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ON M-PROJECTIVELY FLAT LP-SASAKIAN MANIFOLDS

ПРО M-ПРОЕКТИВНО ПЛОСКІ LP-МНОГОВИДИ САСАКЯНА

The object of the present paper is to study the nature of LP-Sasakian manifolds admitting the M-projective curvature tensor. It is examined whether this manifold satisfies the condition $W(X,Y).R = 0$. Moreover, it is proved that, in the M-projectively flat LP-Sasakian manifolds, the conditions $R(X,Y).R = 0$ and $R(X,Y).S = 0$ are satisfied. In the last part of our paper, M-projectively flat space-time is introduced and some properties of this space are obtained.

Вивчається природа многовидів Сасакяна, що допускають M-проективний тензор кривизни. Перевірено, чи задовольняє цей многовид умову $W(X,Y).R = 0$. Більш того, доведено, що умови $R(X,Y).R = 0$ та $R(X,Y).S = 0$ виконуються для M-проективно плоских LP-многовидів Сасакяна. В останній частині роботи введено M-проективно плоский простір-час та встановлено деякі властивості цього простору.

1. Introduction. A Riemannian manifold (M,g) is called a Sasakian manifold if there exists a Killing vector field ξ of unit length on M so that tensor field Φ of type (1,1), defined by $\Phi(X) = -\nabla_X \xi$, satisfies the condition $(\nabla_X \Phi)(Y) = g(X,Y)\xi - g(\xi,Y)X$ for any pair of vector fields X and Y on M. This is a curvature condition which can be easily expressed in terms the Riemann curvature tensor as $R(X,\xi)Y = g(\xi,Y)X - g(X,Y)\xi$. Equivalently, the Riemannian cone defined by $(C(M),\bar{g},\Omega) = (R_+ XM, dr^2 + r^2 g, d(r^2\eta))$ is Kähler with the Kähler form $\Omega = d(r^2\eta)$, where η is the dual 1-form of ξ. The 4-tuple $s = (\xi,\eta,\Phi,g)$ is commonly called a Sasakian structure on M and ξ is its characteristic or Reeb vector field.

Sasakian geometry is a special kind of contact metric geometry such that the structure transverse to the Reeb vector field ξ is Kähler and invariant under the flow of ξ. On the analogy of Sasakian manifolds, in 1989 Matsumoto [1, 2], introduced the notion of LP-Sasakian manifolds. Again the same notion is introduced by Mihai and Rosca [3] and obtained many interesting results. LP-Sasakian manifolds are also studied by De et al. [4], Shaikh et al. [5 – 8], Taleshian and Asghari [9], Venkatesha and Bagewadi [10] and many others.

The M-projective curvature tensor of a Riemannian manifold M defined by Pokhariyal and Mishra [11] is in the following form:

$$W(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}\big(S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\big), \tag{1.1}$$

where $R(X,Y)Z$ and $S(X,Y)$ are the curvature tensor and the Ricci tensor of M, respectively and Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$. Some properties of this tensor in Sasakian and Kähler manifolds have been studied before [12, 13]. In 2010, Chaubey and Ojha [14] investigated the M-projective curvature tensor of a Kenmotsu manifold.

The object of the present paper is to study LP-Sasakian manifolds admitting M-projective curvature tensor. The paper is organized as follows. Section 2 is concerned with some preliminaries about LP-Sasakian manifolds. Section 3 deals with LP-Sasakian manifolds with M-projective curvature tensor. Section 4 is devoted to M-projectively flat LP-Sasakian manifolds. In Section 5, M-projectively flat LP-Sasakian spacetimes are introduced.

2. Preliminaries. An n-dimensional differentiable manifold M is called an LP-Sasakian manifold [1, 2] if it admits a $(1,1)$ tensor field φ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy:

$$\varphi^2 = I + \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = -1, \tag{2.2}$$

$$g(\varphi X, \varphi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$\nabla_X \xi = \varphi X, \qquad g(X,\xi) = \eta(X), \tag{2.4}$$

$$(\nabla_X \varphi)Y = g(X,Y)\xi + 2\eta(X)\eta(Y)\xi, \tag{2.5}$$

where ∇ denotes the operator of the covariant differentiation with respect to the Lorentzian metric g.

It can be easily seen that in an LP-Sasakian manifold, the following relations hold:

$$\varphi\xi = 0, \qquad \eta(\varphi X) = 0,$$

$$\operatorname{rank}\varphi = n-1.$$

Again if we put

$$\Omega(X,Y) = g(X,\varphi Y)$$

for any vector fields X and Y, then $\Omega(X,Y)$ is symmetric $(0,2)$ tensor field [1]. Also since the 1-form η is closed in an LP-Sasakian manifold, we have [1, 4]

$$(\nabla_X \eta)(Y) = \Omega(X,Y), \qquad \Omega(X,\xi) = 0$$

for any vector fields X and Y.

Also, in an LP-Sasakian manifold, the following conditions hold [2, 4]:

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.6}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.7}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.8}$$

$$R(\xi,X)\xi = X + \eta(X)\xi, \tag{2.9}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{2.10}$$

$$S(\varphi X, \varphi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y) \tag{2.11}$$

for any vector fields X, Y, Z where $R(X,Y)Z$ is the curvature tensor and $S(X,Y)$ is the Ricci tensor.

3. LP-Sasakian manifold satisfying $W(X,Y).S=0$. Let us consider an LP-Sasakian manifold (M,g) satisfying the condition

$$W(X,Y).S=0. \tag{3.1}$$

Now, we have

$$S(W(\xi,X)Y,Z)+S(Y,W(\xi,X)Z)=0. \tag{3.2}$$

From (1.1), (2.7) and (2.10), we get

$$W(\xi,X)Y=\frac{1}{2}g(X,Y)\xi-\frac{1}{2}\eta(Y)X-\frac{1}{2(n-1)}S(X,Y)\xi+\frac{1}{2(n-1)}\eta(Y)QX. \tag{3.3}$$

By using (2.10) and (3.3), (3.2) takes the form

$$\frac{1}{2}(n-1)g(X,Y)\eta(Z)+\frac{1}{2}(n-1)g(X,Z)\eta(Y)-S(X,Z)\eta(Y)-$$

$$-S(X,Y)\eta(Z)+\frac{1}{2(n-1)}S(QX,Z)\eta(Y)+\frac{1}{2(n-1)}S(QX,Y)\eta(Z)=0. \tag{3.4}$$

Let λ be the eigenvalue of the endomorphism Q corresponding to an eigenvector X. Then

$$QX=\lambda X. \tag{3.5}$$

By using (3.5) in (3.4), we obtain

$$\frac{1}{2}(n-1)g(X,Y)\eta(Z)+\frac{1}{2}(n-1)g(X,Z)\eta(Y)-S(X,Z)\eta(Y)-$$

$$-S(X,Y)\eta(Z)+\frac{\lambda}{2(n-1)}S(X,Z)\eta(Y)+\frac{\lambda}{2(n-1)}S(X,Y)\eta(Z)=0. \tag{3.6}$$

Remembering that $g(QX,Y)=S(X,Y)$ and using (3.6), we have

$$g(QX,Y)=g(\lambda X,Y)=\lambda g(X,Y)=S(X,Y). \tag{3.7}$$

Thus, from (3.6) and (3.7), taking $Z=\xi$ in (3.6) and using (2.2), it can be easily seen that

$$\left(\frac{\lambda^2}{2(n-1)}-\lambda+\frac{n-1}{2}\right)(g(X,Y)-\eta(X)\eta(Y))=0. \tag{3.8}$$

Finally, taking $Y=\xi$ in (3.8) and using the properties (2.2) and $(2.4)_2$, we obtain

$$\left(\frac{\lambda^2}{2(n-1)}-\lambda+\frac{n-1}{2}\right)\eta(X)=0. \tag{3.9}$$

In this case, as $\eta(X)\neq 0$, we have from (3.9)

$$\lambda^2-2(n-1)\lambda+(n-1)^2=0. \tag{3.10}$$

From (3.10), it follows that the non-zero eigenvalues of the endomorphism Q are congruent such as $(n-1)$. Thus we can state the following theorem.

Theorem 3.1. *If an n-dimensional $(n \geq 3)$ LP-Sasakian manifold admitting M-projective curvature tensor and with non-zero Ricci tensor S satisfies*

$$W(X,Y).S = 0,$$

then the non-zero eigenvalues of the symmetric endomorphism Q of the tangent space corresponding to S are congruent such as $(n-1)$.

4. M-projectively flat LP-sasakian manifolds. Let us consider that M be an M-projectively flat LP-Sasakian manifold. Thus, we have $W(X,Y)Z = 0$ for all vector fields X, Y, Z. Then, we get from (1.1)

$$R(X,Y)Z = \frac{1}{2(n-1)}\big(S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\big). \tag{4.1}$$

Taking $Z = \xi$ in (4.1) and using the relations (2.4), (2.8) and (2.10), we find

$$\eta(Y)X - \eta(X)Y = \frac{1}{n-1}\big[\eta(Y)QX - \eta(X)QY\big]. \tag{4.2}$$

Again taking $Y = \xi$ in (4.2) and applying (2.2), (4.2) reduces to

$$QX = (n-1)X. \tag{4.3}$$

Hence in view of (2.7), (4.1) and (4.3), we get

$$S(X,Y)\xi = (n-1)g(X,Y)\xi. \tag{4.4}$$

Taking the inner product of both sides (4.4) with ξ and using (2.2), we have

$$S(X,Y) = (n-1)g(X,Y). \tag{4.5}$$

Next, we have the following theorem.

Theorem 4.1. *Let M be an n-dimensional M-projectively flat LP-Sasakian manifold. Then M is an Einstein manifold and the Ricci tensor of M is in the form $S(X,Y) = (n-1)g(X,Y)$.*

In this case, by the use of (4.3) and (4.5) in (4.1), we obtain

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{4.6}$$

According to Karcher [15], a Lorentzian manifold is called infinitesimally spatially isotropic relative to a unit timelike vector field U if its Riemann curvature tensor R satisfies the relation

$$R(X,Y)Z = \delta\big[g(Y,Z)X - g(X,Z)Y\big]$$

for all $X, Y, Z \in U^{\perp}$ and $R(X,U)U = \gamma X$ for $X \in U^{\perp}$ where δ, γ are real valued functions on the manifold. Hence, we can obtain the following theorem.

Theorem 4.2. *An n-dimensional M-projectively flat LP-Sasakian manifold is infinitesimally spatially isotropic relative to the unit timelike vector field ξ.*

Theorem 4.3. *Let M be an n-dimensional M-projectively flat LP-Sasakian manifold. Then M is semisymmetric, i.e., the condition $R(X,Y).R = 0$ holds.*

Proof. Let M be an n-dimensional M-projectively flat LP-Sasakian manifold. Thus, we can write

$$R(X,Y).R = R(X,Y)R(Z,U)V - R(R(X,Y)Z,U)V-$$
$$-R(Z,R(X,Y)U)V - R(Z,U)R(X,Y)V \tag{4.7}$$

for all vector fields X, Y, Z, U, V on M. So from (4.6), we get

$$R(R(X,Y)Z,U)V = g(U,V)g(Y,Z)X - g(Y,Z)g(X,V)U-$$
$$-g(X,Z)g(U,V)Y + g(X,Z)g(Y,V)U. \tag{4.8}$$

Again, we obtain

$$R(Z,R(X,Y)U)V = g(U,Y)g(X,V)Z - g(U,Y)g(Z,V)X-$$
$$-g(U,X)g(Y,V)Z + g(X,U)g(Z,V)Y \tag{4.9}$$

and finally

$$R(Z,U)R(X,Y)V = g(U,X)g(Y,V)Z - g(X,Z)g(Y,V)U-$$
$$-g(X,V)g(U,Y)Z + g(X,V)g(Z,Y)U. \tag{4.10}$$

So from (4.7) – (4.10), one can easily get

$$R(X,Y).R = 0.$$

Theorem4.3 is proof is proved.

***Corollary* 4.1.** *Let M be an n-dimensional M-projectively flat LP-Sasakian manifold. Then M is Ricci semisymmetric, i.e., the condition $R(X,Y).S = 0$ holds.*

Proof. Let M be an n-dimensional M-projectively flat LP-Sasakian manifold. Since a semisymmetric manifold is also Ricci semisymmetric, [16], from Theorem 4.2, the proof is clear.

5. M-projectively flat LP-Sasakian spacetimes. In this section, we consider that M is an M-projectively flat LP-Sasakian spacetime (M^4, g) satisfying the Einstein's equations with a cosmological constant. Further let ξ be the unit time-like velocity vector of the fluid. It is known that the Einstein's equations with a cosmological constant can be written as [17]

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = kT(X,Y) \tag{5.1}$$

for all vector fields X and Y. Here, $S(X,Y)$ and $T(X,Y)$ denote the Ricci tensor and the energy-momentum tensor, respectively. In addition, λ is the cosmological constant and k is the non-zero gravitational constant.

Hence by use of (4.5), (5.1) forms into

$$T(X,Y) = \left(\frac{\lambda - 3}{k}\right) g(X,Y). \tag{5.2}$$

Thus, we have the following theorem.

Theorem 5.1. *Let M^4 be an M-projectively flat LP-Sasakian spacetime satisfying the Einstein's equations with a cosmological constant. Then the energy momentum tensor of this space is found as in* (5.2).

In a perfect fluid spacetime, the energy momentum tensor is in the form

$$T(X,Y) = (\sigma + p)u(X)u(Y) + pg(X,Y), \tag{5.3}$$

where σ is the energy density, p is the isotropic pressure and $u(X)$ is a non-zero 1-form such that $g(X,V) = u(X)$ for all X, V being the velocity vector field of the flow, that is, $g(V,V) = -1$. Also, $\sigma + p \neq 0$.

With the help of (5.2) and (5.3), we obtain

$$(\lambda - 3 - kp)g(X,Y) = k(\sigma + p)u(X)u(Y). \tag{5.4}$$

Contraction of (5.4) over X and Y leads to

$$\lambda = 3 - \frac{k}{4}(\sigma - 3p). \tag{5.5}$$

If we put $X = Y = V$ in (5.4) then we find

$$\lambda = 3 - k\sigma. \tag{5.6}$$

Combining the equations (5.5) and (5.6), we get

$$\sigma + p = 0. \tag{5.7}$$

Hence we have the following theorem.

Theorem 5.2. *In an M-projectively flat LP-Sasakian spacetime M^4 satisfying the Einstein's field equations with a cosmological term then the matter contents of M^4 satisfy the vacuum-like equation of state.*

If we assume a dust in a perfect fluid, we have

$$\sigma = 3p. \tag{5.8}$$

By putting (5.8) in (5.7), we get

$$p = 0.$$

Thus, we can state the following theorem.

Theorem 5.3. *The M-projectively flat LP-Sasakian spacetime admitting a dust for a perfect fluid is filled with radiation.*

In a relativistic spacetime, the energy-momentum tensor is in the form

$$T(X,Y) = \mu u(X)u(Y). \tag{5.9}$$

From (5.2), (5.9) takes the form

$$(\lambda - 3)g(X,Y) = k\mu u(X)u(Y). \tag{5.10}$$

Contraction of (5.10) over X and Y leads to

$$\lambda = 3 - \frac{1}{4}k\mu. \tag{5.11}$$

And, if we put $X = Y = V$ in (5.10), we get

$$\lambda = 3 - k\mu. \tag{5.12}$$

Thus, combining the equations (5.11) and (5.12), we finally get that $\mu = 0$. From this relation and (5.9), we find $T(X, Y) = 0$. This means that the spacetime is devoid of the matter. In this case, we can give the following theorem.

Theorem 5.4. *A relativistic M-projectively flat LP-Sasakian manifold satisfying the Einstein's field equations with a cosmological term is vacuum.*

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