G| = mn$, where $m = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$, $(m,n) = 1$. Let $\pi = \{p_1, \ldots, p_r\}$ and h_m be the number of π-Hall subgroups of G. Then $h_m = q_1^{\beta_1} \ldots q_s^{\beta_s}$ satisfies the following conditions for all $i \in \{1, 2, \ldots, s\}$:*

1) $q_i^{\beta_i} \equiv 1 (\mathrm{mod}\, p_j)$ *for some p_j;*

2) *the order of some chief factor of G is divisible by $q_i^{\beta_i}$.*

Lemma 2.2 [8]. *If G is a simple K_3-group, then G is isomorphic to one of the following groups: A_5, A_6, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,17)$, $\mathrm{PSL}(3,3)$, $\mathrm{PSU}(3,3)$ or $\mathrm{PSU}(4,2)$.*

Lemma 2.3 [14]. *Let G be a simple group of order $2^a 3^b 5 p^c$, where $p \neq 2, 3, 5$ is a prime, and $abc \neq 0$. Then G is isomorphic to one of the following groups: A_7, A_8, A_9; M_{11}, M_{12}; $\mathrm{PSL}(2,q)$, $q = 11, 16, 19, 31, 81$; $\mathrm{PSL}(3,4)$, $\mathrm{PSL}(4,3)$, $S_6(2)$, $\mathrm{PSU}(4,3)$, or $\mathrm{PSU}(5,2)$. In particular, if $p = 11$, then $G \cong M_{11}$, M_{12}, $\mathrm{PSU}(5,2)$, or $\mathrm{PSL}(2,11)$; if $p = 7$, then $G \cong A_7$, A_8, A_9, A_{10}, $\mathrm{PSL}(2,49)$, $\mathrm{PSL}(3,4)$, $S_4(7)$, $S_6(2)$, $\mathrm{PSU}(3,5)$, $\mathrm{PSU}(4,3)$, J_2, or $O_8^+(2)$.*

Lemma 2.4 [11]. *Let G be a finite group, $P \in \mathrm{Syl}_p(G)$, where $p \in \pi(G)$. Let G have a normal series $K \trianglelefteq L \trianglelefteq G$. If $P \leq L$ and $p \nmid |K|$, then the following hold*:

(1) $N_{G/K}(PK/K) = N_G(P)K/K$;

(2) $|G \colon N_G(P)| = |L \colon N_L(P)|$, *that is,* $n_p(G) = n_p(L)$;

(3) $|L/K \colon N_{L/K}(PK/K)|t = |G \colon N_G(P)| = |L \colon N_L(P)|$, *that is,* $n_p(L/K)t = n_p(G) = n_p(L)$ *for some positive integer t, and $|N_K(P)|t = |K|$.*

Lemma 2.5 [7] (9.1.2). *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m \mid |L_m(G)|$.*

Lemma 2.6 [13]. *Let G be a group containing more than two elements. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. If $s = \sup\{m_k | k \in \pi_e(G)\}$ is finite, then G is finite and $|G| \leq s(s^2 - 1)$.*

Lemma 2.7 [10]. *Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$, where $(p,m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s.*

Let G be a group such that $\mathrm{nse}\,(G) = \mathrm{nse}\,(\mathrm{PSL}(2,q))$, where $q \in \{16, 17, 19, 23\}$. By Lemma 2.6, we can assume that G is finite. Let m_n be the number of elements of order n. We note that $m_n = k\phi(n)$, where k is the number of cyclic subgroups of order n in G. Also we note that if $n > 2$, then $\phi(n)$ is even. If $n \in \pi_e(G)$, then by Lemma 2.5 and the above notation we have

$$\phi(n) \mid m_n,$$
$$n \mid \sum_{d \mid n} m_d. \tag{2.1}$$

In the proof of the main theorem, we often apply (2.1) and the above comments.

3. Proof of the main theorem. In this section, first we prove that the group $\mathrm{PSL}(2, 16)$ is characterizable by nse.

***3.1. Characterizability of the group* $\mathbf{PSL(2, 16)}$ *by* NSE.** Let G be a group such that $\mathrm{nse}\,(G) = \mathrm{nse}\,(\mathrm{PSL}(2, 16)) = \{1, 255, 272, 544, 1088, 1920\}$. First, we prove that $\pi(G) \subseteq \{2, 3, 5, 17\}$. Since $255 \in \mathrm{nse}\,(G)$, it follows that by (2.1), $2 \in \pi(G)$ and $m_2 = 255$. Let $2 \neq p \in \pi(G)$, by (2.1), $p \mid (1 + m_p)$, and $(p - 1) \mid m_p$, which implies that $p \in \{3, 5, 17\}$. If 3, 5 and $17 \in \pi(G)$, then $m_3 = 272$, $m_5 = 544$ and $m_{17} = 1920$, by (2.1). Also we can see easily that G does not contain any elements of order 9, 25, 51, 85, 289, and 512. Similarly, we can see that if $10 \in \pi_e(G)$, then $m_{10} = 1920$.

Let $3 \in \pi(G)$. Since $9 \notin \pi_e(G)$, $\exp(P_3) = 3$. Considering $m = |P_3|$ in Lemma 2.5, we have $|P_3| \mid (1 + m_3) = 273$. Hence $|P_3| = 3$, and $n_3 = m_3/\phi(3) = 136 \mid |G|$. Thus if $3 \in \pi(G)$, then $17 \in \pi(G)$.

Let $17 \in \pi(G)$. Since $289 \notin \pi_e(G)$, $\exp(P_{17}) = 17$. By Lemma 2.5, $|P_{17}| \mid (1 + m_{17}) = 1921$. Hence $|P_{17}| = 17$ and $n_{17} = m_{17}/\phi(17) = 120 \mid |G|$. Thus if $17 \in \pi(G)$, then 3 and $5 \in \pi(G)$.

Now let $5 \in \pi(G)$. Since $25 \notin \pi_e(G)$, $\exp\,(P_5) = 5$. By Lemma 2.5, $|P_5| \mid (1 + m_5) = 545$. Thus $|P_5| = 5$ and $n_5 = m_5/\phi(5) = 136 \mid |G|$. Hence if $5 \in \pi(G)$, then $17 \in \pi(G)$. By the above discussion if 3 or 5, or $17 \in \pi(G)$, then $\pi(G) = \{2, 3, 5, 17\}$. In follow, we show that $\pi(G)$ could not be the set $\{2\}$, and hence $\pi(G)$ must be equal to $\{2, 3, 5, 17\}$.

If $\pi(G) = \{2\}$, then $\pi_e(G) \subseteq \{1, 2, 2^2, \ldots, 2^8\}$. Hence $|G| = 2^m = 4080 + 272k_1 + 544k_2 + 1088k_3 + 1920k_4$, where m, k_1, k_2, k_3 and k_4 are nonnegative integers and $0 \leq k_1 + k_2 + k_3 + k_4 \leq 3$. It is clear that $4080 \leq |G| \leq 9840$. Hence $m = 12$, or 13.

If $m = 12$, then $16 = 272k_1 + 544k_2 + 1088k_3 + 1920k_4$, which is a contradiction. Hence $m = 13$, and we have $4112 = 272k_1 + 544k_2 + 1088k_3 + 1920k_4$. Using an easy computer calculation, $k_1 = 1$, $k_2 = 0$, $k_3 = 0$, and $k_4 = 2$. Therefore, $\pi_e(G) = \{1, 2, 2^2, \ldots, 2^8\}$, and $|G| = 8192$. We prove that no such group G exists.

By (2.1), clearly $m_{256} = 1920$. Since $m_{256} = k\ \phi(256)$, where k is the number of cyclic subgroups of order 256 in G, then G have 15 cyclic subgroups of order 256. Set $\Omega = \{H_i \mid |H_i| = 256, 1 \leq i \leq 15\}$, where H_i is a cyclic subgroup of order 256 of G and we know that G acts on Ω by conjugation. If $\Delta(H_i)$ is an orbit of this action, then $|\Delta(H_i)| = |G \colon N_G(H_i)|$, and hence by Orbit – Stabilizer theorem, the size of each orbit is a power of 2. On the other hand, the sizes of these orbits add up to 15, which is odd, so at least one of the orbits must have size 1. Hence for one of H_i, we have $|\Delta(H_i)| = |G \colon N_G(H_i)| = 1$, that is, $G = N_G(H_i)$. Therefore, G has a normal cyclic subgroup of order 256, say N. The index of N in G is $|G \colon N| = 8192/256 = 32$. Let h be a generator of N, then N has only one element of order 2, namely $z = h^{128}$. Now we consider any other coset of N in G. If this contains an element of order 2, say x, then the coset is Nx, and we can proceed as follows. Since x normalizes N and $N = \langle h \rangle$, then x conjugates h to h^r for a some odd number r, and since x has order 2, then by relation $h^x = h^r$, we conclude that $h^{rx} = h^{r^2}$. Hence $(h^r)^x = h^{r^2}$, then $(h^x)^x = h^{x^2} = h = h^{r^2}$. Therefore, $r^2 \equiv 1$ (mod 256), then $r \equiv 1$, 127, 129 or 255 (mod 256). Let L be a subgroup of G generated by N and x which is a semidirect product of order $2|N| = 512$, and is also the union of the two coset N and Nx. Now we can count the number of elements of order 2 in the coset Nx. If $r = 1$, then L is isomorphic to a direct product of C_{256} by C_2 and so it has exactly three elements of order 2 one of them is z, and the other two are x and zx, lying in Nx. If $r = 127$, then $(h^i x)^2 = h^i(xh^i x) = h^{i+127i} = h^{128i}$ and so $h^i x$ has order 2 when i is even and order 4 when i is odd, hence in this case, Nx contains exactly 128 elements of order 2. If $r = 129$, then $(h^i x)^2 = h^i(xh^ix) = h^{i+129i} = h^{130i}$ and so h^ix has order 2 precisely when i is divisible by 128, and so in this case, Nx has exactly two elements of order 2, x and zx. If $r = 255$, then $(h^ix)^2 = h^i(xh^ix) = h^{i-i} = 1$, for all i, and so all 256 elements of Nx has order 2. It follows that every coset Ng of N in G contains 0, 1, 2, 128 or 256 elements of order 2. Finally, let t_i be the number of elements of order 2 in the j the coset of N in G. The number of t_i must be 32 and the sum of t_i's is equal to 255, since we have $m_2 = 255$, which is impossible. Because by choose 32 integers from the set $\{0, 1, 2, 128, 256\}$, we don't obtain the number 255. Hence no such group G exists.

Therefore, $\pi(G) = \{2, 3, 5, 17\}$. Since $51 \notin \pi_e(G)$, the group P_{17} acts fixed point freely on the set of elements of order 3, and so $|P_{17}| \mid m_3 = 272$, which implies that $|P_{17}| = 17$. Similarly, we conclude that $|P_3| = 3$. In addition, since $85 \notin \pi_e(G)$, the group P_5 acts fixed point freely on the set of elements of order 17, which implies that $|P_5| = 5$.

Now we prove that $10 \notin \pi_e(G)$. Suppose that $10 \in \pi_e(G)$. We know that if P and Q are Sylow 5-subgroups of G, then P and Q are conjugate, which implies that $C_G(P)$ and $C_G(Q)$ are conjugate in G. Therefore, $m_{10} = \phi(10)n_5k$, where k is the number of cyclic subgroups of order 2 in $C_G(P_5)$. Since $n_5 = m_5/\phi(5) = 136$, $544 \mid m_{10}$. On the other hand, $m_{10} = 1920$, which is a contradiction. Hence $10 \notin \pi_e(G)$.

Since $10 \notin \pi_e(G)$, the group P_2 acts fixed point freely on the set of elements of order 5. Thus $|P_2| \mid m_5 = 544$, which implies that $|P_2| \mid 2^5$. Since $4080 \leq |G|$, $|G| = 2^5 \times 3 \times 5 \times 17$ or $|G| = 2^4 \times 3 \times 5 \times 17$. If $|G| = 2^5 \times 3 \times 5 \times 17$, then since $m_5 = 544$, $n_5 = 136$.

We claim that G is a nonsolvable group. Suppose G is a solvable group. By Lemma 2.1, $2^3 \equiv 1 \pmod 5$, which is a contradiction. Hence G is a nonsolvable group. Since G is a nonsolvable group, and $p \parallel |G|$ for $p \in \{3, 5, 17\}$, G has a normal series $1 \trianglelefteq N \trianglelefteq H \trianglelefteq G$, such that N is a maximal solvable normal subgroup of G, and H/N is a nonsolvable minimal normal subgroup of G/N. Then H/N is a non-Abelian simple K_3-group, or K_4-group. Let H/N be a non-Abelian simple K_3-group. By Lemma 2.2, $H/N \cong A_5$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5N/N \in \mathrm{Syl}_5(H/N)$, and $n_5(H/N)t = n_5(G)$ for some positive integer t. By Lemma 2.4, $5 \nmid t$. Since $n_5(H/N) = n_5(A_5) = 6$, $136 = 6t$, which is a contradiction. Therefore, H/N is a non-Abelian simple K_4-group. By Lemma 2.3, $H/N \equiv \mathrm{PSL}(2, 16)$. Now set $\overline{H} := H/N \cong \mathrm{PSL}(2, 16)$ and $\overline{G} := G/N$. On the other hand, we have

$$\mathrm{PSL}(2,16) \cong \overline{H} \cong \overline{H}C_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \overline{G}/C_{\overline{G}}(\overline{H}) = N_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \mathrm{Aut}(\overline{H}).$$

Let $K = \{x \in G \mid xN \in C_{\overline{G}}(\overline{H})\}$, then $G/K \cong \overline{G}/C_{\overline{G}}(\overline{H})$. Hence $\mathrm{PSL}(2,16) \leq G/K \leq$ $\leq \mathrm{Aut}(\mathrm{PSL}(2,16))$, then $G/K \cong \mathrm{PSL}(2,16)$, $\mathrm{PGO}^-(4,4)$, or $\mathrm{P\Gamma L}(2,16)$. If $G/K \cong \mathrm{P\Gamma L}(2,16)$, then since $|G| = 2|\mathrm{PSL}(2,16)|$, which is a contradiction.

If $G/K \cong \mathrm{PGO}^-(4,4)$, then since $|G| = 2|\mathrm{PSL}(2,16)|$, $|K| = 1$, and $G \cong \mathrm{PGO}^-(4,4)$. On the other hand, $\mathrm{nse}\,(G) \neq \mathrm{nse}\,(\mathrm{PGO}^-(4,4))$, we get a contradiction.

If $G/K \cong \mathrm{PSL}(2,16)$, then $|K| = 2$. Since $N \leq K$, and N is a maximal solvable normal subgroup of G, $N = K$. Now we know that $H/N \cong \mathrm{PSL}(2,16)$, where $|N| = 2$, so G has a normal subgroup N of order 2, generated by a central involution z. Let x be an element of order 5 in G. Since $xz = zx$ and $(o(x), o(z)) = 1$, $o(xz) = 10$. Hence $10 \in \pi_e(G)$, this gives a contradiction. Therefore, $|G| = 2^4 \times 3 \times 5 \times 17$. Now we have $|G| = |\mathrm{PSL}(2,16)|$, and $\mathrm{nse}\,(G) = \mathrm{nse}\,(\mathrm{PSL}(2,16))$. By [11], since $\mathrm{PSL}(2,16)$ is simple K_4-group, $G \cong \mathrm{PSL}(2,16)$.

3.2. Characterizability of the group $\mathbf{PSL(2,17)}$ ***by*** **NSE.** Let G be a group such that $\mathrm{nse}\,(G) =$ $= \mathrm{nse}\,(\mathrm{PSL}(2,17)) = \{1, 153, 272, 288, 306, 612, 816\}$. First, we prove that $\pi(G) \subseteq \{2, 3, 17\}$.

Since $153 \in \mathrm{nse}\,(G)$, it follows that by (2.1), $2 \in \pi(G)$ and $m_2 = 153$. Let $2 \neq p \in \pi(G)$, by (2.1), $p \mid (1 + m_p)$, and $(p-1) \mid m_p$, which implies that $p \in \{3, 17, 307, 613\}$.

We will show $307, 613 \notin \pi(G)$. If $307 \in \pi(G)$, then by (2.1), $m_{307} = 306$. On the other hand, by (2.1), we conclude that if $614 \in \pi_e(G)$, then $m_{614} = 306$, or 612, and $614 \mid (1+m_2+m_{307}+m_{614})$. Hence $614 \mid 766$, or $614 \mid 1071$, which is a contradiction. Therefore, $614 \notin \pi_e(G)$. Since $614 \notin \pi_e(G)$, the group P_{307} acts fixed point freely on the set of elements of order 2, and $|P_{307}| \mid m_2$, which is a contradiction. Hence $307 \notin \pi(G)$. Similar to the above discussion $613 \notin \pi(G)$.

Therefore, $\pi(G) \subseteq \{2, 3, 17\}$. If 3 and $17 \in \pi(G)$, then $m_3 = 272$, $m_{17} = 288$, by (2.1). Also we can see easily that G does not contain any elements of order 17^2, $2^2 \times 17$, 3×17 and 3^4. Similarly, we can see that if $9 \in \pi_e(G)$, then $m_9 = 816$, and if $27 \in \pi_e(G)$, then $m_{27} \in \{612, 288\}$. Let $3 \in \pi(G)$, since $81 \notin \pi_e(G)$, then $\exp(P_3) = 3$ or 9, or 27. We will show, in all of these cases if $3 \in \pi(G)$, then $17 \in \pi(G)$.

If $\exp\,(P_3) = 3$, then $|P_3| \mid (1 + m_3) = 273$. Hence $|P_3| = 3$ and $n_3 = m_3/\phi(3) = 136 \mid |G|$. Hence in this case if $3 \in \pi(G)$, then $17 \in \pi(G)$.

If $\exp\,(P_3) = 9$, then $|P_3| \mid (1 + m_3 + m_9) = 1089$. Hence $|P_3| = 9$ and $n_3 = m_9/\phi(9) = 136 \mid |G|$. Hence if $3 \in \pi(G)$, then $17 \in \pi(G)$.

If $\exp\,(P_3) = 27$, then $|P_3| \mid (1 + m_3 + m_9 + m_{27})$. Since $m_{27} \in \{612, 288\}$, hence $|P_3| \mid 81$, or 243. If $|P_3| = 27$, then $n_3 = m_{27}/\phi(27) = 16$, or 34. If $n_3 = 34$, then $17 \in \pi(G)$. If $n_3 = 16$, then since every element of order 3 lying in the Sylow 3-subgroup and every Sylow 3-subgroup has at most 2 elements of order 3, $m_3 \leq 2 \times 16 = 32$, which is a contradiction. If $|P_3| \geq 81$ by Lemma 2.7, m_{27} is a multiple of 27, which is a contradiction. Hence if $3 \in \pi(G)$, then $17 \in \pi(G)$.

Now let $17 \in \pi(G)$. Since $2^2 \times 17 \notin \pi_e(G)$ and $4 \in \pi_e(G)$, the group P_{17} acts fixed point freely on the set of elements of order 4. Hence $|P_{17}| \mid m_4 = 306$, which implies that $|P_{17}| = 17$, then $n_{17} = m_{17}/\phi(17) = 18$. Therefore, if $17 \in \pi(G)$, then $3 \in \pi(G)$. By the above discussion if 3 or $17 \in \pi(G)$, then $\pi(G) = \{2, 3, 5, 17\}$. In follow, we show that $\pi(G)$ could not be the set $\{2\}$, and hence $\pi(G)$ must be equal to $\{2, 3, 17\}$.

Let $\pi(G) = \{2\}$. Hence $\pi_e(G) \subseteq \{1, 2, 2^2, \ldots, 2^6\}$. Thus we have $|G| = 2^m = 2448 + 272k_1 + 288k_2 + 816k_3 + 612k_4 + 306k_5$, where m, k_1, k_2, k_3, k_4 and k_5 are nonnegative integers and $k_1 + k_2 + k_3 + k_4 + k_5 = 0$, which is a contradiction.

Therefore, $\pi(G) = \{2, 3, 17\}$. Since $51 \notin \pi_e(G)$, then P_{17} acts fixed point freely on the set of elements of order 3. Hence $|P_{17}| \mid m_3 = 272$, which implies that $|P_{17}| = 17$. Similarly, we conclude that $|P_3| = 3$, or 3^2. We have $|G| = 2^m \times 3^n \times 17 = 2448 + 272k_1 + 288k_2 + 306k_3 + 612k_4 + 816k_5$, where m, n, k_1, k_2, k_3, k_4 and k_5 are nonnegative integers $1 \leq n \leq 2$, and $0 \leq k_1+k_2+k_3+k_4+k_5 \leq 20$. Let $34 \in \pi_e(G)$, then $m_{34} = \phi(34)n_{17}k$, where k is the number of cyclic subgroups of order 2 in $C_G(P_{17})$. Since $n_{17} = m_{17}/\phi(17) = 18$, $288 \mid m_{34}$, and so by $\mathrm{nse}\,(G)$ we have $m_{34} = 288$. On the other hand, by (2.1), $34 \mid (1 + m_2 + m_{17} + m_{34}) = 730$, which is a contradiction. Hence $34 \notin \pi_e(G)$.

Since $34 \notin \pi_e(G)$, the group P_2 acts fixed point freely on the set of elements of order 17, and so $|P_2| \mid m_{17} = 288$, which implies that $1 \leq m \leq 5$. Since $2448 \leq |G|$, and $1 \leq n \leq 2$, $m \geq 4$. Therefore, $|G| = 2^4 \times 3^2 \times 17$, or $|G| = 2^5 \times 3^2 \times 17$. If $|G| = 2^5 \times 3^2 \times 17$, then we show that G is a nonsolvable group. Suppose that G is a solvable group. Since $n_{17} = 18$, then by Lemma 2.1, $3^2 \equiv 1 \pmod{17}$, which is a contradiction. Hence G is a nonsolvable group. Since G is a nonsolvable group, and $17^2 \nmid |G|$, G has a normal series: $1 \trianglelefteq N \trianglelefteq H \trianglelefteq G$, such that N is a maximal solvable normal subgroup of G and H/N is a nonsolvable minimal normal subgroup of G/N. Then H/N is a non-Abelian simple K_3-group. Hence by Lemma 2.2, $H/N = \mathrm{PSL}(2, 17)$. Now set

$\overline{H} := H/N \cong \mathrm{PSL}(2,17)$ and $\overline{G} := G/N$. On the other hand, we have

$$\mathrm{PSL}(2,17) \cong \overline{H} \cong \overline{H}C_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \overline{G}/C_{\overline{G}}(\overline{H}) = N_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \mathrm{Aut}(\overline{H}).$$

Let $K = \{x \in G \mid xN \in C_{\overline{G}}(\overline{H})\}$, then $G/K \cong \overline{G}/C_{\overline{G}}(\overline{H})$. Hence $\mathrm{PSL}(2,17) \leq G/K \leq \leq \mathrm{Aut}(\mathrm{PSL}(2,17))$, then $G/K \cong \mathrm{PGL}(2,17)$, or $\mathrm{PSL}(2,17)$. If $G/K \cong \mathrm{PGL}(2,17)$. Since $|G| = = 2|\mathrm{PSL}(2,17)|$, $|K| = 1$, and $G \cong \mathrm{PGL}(2,17)$. On the other hand, $\mathrm{nse}\,(G) \neq \mathrm{nse}\,(\mathrm{PGL}(2,17))$, we get a contradiction. If $G/K \cong \mathrm{PSL}(2,17)$, then $|K| = 2$. Since $N \leq K$, and N is a maximal solvable normal subgroup of G, $N = K$. Now we know that $H/N \cong \mathrm{PSL}(2,17)$, where $|N| = 2$, so G has a normal subgroup N of order 2, generated by a central involution z. Let x be an element of order 17 in G. Since $xz = zx$ and $(o(x), o(z)) = 1$, $o(xz) = 34$. Hence $34 \in \pi_e(G)$, this gives a contradiction. If $|G| = 2^4 \times 3^2 \times 17$, then $|G| = |\mathrm{PSL}(2,17)|$. Since $|G| = |\,\mathrm{PSL}(2,17)|$ and $\mathrm{nse}\,(G) = \mathrm{nse}\,(\mathrm{PSL}(2,17))$. By [12], $G \cong \mathrm{PSL}(2,17)$.

***3.3. Characterizability of the group* $\mathbf{PSL(2,19)}$ *by* NSE.** Let G be a group such that $\mathrm{nse}\,(G) = = \mathrm{nse}\,(\mathrm{PSL}(2,19)) = \{1, 171, 360, 380, 684, 1140\}$. First, we prove that $\pi(G) \subseteq \{2,3,5,19\}$. Since $171 \in \mathrm{nse}\,(G)$, it follows that by (2.1), $2 \in \pi(G)$ and $m_2 = 171$. If $2 \neq p \in \pi(G)$, then $p \in \{3,5,7,19\}$.

We will show $7 \notin \pi(G)$. Let $7 \in \pi(G)$, then by (2.1), $m_7 = 1140$. On the other hand, by (2.1), we conclude that if $14 \in \pi_e(G)$, then $m_{14} \in \{360, 684, 1140\}$, and $14 \mid (1+m_2+m_7+m_{14})$, which is a contradiction. Hence $14 \notin \pi_e(G)$. Thus the group P_7 acts fixed point freely on the set of elements of order 2, and $|P_7| \mid m_2$, which is a contradiction. Hence $7 \notin \pi(G)$ and so $\pi(G) \subseteq \{2,3,5,19\}$.

If 3, 5 and $19 \in \pi(G)$, then $m_3 = 380$, $m_5 = 684$, and $m_{19} = 360$. Also we can see easily that G does not contain any elements of order 27, 32, 95, 125, and 361. Similarly, we can see that if $9 \in \pi_e(G)$, then $m_9 = 1140$, and if $25 \in \pi_e(G)$, then $m_{25} = 1140$.

Let $3 \in \pi(G)$. Since $27 \notin \pi_e(G)$, $\exp(P_3) = 3$, or 9.

If $\exp\,(P_3) = 3$, then $|P_3| \mid (1+m_3) = 381$. Thus $|P_3| = 3$ and $n_3 = m_3/\phi(3) = 190 \mid |G|$. Hence in this case if $3 \in \pi(G)$, then 5 and $19 \in \pi(G)$.

If $\exp\,(P_3) = 9$, then $|P_3| \mid (1+m_3+m_9) = 1521$ and so $|P_3| = 9$, and $n_3 = m_9/\phi(9) = 190 \mid |G|$. Hence if $3 \in \pi(G)$, then 5 and $19 \in \pi(G)$.

Now let $5 \in \pi(G)$. Since $125 \notin \pi_e(G)$, $\exp(P_5) = 5$, or 25.

If $\exp\,(P_5) = 5$, then $|P_5| \mid (1+m_5) = 685$, and so $|P_5| = 5$ and $n_5 = m_5/\phi(5) = 171 \mid |G|$. Hence in this case if $5 \in \pi(G)$, then 3 and $19 \in \pi(G)$.

If $\exp\,(P_5) = 25$, then $|P_5| \mid (1+m_5+m_{25}) = 1825$. Hence $|P_5| = 25$ and $n_5 = m_{25}/\phi(25) = = 157 \mid |G|$. Therefore, if $5 \in \pi(G)$, then 3 and $19 \in \pi(G)$.

In follow, we show that $\pi(G)$ could not be the sets $\{2\}$ and $\{2,19\}$, and $\pi(G)$ must be equal to $\{2,3,5,19\}$.

Let $\pi(G) = \{2\}$, then $\pi_e(G) \subseteq \{1, 2, 2^2, 2^3, 2^4\}$. Since nse (G) has six elements, this case is impossible.

Let $\pi(G) = \{2,19\}$. Since $361 \notin \pi_e(G)$, $\exp\,(P_{19}) = 19$. Hence $|P_{19}| \mid (1+m_{19}) = 361$ and so $|P_{19}| = 19$, or 361.

If $|P_{19}| = 19$, then $n_{19} = m_{19}/\phi(19) = 20| \mid G|$; a contradiction.

If $|P_{19}| = 361$, then $|G| = 2^m \times 19^2 = 2736 + 360k_1 + 380k_2 + 684k_3 + 1140k_4$, where m, k_1, k_2, k_3, and k_4 are nonnegative integers, and $0 \leq k_1+k_2+k_3+k_4 \leq 2$, by $|\pi_e(G)| \leq 8$. On the other hand, $2736 \leq |G| \leq 5016$, so $m = 3$ and $|G| = 2^3 \times 19^2$. Then $152 = 2736 + 360k_1 + 380k_2 + + 684k_3 + 1140k_4$. It is easy to check this equation has no solution. Hence this case is impossible.

Therefore, $\pi(G) = \{2, 3, 5, 19\}$. Since $95 \notin \pi_e(G)$, the group P_{19} acts fixed point freely on the set of elements of order 5, and so $|P_{19}| \mid m_5 = 684$, which implies that $|P_{19}| = 19$, then $n_{19} = m_{19}/\phi(19) = 20 \mid |G|$. Similarly, we conclude that $|P_5| = 5$. If $15 \in \pi_e(G)$, then $m_{15} = \phi(15)n_5k$, where k is the number of cyclic subgroups of order 3 in $C_G(P_5)$. Since $n_5 = m_5/\phi(5) = 171$, $2 \times 684 \mid m_{15}$. On the other hand, we know that $m_{15} = 360$, which is a contradiction. Hence $15 \notin \pi_e(G)$. Similarly, we can see easily $38 \notin \pi_e(G)$.

Since $15 \notin \pi_e(G)$, the group P_3 acts fixed point freely on the set of elements of order 5. So $|P_3| \mid m_5$, which implies that $|P_3| \mid 9$. In addition, since $38 \notin \pi_e(G)$, $|P_2| \mid m_{19} = 360$, and so $|P_2| \mid 2^4$. Because $2736 \leq |G|$ and $20 \mid |G|$, $|G| = 2^2 \times 3^2 \times 5 \times 19$, or $|G| = 2^3 \times 3^2 \times 5 \times 19$. If $|G| = 2^3 \times 3^2 \times 5 \times 19$, then similar to the case $\mathrm{PSL}(2, 16)$, or $\mathrm{PSL}(2, 17)$, we get a contradiction. Now $\mathrm{nse}\,(G) = \mathrm{nse}\,(\mathrm{PSL}(2, 19))$, and $|G| = |\mathrm{PSL}(2, 19)|$. By [11], since $\mathrm{PSL}(2, 19)$ is a simple K_4-group, $G \cong \mathrm{PSL}(2, 19)$.

3.4. Characterizability of the group $\mathbf{PSL(2, 23)}$ by NSE. Let G be a group, such that $\mathrm{nse}\,(G) = \mathrm{nse}\,(\mathrm{PSL}(2, 23)) = \{1, 253, 506, 528, 1012, 2760\}$. First, we prove that $\pi(G) \subseteq \{2, 3, 11, 23\}$. Since $253 \in \mathrm{nse}\,(G)$, it follows that by (2.1), $2 \in \pi(G)$ and $m_2 = 253$. Let $2 \neq p \in \pi(G)$, by (2.1), $p \mid (1 + m_p)$, and $(p - 1) \mid m_p$, which implies that $p \in \{3, 11, 23, 1013\}$.

We will show $1013 \notin \pi(G)$. Let $1013 \in \pi(G)$. Then $m_{1013} = 1012$. On the other hand, by (2.1) we conclude that if $2026 \in \pi_e(G)$, then $m_{2026} = 1012$ and $2026 \mid (1 + m_2 + m_{1013} + m_{2026}) = 2278$, which is a contradiction. Hence $2026 \notin \pi_e(G)$. Thus the group P_{1013} acts fixed point freely on the set of elements of order 2. Then $|P_{1013}| \mid m_2$, which is a contradiction. Hence $1013 \notin \pi(G)$ and so $\pi(G) \subseteq \{2, 3, 11, 23\}$. If 3, 11, and $23 \in \pi(G)$, then $m_3 = 506$, $m_{11} = 2760$, and $m_{23} = 528$. Also we can see easily that G does not contain any elements of order 22, 27, 33, 64, 121, 184, 253, and 529. If $9 \in \pi_e(G)$, then $m_9 \in \{528, 2760\}$.

Let $3 \in \pi(G)$. Since $27 \notin \pi_e(G)$, $\exp\,(P_3) = 3$, or 9. If $\exp(P_3) = 3$, then $|P_3| \mid (1 + m_3) = 507$. Hence $|P_3| = 3$ and $n_3 = m_3/\phi(3) = 253 \mid |G|$. Hence in this case if $3 \in \pi(G)$, then 11 and $23 \in \pi(G)$.

Let $\exp(P_3) = 9$. Then $|P_3| \mid (1 + m_3 + m_9) = 1035$, or 3267. Hence $|P_3| = 9$, or 27.

If $|P_3| = 9$, then $n_3 = m_9/\phi(9) = 88$, or 460. If $n_3 = 460$, then since $5 \notin \pi(G)$, we get a contradiction. Let $n_3 = 88$. Since every element of order 3 lying in the Sylow 3-subgroup and we have every Sylow 3-subgroup has at most 2 elements of order 3, $m_3 \leq 2 \times 88 = 176$, which is a contradiction.

If $|P_3| = 27$, by Lemma 2.7, m_9 is a multiple of 9, which is a contradiction. Therefore, if $3 \in \pi(G)$, then 11 and $23 \in \pi(G)$.

Now let $11 \in \pi(G)$. We have $\exp(P_{11}) = 11$, hence $|P_{11}| \mid (1 + m_{11}) = 2761$ and so $|P_{11}| = 11$. Then $n_{11} = m_{11}/\phi(11) = 276$. Since $n_{11} \mid |G|$, $3 \in \pi(G)$ and so $23 \in \pi(G)$. In follow, we show that $\pi(G)$ could not be the sets $\{2\}$ and $\{2, 23\}$, and $\pi(G)$ must be equal to $\{2, 3, 11, 23\}$.

Let $\pi(G) = \{2\}$, then $\pi_e(G) \subseteq \{1, 2, 2^2, 2^3, 2^4, 2^5\}$. Therefore, $|G| = 2^m = 5060 + 506k_1 + 528k_2 + 1012k_3 + 2760k_4$, where m, k_1, k_2, k_3, k_4 are nonnegative integers and $k_1 + k_2 + k_3 + k_4 = 0$, which is impossible.

Let $\pi(G) = \{2, 23\}$. Since $529 \notin \pi_e(G)$, $\exp(P_{23}) = 23$. Then $|P_{23}| \mid (1 + m_{23}) = 529$. Hence $|P_{23}| = 23$, or 529. If $|P_{23}| = 23$, then $n_{23} = m_{23}/\phi(23) = 24 \mid |G|$; a contradiction. If $|P_{23}| = 529$, then $|G| = 2^m \times 23^2 = 5060 + 506k_1 + 528k_2 + 1012k_3 + 2760k_4$, where m, k_1, k_2, k_3, and k_4 are nonnegative integers, and $0 \leq k_1 + k_2 + k_3 + k_4 \leq 3$, by $|\pi_e(G)| \leq 9$. On the other hand, $5060 \leq |G| \leq 13340$. Hence $m = 4$ and $|G| = 2^4 \times 23^2$, and so $3404 = 506k_1 + 528k_2 + 1012k_3 + 2760k_4$.

Clearly $23 \mid k_2$. Then $k_2 = 0$. Therefore, $74 = 11k_1 + 22k_3 + 60k_4$, where $0 \le k_1 + k_3 + k_4 \le 3$. It is easy to check this equation has no solution. Hence this case is impossible.

Therefore, $\pi(G) = \{2, 3, 11, 23\}$. Since $23 \times 11 \notin \pi_e(G)$, the group P_{23} acts fixed point freely on the set of elements of order 11, and so $|P_{23}| \mid m_{11} = 2760$, which implies that $|P_{23}| = 23$. Similarly, $|P_{11}| = 11$. In addition, since $33 \notin \pi_e(G)$, $|P_3| \mid m_{11}$. Then $|P_3| = 3$. As well $22 \notin \pi_e(G)$, hence $|P_2| \mid 2^3$. On the other hand, $5060 \le |G|$, so $|G| = 2^3 \times 3 \times 11 \times 23 = |\mathrm{PSL}(2, 23)|$. Now we have $\mathrm{nse}\,(G) = \mathrm{nse}\,(\mathrm{PSL}(2, 23))$, and $|G| = |\mathrm{PSL}(2, 23)|$. By [11], since $\mathrm{PSL}(2, 23)$ is a simple K_4-group, $G \cong \mathrm{PSL}(2, 23)$, and the proof is completed.

4. Remark. In this paper, we used from Lemma 2.5 for find the order of group G while the neither in [9] nor in [13] it did not use. The authors in [9] and [13] for find the order of the group G being forced to provide proof in several cases and used from Gap program for many cases, but we did not use from Gap program. By the method [9] and [13] we can not characterized the group with order more 2000, because they used the GAP program and in the library of GAP, there are only the groups with order less than 2000. But our method, can work on the groups with order more than 2000. Therefore, this technique can work for the group $\mathrm{PSL}(2, p)$, for a given prime number p.

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