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SMOOTHING PROBLEM IN ANTICIPATING SCENARIO*
ЗАДАЧА ІНТЕРПОЛЯЦІЇ ДЛЯ НЕУЗГОДЖЕНИХ ШУМІВ

We consider a smoothing problem for stochastic processes satisfying stochastic differential equations with Wiener processes which can have not a semimartingale property with respect to the joint filtration.

Розглядається задача інтерполяції для випадкових процесів, що задовольняють стохастичні диференціальні рівняння з вінеровими процесами, які не є семімартингалами відносно спільної фільтрації.

0. Introduction. This article is devoted to the stochastic anticipating equations with the extended stochastic integral with respect to the Gaussian processes of a special type and its application to the smoothing problem in the case when noise is represented by the two jointly Gaussian Wiener processes, which can have not a semimartingale property with respect to the joint filtration. In order to describe the objects of our consideration more explicitly, consider the following example.

Example 0.1. Consider the ordinary stochastic differential equation in $\mathbb{R}$

$$dx(t) = a(x(t))dt + b(x(t))dw(t)$$

with the smooth enough coefficients a and b. Denote by $x(r, s, t)$ the solution which starts at the moment s from the point r. Let the function $\varphi \in C^2(\mathbb{R})$ have bounded derivatives. For $r \in \mathbb{R}, s \in [0; T]$, define

$$\Phi(r, s) = \Gamma(\varphi(r, s, T)), \tag{0.1}$$

where Γ is the certain operator of the second quantization [1, 2]. In particular, Γ can be a mathematical expectation. Then it can be proved that Φ satisfies the following partial stochastic differential equation:

$$d\Phi(s, r) = -\left[\frac{1}{2}b^2(r)\frac{\partial^2}{\partial r^2}\Phi(r, s) + a(r)\frac{\partial}{\partial r}\Phi(r, s)\right] ds+$$
$$+ b(r)\frac{\partial}{\partial r}\Phi(r, s)d\gamma(s).$$

Here, $\gamma(s) = \Gamma w(s)$ and the last differential is treated in the sense of anticipating stochastic integration. When Γ is the mathematical expectation, the last term vanishes.

This example shows the main goal of this article. Namely, there exist situations when the naturally arising Wiener functionals satisfy the anticipating stochastic differential equations and can be described with the using of stochastic calculus. Here we propose the appropriate machinery and derive the correspondent equations.

We will consider the second quantization transformation of the different Wiener functionals. For such transformed functionals we will get the anticipating stochastic equations with the extended stochastic integral. Accordingly to this aim the article is organized as follows. The first part contains the properties of the second quantization operators in connection with the extended stochastic integral or, more generally, with the Gaussian strong random operators [3]. Sections 2 and 3 are devoted to the following pair of equations:

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$$dx_1(t) = a_1(x_1(t)) + dw_1(t),$$

$$dx_2(t) = a_2(x_1(t)) + dw_2(t),$$

where w_1, w_2 are jointly Gaussian Wiener processes, which can have not a semimartingale property with respect to the joint filtration. Here we will look for the equation for $E(f(x_1(t))/x_2)$.

1. Second quantization and integrators. The material of this section is partially based on the works [3–5]. Corresponding facts are placed here for the completeness of the exposition but their proofs are omitted. New claims are presented with the proofs.

We will start here with the abstract picture, when the "white noise" generated by the Wiener process is substituted by the generalized Gaussian random element in the Hilbert space. Let H be a separable real Hilbert space with the norm $\|\cdot\|$ and inner product $(\cdot\,,\cdot)$. Suppose that ξ is the generalized Gaussian random element in H with zero mean and identical covariation. In other words ξ is the family of jointly Gaussian random variables denoted by $(\varphi, \xi), \varphi \in H$ with the properties:

1) (φ, ξ) has the normal distribution with zero mean and variance $\|\varphi\|^2$ for every $\varphi \in H$;

2) (φ, ξ) is linear with respect to φ.

During this section we suppose that all the random variables and elements are measurable with respect to $\sigma(\xi) = \sigma((\varphi, \xi), \varphi \in H.)$ If the random variable α has the finite second moment, then α has an Ito–Wiener expansion [6]

$$\alpha = \sum_{k=0}^{\infty} A_k(\xi, \ldots, \xi). \tag{1.1}$$

Here, for every $k \geq 1$, $A_k(\xi, \ldots, \xi)$ is the infinite-dimensional generalization of the Hermite polinomial from ξ, correspondent to the k-linear symmetric Hilbert–Schmidt form A_k on H. Moreover, now the following relation holds:

$$\mathrm{E}\alpha^2 = \sum_{k=0}^{\infty} k! \|A_k\|_k^2. \tag{1.2}$$

Here $\|\cdot\|_k$ is the Hilbert–Schmidt form in $H^{\otimes k}$. The same expansion for H-valued random elements will be necessary. Let x be a random element in H such that

$$\mathrm{E}\|x\|^2 < +\infty.$$

Then, for every $\varphi \in H$,

$$(x, \varphi) = \sum_{k=0}^{\infty} A_k(\varphi; \xi, \ldots, \xi). \tag{1.3}$$

It can be easily checked using (1.2) that now A_k is the $(k+1)$-linear (not necessary symmetric) Hilbert–Schmidt form. So one can write now

$$x = \sum_{k=0}^{\infty} \widetilde{A}_k(\xi, \ldots, \xi), \tag{1.4}$$

where

$$\widetilde{A}_k(\varphi_1, \ldots, \varphi_k) := \sum_{j=1}^{\infty} A_k(e_j; \varphi_1, \ldots, \varphi_k) e_j$$

for the arbitrary orthonormal basis $\{e_j;\ j \geq 1\}$ in H and the series (1.4) converges in H in the square mean. The relation (1.2) remains to be true

$$\mathrm{E}\|x\|^2 = \sum_{k=0}^{\infty} k!\|\widetilde{A}_k\|_k^2, \tag{1.5}$$

where $\|\widetilde{A}_k\|_k$ is the Hilbert – Schmidt norm in the space of H-valued k-linear forms on H.

Now recall the definition of the operators of the second quantization. Let C be a continuous linear operator in H. Suppose that the operator norm $\|C\| \leq 1$. Then for α and x from (1.1) and (1.4) define

$$\begin{aligned} \Gamma(C)\alpha &= \sum_{k=0}^{\infty} A_k(C\xi, \ldots, C\xi), \\ \Gamma(C)x &= \sum_{k=0}^{\infty} \widetilde{A}_k(C\xi, \ldots, C\xi), \end{aligned} \tag{1.6}$$

where for $k \geq 1$ $A_k(C\cdot, C\cdot, \ldots, C\cdot)$ and $\widetilde{A}_k(C\cdot, C\cdot, \ldots, C\cdot)$ are new Hilbert – Schmidt forms.

Using the estimation

$$\|A_k(C\cdot, C\cdot, \ldots, C\cdot)\|_k \leq \|C\|^k \|A_k\|_k,$$

it is easy to prove [2], that $\Gamma(C)$ is a continuous linear operator in the space of square integrable random variables or elements in H.

Definition 1.1 [2]. *Operator $\Gamma(C)$ is the operator of the second quantization corresponding to the operator C.*

Before to consider some examples, we will present the useful representation of the second quantization operators. Let ξ' be the generalized Gaussian random element in H, independent and equidistributed with ξ.

Consider the following generalized Gaussian random element in H :

$$\eta = \sqrt{1 - CC^*}\xi' + C\xi. \tag{1.7}$$

This element can be properly defined by the formula

$$\forall \varphi \in H: \quad (\varphi, \eta) := (\sqrt{1 - CC^*}\varphi, \xi') + (C^*\varphi, \xi). \tag{1.8}$$

Note that η has zero mean and identity covariation. In order to check this, it is sufficient to note the relation

$$\|\varphi\|^2 = \|\sqrt{1 - CC^*}\varphi\|^2 + \|C^*\varphi\|^2.$$

For every random variable α with an expansion (1.1), define

$$\alpha(\eta) := \sum_{k=0}^{\infty} A_k(\eta, \ldots, \eta).$$

The following representation will be useful.

Lemma 1.1. *For arbitrary $\alpha \in L_2(\Omega, \sigma(\xi), \mathrm{P})$ and operator C in H with $\|C\| \leq 1$,*

$$\Gamma(C)\alpha = \mathrm{E}(\alpha(\eta)/\xi). \tag{1.9}$$

Proof. Note that the both parts of (1.9) are continuous with respect to α in the square mean. So, it is enough to check (1.9) for the following random variables

$$e^{(\varphi,\xi)-\frac{1}{2}\|\varphi\|^2},\ \varphi\in H. \tag{1.10}$$

Really, the random variable of this kind has the Ito – Wiener expansion of the form

$$e^{(\varphi,\xi)-\frac{1}{2}\|\varphi\|^2}=\sum_{k=0}^{\infty}\frac{1}{k!}\varphi^{\otimes k}(\xi,\ldots,\xi).$$

Here, $\varphi^{\otimes k}$ is k-th tensor power of φ which acts on H by the rule

$$\varphi^{\otimes k}(\psi_1,\ldots,\psi_k)=\prod_{j=1}^{k}(\varphi,\psi_j).$$

So, for $\alpha,$ which has an expansion (1.1),

$$\mathrm{E}\alpha e^{(\varphi,\xi)-\frac{1}{2}\|\varphi\|^2}=\sum_{k=0}^{\infty}A_k(\varphi,\ldots,\varphi).$$

Hence, the set of all linear combinations of the variables (1.10) is dense in L_2. Now [2] the following equality holds:

$$\Gamma(C)e^{(\varphi,\xi)-\frac{1}{2}\|\varphi\|^2}=e^{(C^*\varphi,\xi)-\frac{1}{2}\|C^*\varphi\|^2}. \tag{1.11}$$

On the other hand,

$$e^{(\varphi,\eta)-\frac{1}{2}\|\varphi\|^2}=e^{(C^*\varphi,\xi)-\frac{1}{2}\|C^*\varphi\|^2}\,e^{(\sqrt{1-CC^*}\varphi,\xi')-\frac{1}{2}\|\sqrt{1-CC^*}\varphi\|^2}.$$

In order to finish the proof, it is sufficient now to note that

$$\mathrm{E}e^{(\sqrt{1-CC^*}\varphi,\xi')-\frac{1}{2}\|\sqrt{1-CC^*}\varphi\|^2}=1$$

and that ξ' and ξ are independent. It follows from here that

$$\mathrm{E}\left(e^{(\varphi,\eta)-\frac{1}{2}\|\varphi\|^2}/\xi\right)=e^{(C^*\varphi,\xi)-\frac{1}{2}\|C^*\varphi\|^2}.$$

The lemma is proved.

This lemma has the following useful application for us.

***Corollary* 1.1.** *Let $\Gamma(C)$ be an operator of the second quantization. Let x be a random element in the complete separable metric space $\mathfrak{X}$ measurable with respect to ξ. Then there exists the random probability measure μ on $\mathfrak{X}$ such that for every bounded measurable function $f:\mathfrak{X}\to\mathbb{R}$ the following equality holds:*

$$\int\limits_{\mathfrak{X}}fd\mu=\Gamma(C)f(x).$$

Proof. Let us define μ as a conditional distribution of $x(\eta)$ with respect to ξ. Then, for every bounded measurable $f:\mathfrak{X}\to\mathbb{R}$,

$$\int\limits_{\mathfrak{X}}fd\mu=\mathrm{E}(f(x(\eta))/\xi)=\Gamma(C)f(x).$$

The unique difficulty on this way lies in the proper definition of $x(\eta)$ (remind that ξ and η are not usual random elements). In order to break this difficulty, we will use the following analog of the Levy theorem. Let $\{e_j; j \geq 1\}$ be an orthonormal basis in H. Define the sequences of random elements in H by the rule

$$\xi_n = \sum_{j=1}^{n}(e_j, \xi)e_j,$$

$$\eta_n = \sum_{j=1}^{n}(e_j, \eta)e_j,\ n \geq 1.$$

Note that the sequences $\{\xi_n; n \geq 1\}$ and $\{\eta_n; n \geq 1\}$ are equidistributed. Now for every $n \geq 1$ consider the random measure ν_n in $\mathfrak{X}$, which is constructed in the following way:

$$\nu_n(\Delta) = \mathrm{E}\{1\!\!1_{\Delta}(x)/\xi_n\}.$$

Here, Δ is an arbitrary Borel subset of $\mathfrak{X}$. This random measures have two important properties. First of all, for every $n \geq 1,\ \nu_n$ can be viewed as $\widetilde{\nu}_n(\xi_n)$, where $\widetilde{\nu}_n$ is a Borel function from H to the space of all probability measures on $\mathfrak{X}$ equipped with the distance of weak convergence. Secondly, with probability one, ν_n weakly converge to δ_x as n tends to infinity. The last assertion follows from the usual Levy theorem [7]. More precisely, for arbitrary continuous bounded function $f : \mathfrak{X} \to \mathbb{R}$,

$$f(x) = \mathrm{E}(f(x)/\xi) = \lim_{n\to\infty} \mathrm{E}(f(x)/\xi_n) =$$

$$= \lim_{n\to\infty} \int_{\mathfrak{X}} f(u)\nu_n(du) \quad \text{a.s.}$$

Taking f from the countable set which define the weak convergence [8], we get the required statement. Now note that the sequence of random measures $\{\widetilde{\nu}_n(\eta_n); n \geq 1\}$ is equidistributed with $\{\widetilde{\nu}_n(\xi_n); n \geq 1\}$. Hence, with probability one, there exists the weak limit of $\widetilde{\nu}_n(\eta_n)$ which is a delta-measure concentrated in the certain random point y. This random point y is by definition $x(\eta)$. The correctness of this definition can be easily checked.

The lemma is proved.

Consider the examples of the random measures, which arise in the application of the Corollary 1.1 and will be important for us.

Example **1.1.** Suppose that $H = L_2([0; T], \mathbb{R}^d)$. Define the generalized Gaussian random element ξ in H with the help of the d-dimensional Wiener process W on $[0; T]$. Namely, for $\varphi = (\varphi_1, \ldots, \varphi_d) \in L_2([0; 1], \mathbb{R}^d)$, define

$$(\varphi, \xi) := \sum_{j=1}^{d} \int_0^T \varphi_j(s)dW_j(s). \tag{1.12}$$

Now consider the functions $a : \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ which satisfy the Lipschitz condition and the domain G in $\mathbb{R}^d$ with the C^1-boundary Γ. Let for every $s \in [0; T]$ and $u \in \mathbb{R}^d$ $x(u, s, T)$ denote the solution at time T of the following Cauchy problem:

$$dx(t) = a(x(t))dt + b(x(t))dW(t),$$
$$x(s) = u. \tag{1.13}$$

Denote by $\nu_{u,s}$ the random measure obtained from $x(u,s,T)$ via Corollary 1.1 with the help of the certain operator of the second quantization $\Gamma(C)$. In the next section, we will obtain the stochastic variant of the Kolmogorov equation for $\nu_{u,s}$. Note that in the case $C=0$ measures $\nu_{u,s}$ became to be deterministic and satisfy the usual Kolmogorov equation [1].

Now let us define for every $u \in G$ the random moment

$$\tau_{u,s} = \inf\{t, t \le T : x(u,s,t) \in \Gamma\}.$$

Let $\mu_{u,s}$ be the random measure obtained from $x(u,s,\tau_{u,s})$ via Corollary 1.1. It occurs that measures $\mu_{u,s}$ satisfy certain anticipating boundary-value problem.

In order to describe the anticipating SPDE for the random measures from above mentioned example, we need in the relation between the operators of the second quantization and extended stochastic integral. We will study this connection in the more general situation when the extended stochastic integral is substituted by the general Gaussian strong random operator (GSRO in the sequel). Let us recall the following definition.

Definition 1.2 [3]. *The Gaussian strong random linear operator (GSRO) A in H is the mapping, which maps every element x of H into the jointly Gaussian with ξ random element in H and is continuous in the square mean.*

As an example of GSRO the integral with respect to Wiener process can be considered.

***Example* 1.2.** Consider H and ξ from Example 1.1. Let for simplicity $d=1$. Define GSRO A in the following way

$$\forall \varphi \in H: \qquad (A\varphi)(t) = \int_0^t \varphi(s)dw(s), \quad t \in [0;T].$$

It can be easily seen that $A\varphi$ now is a Gaussian random element in H, and A is continuous in square mean.

In order to include in this picture the integration with respect to another Gaussian processes (for example, with respect to the fractional Brownian motion), consider more general GSRO. Suppose that K is a bounded linear operator, which acts from $L_2([0;T])$ to $L_2([0;T]^2)$. Define

$$\forall \varphi \in H: \qquad (A\varphi)(t) = \int_0^T (K\varphi)(t,s)dw(s).$$

It can be checked that A is GSRO in H. Making an obvious changes, one can define the GSRO acting from the different Hilbert space H_1 into H. For example, consider for $\alpha \in \left(\frac{1}{2};1\right)$ the covariation function of the fractional Brownian motion [9] with Hurst parameter α

$$R(s,t) = \frac{1}{2}(t^{2\alpha} + s^{2\alpha} - |t-s|^{2\alpha}).$$

Define the space H_1 as a completion of the set of step functions on $[0;T]$ with respect to the inner product under which

$$(1\!\!1_{[0;s]}, 1\!\!1_{[0;t]}) = R(s,t).$$

Consider the kernel K^α from the integral representation of the fractional Brownian motion B^α [10]

$$B^{\alpha}(t) = \int_0^t K^{\alpha}(t,s)dw(s)$$

and

$$\frac{\partial K^{\alpha}}{\partial t}(t,s) = c_{\alpha}\left(\alpha - \frac{1}{2}\right)(t-s)^{\alpha} - \frac{3}{2}\left(\frac{s}{t}\right)^{\frac{1}{2}-\alpha}.$$

Define for $\varphi \in H_1$

$$(K\varphi)(t,s) = \int_s^t \varphi(r)\frac{\partial K^{\alpha}}{\partial r}(r,s)dr\,1\!\!1_{[0;t]}(s).$$

Now let

$$(A\varphi)(t) = \int_0^T (K\varphi)(t,s)dw(s) = \int_0^t (K\varphi)(t,s)dw(s).$$

Then

$$(A\varphi)(t) = \int_0^t \varphi(s)dB^{\alpha}(s).$$

We will consider the action of GSRO on the random elements in H. The corresponding definition was proposed in [3, 11]. Consider arbitrary GSRO A in H. Then for every $\varphi \in H$ the Ito – Wiener expansion of $A\varphi$ contains only two terms:

$$A\varphi = \alpha_0\varphi + \alpha_1(\varphi)(\xi). \tag{1.14}$$

Here, α_0 is a continuous linear operator in H and α_1 is a continuous linear operator from H to the space of Hilbert – Schmidt operators in H. Now let x be a random element in H with the finite second moment. Then $\alpha_1(x)$ has a finite second moment in the space of Hilbert – Schmidt operators. So, for every $\varphi \in H$,

$$\alpha_1(x)(\varphi) = \sum_{k=0}^{\infty} B_k(\varphi;\xi,\ldots,\xi).$$

It can be easily verified that B_k is $(k+1)$-linear H-valued Hilbert – Schmidt form on H. Define ΛB_k as a symmetrization of B_k with respect to all $k+1$ variables.

Definition 1.3 [3, 11]**.** *The random element x belongs to the domain of definition of GSRO A if the series*

$$\sum_{k=0}^{\infty} \Lambda B_k(\xi,\ldots,\xi)$$

converges in H in the square mean and in this case

$$Ax = \alpha_0 x + \sum_{k=0}^{\infty} \Lambda B_k(\xi,\ldots,\xi). \tag{1.15}$$

In the partial cases, this definition gives us the definition of the extended stochastic integral [6, 12, 13]. We will define this integral for the special class of Gaussian processes. Suppose that $H = L_2([0;T])$ and ξ is generated by the Wiener process w as above. Consider the jointly Gaussian with w process $\{\gamma(t); t \in [0;T]\}$ with zero mean.

Definition 1.4 [14]. *Process γ is an integrator if there exists the constant C such that, for every step function φ on $[0;T]$ of the form*

$$\varphi = \sum_{k=0}^{n-1} a_k \mathbb{I}_{[t_k;t_{k+1})},$$

the following unequality holds:

$$\mathrm{E}\left(\sum_{k=0}^{n-1} a_k(\gamma(t_{k+1}) - \gamma(t_k))\right)^2 \leq C\sum_{k=0}^{n-1} a_k^2(t_{k+1} - t_k). \tag{1.16}$$

The good examples of the integrators can be obtained via the following simple statement.

Lemma 1.2. *Let $\Gamma(C)$ be an operator of the second quantization. Then*

$$\gamma(t) = \Gamma(C)w(t), \quad t \in [0;T],$$

is an integrator.

The proof of this lemma easily follows from the properties of $\Gamma(C)$.

Note that the integrator can have the unbounded quadratic variation and consequently can have not the semimartingale properties (see [14]). It is easy to see from (1.16) that, for every integrator γ and $\varphi \in L_2([0;T])$, the stochastic integral

$$\int_0^t \varphi d\gamma$$

exists as a limit of the integrals from the step functions and

$$\mathrm{E}\left(\int_0^t \varphi d\gamma\right)^2 \leq C\int_0^t \varphi^2(s)ds.$$

So, one can define GSRO A_γ associated with the integrator γ by the rule

$$\forall\varphi \in L_2([0;T]): \quad (A_\gamma\varphi)(t) = \int_0^t \varphi d\gamma.$$

In this situation, Definition 1.3 became to be a definition of the extended stochastic integral with respect to γ. Note that in the case $\gamma = w$ it will be a usual extended integral.

Now let us consider the relation between the action of GSRO and the operators of the second quantization.

Theorem 1.1 [4]. *Let A be a GSRO in H and $\Gamma(C)$ be an operator of the second quantization. Suppose that the random element x lies in the domain of definition of A in the sence of Definition* 1.3. *Then $\Gamma(C)x$ belongs to the domain of definition of GSRO $\Gamma(C)A$ and the following equality holds:*

$$\Gamma(C)(Ax) = \Gamma(C)A(\Gamma(C)x). \tag{1.17}$$

Here, $\Gamma(C)A$ is the GSRO which acts by the rule

$$\forall\varphi \in H: \quad \Gamma(C)A\varphi = \Gamma(C)(A\varphi).$$

The proof of this theorem is placed in [4] and so is omitted. Instead the proof consider the following important example of application of Theorem 1.1 to the stochastic integration.

***Example* 1.3.** Consider in the situation of Example 1.2 GSRO of integration with respect to Wiener process w. Suppose that random function x in $L_2([0;T])$ with the finite second moment is adapted to the flow of σ-fields generated by w. It is well known [12, 13] that, in this case, the extended stochastic integral

$$\int_0^t x(s)dw(s), \quad t \in [0;T],$$

exists and coincides with the Ito integral. Now the Theorem 1.1 says us that

$$\Gamma(C)\left(\int_0^t x(s)dw(s)\right) = \int_0^t \Gamma(C)x(s)d\gamma(s),$$

where γ is an integrator of the type $\gamma(t) = \Gamma(C)w(t)$ and the integral in the right part is an extended stochastic integral.

2. Smoothing problem. The last sections of the article are devoted to the following problem. Let (w_1, w_2) be the pair of jointly Gaussian one-dimensional Wiener processes. Let the processes x_1, x_2 are obtained via the relations

$$\begin{gathered} dx_1(t) = a_1(x_1(t))dt + dw_1(t), \\ dx_2(t) = a_2(x_1(t))dt + dw_2(t), \\ x_1(0) = x_2(0) = 0. \end{gathered} \tag{2.1}$$

Note that the second equality is just a definition of x_2 but not an equation. The problem is to find the conditional distribution of $x_1(t)$ for $t \in [0;1]$ under given $\{x_2(s); s \in [0;1]\}$. We will try to get the equation for

$$\mathrm{E}(f(x_1(t))/x_2)$$

for the appropriate functions f.

First, let us study the joint distribution of (w_1, w_2). Note that there exists the bounded linear operator $V : L_2([0;1]) \to L_2([0;1])$ such that

$$\forall \varphi_1, \varphi_2 \in L_2([0;1]): \quad \mathrm{E}\int_0^1 \varphi_1 dw_1 \int_0^1 \varphi_2 dw_2 = \int_0^1 \varphi_1 V\varphi_2 ds.$$

This fact follows from the reason that the left part of the above formula is the continuous bilinear form with respect to φ_1 and φ_2. Moreover, the operator norm $\|V\| \le 1$.

In this section, we consider the density of the distribution (x_1, x_2) with respect to the distribution of (w_1, w_2) and study its properties under the conditional expectation. The problem is that the distribution of (w_1, w_2) is not a Wiener measure in $C([0;1], \mathbb{R}^2)$. So, in order to get the density, we need to adapt the general Gaussian measure setup [15] to the our case. For the future let us denote $C([0;1])$ as C and identify the space $C([0;1], \mathbb{R}^2)$ with the direct sum $C \oplus C$, which is furnished by the sum of the norms. Denote also by H the space

$$L_2([0;1],\mathbb{R}^2) = L_2([0;1]) \oplus L_2([0;1])$$

with the scalar product defined by the formula

$$(\varphi,\psi) = \int_0^1 \varphi_1\psi_1 ds + \int_0^1 \varphi_2\psi_2 ds.$$

With the pair (w_1, w_2) we can associate the generalized Gaussian random element ξ in H by the rule

$$(\varphi,\xi) = \int_0^1 \varphi_1 dw_1 + \int_0^1 \varphi_2 dw_2.$$

Note that ξ has not an identity covariation operator. Really,

$$\mathrm{E}(\varphi,\xi)(\psi,\xi) = \int_0^1 \varphi_1\psi_1 ds + \int_0^1 \varphi_2\psi_2 ds +$$

$$+ \int_0^1 \varphi_1 V\psi_2 ds + \int_0^1 \psi_1 V\varphi_2 ds.$$

Here, V is described above bounded linear operator in $L_2([0;1])$. Denote by S the operator in H which acts by the rule

$$S\varphi = (\varphi_1 + V\varphi_2, V^*\varphi_1 + \varphi_2).$$

Then

$$\mathrm{E}(\varphi,\xi)(\psi,\xi) = (S\varphi,\psi).$$

Our aim is to describe the transformations of the pair (w_1, w_2) in the terms of ξ. Let us start with the deterministic admissible shifts. Denote by i the canonical embedding of H into C^2, i.e.,

$$i(\varphi)(t) = \left(\int_0^t \varphi_1 ds, \ \int_0^t \varphi_2 ds\right).$$

Lemma 2.1. *Let the operator norm* $\|V\| < 1$, *then for every* $h \in H$, $i(h)$ *is admissible shift for* μ_{w_1,w_2} *and the corresponding density has the form*

$$p_h(\xi) = \exp\left\{(S^{-1}h,\xi) - \frac{1}{2}(S^{-1}h,h)\right\}. \qquad (2.2)$$

Remark **2.1.** Due to the condition $\|V\| < 1$, the operator S^{-1} is bounded on H and can be written in the form

$$S^{-1}\varphi = \varphi + (Q_{11}\varphi_1 + Q_{12}\varphi_2, Q_{21}\varphi_1 + Q_{22}\varphi_2) = \varphi + Q\varphi,$$

where $\|Q\| < 1$.

Proof. Note that, by the definition, the operator S is nonnegative. Define

$$\xi' = S^{-\frac{1}{2}}\xi.$$

Then the shift of the distribution of (w_1, w_2) on the vector $i(h)$ is related to the shift of ξ' on the vector $S^{-\frac{1}{2}}h$. Now the statement of the lemma follows from the well-known formula for the density in terms of ξ':

$$p(\xi') = \exp\left\{(\xi', S^{-\frac{1}{2}}h) - \frac{1}{2}(S^{-\frac{1}{2}}h, S^{-\frac{1}{2}}h)\right\}$$

if we rewrite it in terms of ξ.

The lemma is proved.

Remark **2.2.** Formula (2.2) can be rewritten in terms of w_1, w_2. Really, by the definition,

$$\left(S^{-1}h, \xi\right) - \frac{1}{2}\left(S^{-1}h, h\right) =$$

$$= \int_0^1 (S^{-1}h)_1 dw_1 + \int_0^1 (S^{-1}h)_2 dw_2 -$$

$$-\frac{1}{2}\int_0^1 (S^{-1}h)_1 h_1 ds - \frac{1}{2}\int_0^1 (S^{-1}h)_2 h_2 ds =$$

$$= \int_0^1 (h_1 + Q_{11}h_1 + Q_{12}h_2) dw_1 + \int_0^1 (h_2 + Q_{21}h_1 + Q_{22}h_2) dw_2 -$$

$$-\frac{1}{2}\int_0^1 (h_1 + Q_{11}h_1 + Q_{12}h_2) h_1 ds - \frac{1}{2}\int_0^1 (h_2 + Q_{21}h_1 + Q_{22}h_2) h_2 ds. \qquad (2.3)$$

Using the same method, one can find the density of μ_{x_1,x_2} with respect to μ_{w_1,w_2}. First, define the stochastic derivatives of the functionals from w_1, w_2 with respect to ξ and the extended stochastic integral in terms of ξ. Let φ be a differentiable bounded function on $C \oplus C$. Define the stochastic derivative of the random variable $\varphi(w_1, w_2)$ by the formula

$$D\varphi(w_1, w_2) := i^*\nabla\varphi(w_1, w_2).$$

By this definition, for every $t \in [0; 1]$,

$$Dw_1(t) = (1\!\!\mathrm{I}_{[0;t]}, 0),$$

$$Dw_2(t) = (0, 1\!\!\mathrm{I}_{[0;t]}).$$

Note that $\varphi(w_1, w_2)$ can be regarded as a functional from the generalized random element ξ' which was introduced in the proof of Lemma 2.1. Since ξ' has an identity covariation operator, the stochastic derivatives and extended stochastic integral for the functionals from ξ' are connected by the usual relation. Now we will define the extended stochastic integral with respect to ξ. It can be done in the following way. Consider the Gaussian random functional on H of the kind

$$J(\varphi) = (\varphi, \xi).$$

Then, in terms of ξ', J can be rewritten as

$$J(\varphi) = (S^{\frac{1}{2}}\varphi, \xi').$$

So, the action of J on the random element x in H via Definition 1.3 has the form

$$J(x) = I(S^{\frac{1}{2}}x). \tag{2.4}$$

Here, I is the extended stochastic integral with respect to ξ'. Note also that for the stochastic derivatives with respect to ξ' and ξ, we have the obvious relation

$$D_{\xi'}\alpha = S^{-\frac{1}{2}}D_{\xi}\alpha.$$

Hence, on the domain of definition,

$$\mathrm{E}(D_{\xi}\alpha, x) = \mathrm{E}(S^{-\frac{1}{2}}D_{\xi}\alpha, S^{\frac{1}{2}}x) =$$

$$= \mathrm{E}(D_{\xi'}\alpha, S^{\frac{1}{2}}x) = \mathrm{E}\alpha I(S^{\frac{1}{2}}x) = \mathrm{E}\alpha J(x). \tag{2.5}$$

Thus, the relation between the stochastic derivative and extended stochastic integral with respect to ξ is the same as for ξ'. Now let us turn to the nonadapted shifts of the distribution of (w_1, w_2). Consider the pair of random processes x_1, x_2 which are defined by Equations (2.1).

The next lemma is standard.

Lemma 2.2. *Let the functions* a_1, a_2 *be continuously differentiable and have bounded derivatives. Then*:

1) *for every* $t \in [0; 1]$, *the random variables* $x_1(t), x_2(t)$ *have the stochastic derivatives* $Dx_1(t), Dx_2(t)$;

2) *the random element* $(a_1(x_1(\cdot)),\, a_2(x_1(\cdot)))$ *in* H *has the stochastic derivative and*

$$D(a_1(x_1(s)), a_2(x_1(s)))(t) =$$

$$= (a_1'(x_1(s))Dx_1(s)(t), a_2'(x_1(s))Dx_1(s)(t));$$

3) *the stochastic derivative of* x_1 *with respect to* w_1 (*i.e., the first coordinate of* Dx_1) *satisfies the equation*

$$D_1x_1(s)(t) = 1 + \int_t^s a_1'(x_1(r))D_1x_1(r)(t)dr, \quad 0 \le t \le s \le 1,$$

$$D_1x_1(s)(t) = 0, \quad t > s,$$

and

$$Dx_1(s)(t) = (D_1x_1(s)(t), 0).$$

It follows from Lemma 2.2 that $\|D(a_1(x_1(\cdot)), a_2(x_1(\cdot)))\|_H$ can be made small if we take a_1' and a_2' small enough. Since the operator $S^{-\frac{1}{2}}$ is bounded in $H,$ we have due to Theorem 3.2.2 from [15] that the distribution of (x_1, x_2) is absolutely continuous with respect to the distribution (w_1, w_2) for sufficiently small $a_1',$ a_2'. The corresponding density will be denoted by $p.$ Accordingly to [15], p has the form

$$p = \zeta \exp\left\{I(S^{-\frac{1}{2}}h) - \frac{1}{2}(S^{-1}h, h)\right\}, \tag{2.6}$$

where

$$h(t) = (a_1(w_1(t)), \;\; a_2(w_1(t))),$$

and ζ is the corresponding Carleman – Fredholm determinant. Due to (2.4), (2.6) can be rewritten as

$$p = \zeta \exp\left\{J(S^{-1}h) - \frac{1}{2}(S^{-1}h, h)\right\}. \tag{2.7}$$

This expression allows us to conclude that up to the term $\zeta,$ p has the stochastic derivative. We will suppose that this is so in the next section, where the formulas for the conditional expectation and extended stochastic integral will be obtained in non-Gaussian case.

3. Conditional expectation. For the processes (x_1, x_2) from (2.1), let us search for the conditional distribution of $x_1(t)$ under fixed $\{x_2(s); s \in [0;1]\}.$ First note that under our conditions, the distribution of (x_1, x_2) is absolutely continuous with respect to the distribution of (w_1, w_2) and, consequently, the distribution of x_2 is absolutely continuous with respect to the distribution of $w_2.$

Denote for a moment by μ the distribution of the pair (w_1, w_2) on $C([0;1]) \oplus \oplus C([0;1])$ and by μ_1 and μ_2 the distributions of w_1 and w_2 (surely, these are the Wiener measures but on the different copies of $C([0;1])$. It follows from the general theory of integration that the measure μ can be desintegrated with respect to $\mu_2,$ i.e.,

$$\mu(\Delta) = \int\limits_{C([0;1])} \nu(u, \Delta_u)\mu_2(du),$$

for arbitrary Borel Δ in $C([0;1]) \oplus C([0;1]).$ Here, ν is a measurable family of the probability measures and $\Delta_u = \{v \in C([0;1]) : (v, u) \in \Delta\}.$

Define for the measurable bounded function $\varphi : C([0;1]) \to \mathbb{R}$ the function ψ by the rule

$$C([0;1]) \ni u \mapsto \psi(u) = \int\limits_{C([0;1])} \varphi(v)p(v, u)\nu(u, dv)\times$$

$$\times\left(\int\limits_{C([0;1])} p(v, u)\nu(u, dv)\right)^{-1}. \tag{3.1}$$

The following variant of the Bayes formula holds.

Lemma 3.1.

$$\mathrm{E}(\varphi(x_1)/x_2) = \psi(x_2).$$

Proof. First note that $\psi(x_2)$ is correctly defined because the function ψ is defined up to the set of Wiener measure zero, and the distribution of x_2 is equivalent to this measure. Now, for arbitrary bounded and measurable function $\gamma : C([0;1]) \to \mathbb{R}$,

$$\mathrm{E}\varphi(x_1)\gamma(x_2) = \mathrm{E}\varphi(w_1)\gamma(w_2)p(w_1, w_2) =$$

$$= \mathrm{E}\gamma(w_2)\mathrm{E}(\varphi(w_1)p(w_1, w_2)/w_2) =$$

$$= \mathrm{E}\gamma(w_2)\frac{\mathrm{E}(\varphi(w_1)p(w_1, w_2)/w_2)}{\mathrm{E}(p(w_1, w_2)/w_2)}E(p(w_1, w_2)/w_2) =$$

$$= \mathrm{E}\gamma(w_2)\psi(w_2)p(w_1, w_2) = \mathrm{E}\gamma(x_2)\psi(x_2).$$

This finishes the proof.

For arbitrary $t \in [0;1]$, denote by π_t the random measure on $\mathbb{R}$ whose pairing with the bounded measurable function f is defined by the formula

$$\int_{\mathbb{R}} f(r)\pi_t(dr) = \mathrm{E}(f(w_1(t))p(w_1, w_2)/w_2).$$

In view of the previous lemma, it is sufficient to get the equation for π_t. The next lemma contains the necessary facts from the theory of extended stochastic integral.

Lemma 3.2. *Let H be the separable Hilbert space, ξ be a generalized Gaussian random element in H with zero mean and identity covariation. Suppose that the random element x in H has two stochastic derivatives and let I and D be the symbols of the extended stochastic integral and stochastic derivative correspondingly. Then for arbitrary $h \in H$ and stochastically differentiable bounded random variable α, the following formulas hold:*

1) $\alpha I(x) = I(\alpha x) + (x, D\alpha)$;

2) $(DI(x), h) = (x, h) + I((Dx, h))$.

Proof. The first statement is the well-known relation [12]. Let us check 2). Use the integration by part formula. Consider the random variable β which is twice stochastically differentiable. Then, using 1),

$$\mathrm{E}(DI(x), h)\beta = \mathrm{E}(DI(x), \beta h) =$$

$$= \mathrm{E}I(x)I(\beta h) = \mathrm{E}I(x)(\beta I(h) - (D\beta, h)) =$$

$$= \mathrm{E}(x, D(\beta I(h) - (D\beta, h))) =$$

$$= \mathrm{E}[(x, \beta h) + (x, D\beta)I(h) - (x, (D^2\beta, h))] =$$

$$= \mathrm{E}(x, h)\beta + \mathrm{E}(x, I(D\beta h)) =$$

$$= \mathrm{E}\beta((x, h) + I((Dx, h))).$$

The lemma is proved.

***Remark* 3.1.** Note that the statement of the lemma remains to be true in the case when α is not bounded but all terms are well-defined. Also, due to the formula (2.5), the lemma holds in the case when the initial generalized Gaussian random element has not identity covariation.

Now let us turn to our filtration problem.

Take the function $f \in C_0^2(\mathbb{R})$. Then for arbitrary $r \in \mathbb{R}$ from the Ito formula

$$f(r+w_1(t))p(w_1,w_2) =$$

$$= f(r)p(w_1,w_2) + \int_0^t f'(r+w_1(s))dw_1(s)p(w_1,w_2)+$$

$$+\frac{1}{2}\int_0^t f''(r+w_1(s))p(w_1,w_2)ds.$$

Consider the second summand. It contains the Ito integral which coincides with the extended stochastic integral as it was mentioned before. So, we can apply the formula 1) from Lemma 3.2:

$$p(w_1,w_2)\int_0^t f'(r+w_1(s))dw_1(s) =$$

$$= \int_0^t f'(r+w_1(s))p(w_1,w_2)dw_1(s)+$$

$$+\int_0^t f'(r+w_1(s))(SDp(w_1,w_2))_1(s)ds.$$

Here, the index 1 symbolizes the first coordinate of correspondent element from H. Now note that the conditional expectation with respect to w_2 is an operator of the second quantization. So, if we denote

$$\gamma_1(t) = \mathrm{E}(w_1(t)/w_2),$$

then, due to Theorem 1.1,

$$\mathrm{E}(f(r+w_1(t))p(w_1,w_2)/w_2) =$$

$$= \mathrm{E}(f(r)p(w_1,w_2)/w_2) + \int_0^t \mathrm{E}(f'(r+w_1(s))p(w_1,w_2)/w_2)d\gamma(t)+$$

$$+\frac{1}{2}\int_0^t \mathrm{E}(f''(r+w_1(s))p(w_1,w_2)/w_2)ds+$$

$$+\int_0^t \mathrm{E}(f'(r+w_1(s)(SDp(w_1,w_2))_1)(s)/w_2)ds, \tag{3.2}$$

where the integral with respect to γ is an extended stochastic integral. In order to get the stochastic differentiability of $p,$ let us consider the case when the Carleman – Fredholm determinant ζ is equal to one. Denote by P_t the orthogonal projector in $L_2([0;1]) \oplus \oplus L_2([0;1])$ on the subspace $L_2([0;t]) \oplus L_2([0;t])$.

Lemma 3.3. *Suppose that the operator* S *has the property*

$$\forall t \in [0;1]: \quad P_t S = P_t S P_t.$$

Then $\zeta = 1$.

Proof. The value ζ is the Carleman – Fredholm determinant of the operator SDh, where

$$h = (a_1(w_1(\cdot)), a_2(w_1(\cdot))).$$

Now

$$Dh(t,s) = \begin{pmatrix} a_1'(w_1(t))\mathbb{1}_{[0;t]}(s) & 0 \\ a_2'(w_1(t))\mathbb{1}_{[0;t]}(s) & 0 \end{pmatrix}. \tag{3.3}$$

In order to prove that

$$\det_2(Id + SDh) = 1,$$

we will use Theorem 3.6.1 from [15]. Due to this theorem, it is sufficient to check that the operator SDh is quasinilpotent, i.e., that

$$\lim_{n\to\infty} \|(SDh)^n\|^{\frac{1}{n}} = 0. \tag{3.4}$$

It follows from representation (3.3) that

$$\forall \varphi \in H \quad \forall t \in [0;1]: \quad \|P_t Dh\varphi\|^2 \le c \int_0^t \|P_s\varphi\|^2 ds,$$

where c depends on $\sup_{\mathbb{R}}(|a_1'| + |a_2'|)$.

Consequently,

$$\|(SDh)^n(\varphi)\|^2 \le \|S\|^2 c \int_0^1 \|P_{t_1}(SDh)^{n-1}(\varphi)\|^2 dt_1 \le$$

$$\le \|S\|^2 c \int_0^1 \|P_{t_1} S P_{t_1} P_{t_1} Dh (SDh)^{n-2}(\varphi)\|^2 dt_1 \le$$

$$\le \|S\|^4 c^2 \int_0^1 \int_0^{t_1} \|P_{t_2}(SDh)^{n-2}(\varphi)\|^2 dt_2 dt_1 \le \ldots$$

$$\ldots \le \|S\|^{2n} c^n \int_0^1 \int_0^{t_1} \ldots \int_0^{t_{n-1}} \|P_{t_n}\varphi\|^2 dt_n \ldots dt_1 \le$$

$$\le \|\varphi\|^2 \frac{\|S\|^{2n} c^n}{n!}.$$

This means that (3.4) holds and $\zeta = 1$.

The lemma is proved.

Now one can conclude that p has the stochastic derivative and (3.1) is correct. The further concretization of (3.2) can be possible due to the special form of p. As a consequence of (3.2) and (3.4), we have the following theorem.

Theorem 3.1. *Suppose that the coefficients* a_1, a_2 *and the operator* V *satisfy the conditions of Lemma* 3.3. *Then the random function*

$$U(r,t) = \mathrm{E}(f(r + w_1(t))p(w_1, w_2)/w_2)$$

satisfies relation

$$dU(r,t) = \frac{1}{2}\frac{\partial^2}{\partial r^2}U(r,t)dt+$$

$$+\frac{\partial}{\partial r}U(r,t)\gamma(dt) + \mathrm{E}f'(r + w_1(t))(SDp(w_1, w_2))_1(t)dt. \tag{3.5}$$

In some particular case, the last term can be written in a simple form. For example, when $a_2 = 0$, then (3.5) transforms into

$$dU(r,t) = \frac{1}{2}\frac{\partial^2}{\partial r^2}U(r,t)dt + \frac{\partial}{\partial r}U(r,t)\gamma(dt)+$$

$$+ a_1(r)\frac{\partial}{\partial r}U(r,t)dt.$$

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