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GENERA OF THE TORSION FREE POLYHEDRA

РОДИ ПОЛІЕДРІВ БЕЗ СКРУТУ

We study genera of polyhedra (finite cell complexes), i.e., the classes of polyhedra such that all their localizations are stably homotopically equivalent. More procisely, we describe the genera of the torsion free polyhedra of dimensions not greater than 11. In particular, we find the number of the stable homotopy classes in these genera.

Вивчаються роди поліедрів (скінченних клітинних комплексів), тобто класи поліедрів, усі локалізації яких є стабільно гомотопічно еквівалентними. А саме, описано роди поліедрів без скруту розмірності щонайбільше 11. Зокрема, обчислено кількість стабільних гомотопічних класів у цих родах.

1. Stable category and genera. The theory of genera of polyhedra was developed in [11] analogously to the theory of genera of integral representations. This paper discovered relations between two theories and established technique for calculation of genera of polyhedra.

The present paper contains calculations of genera of particular polyhedra. Namely, genera of torsion free polyhedra with integral homologies in dimensions at most 11 are described. Such polyhedra were described in [12, 13] (see also [14]).

In this paper, the number of stable homotopy classes of genera of those torsion free polyhedra is found. This number can only be 1, 2 or 4.

We consider polyhedra (i.e., finite CW-complexes) as objects of the stable homotopy category $\mathscr{S}$. In particular, isomorphism always means stable homotopy equivalence. An important feature of $\mathscr{S}$ is that its homomorphism groups are finitely generated [7].

Let CW_n^m be the full subcategory of $\mathscr{S}$ consisting of $(n-1)$-connected polyhedra of dimension at most $n+m$. The suspension functor maps CW_n^m to CW_{n+1}^m. If $n > m+1$ it is an equivalence of categories. If $n = m+1$, it is an epivalence, i.e., this functor is full, dense and conservative. In particular, it is one-to-one on the isomorphism classes of objects. We set $\mathscr{S}^m = \bigcup_{n=1}^{\infty} \mathsf{CW}_n^m$. So all objects of $\mathscr{S}^m$ are suspensions of the objects from CW_{m+1}^m.

Let $\mathbb{Z}_p = \left\{ \dfrac{a}{b} \middle| a, b \in \mathbb{Z},\ p \nmid b \right\}$, where p is a prime integer. We denote by $\mathscr{S}_p$ the category which has the same objects as $\mathscr{S}$, but its sets of morphisms are $\mathrm{Hos}_p(X,Y) = \mathrm{Hos}(X,Y) \otimes \mathbb{Z}_p$. Actually, $\mathrm{Hos}_p(X,Y)$ coincides with group of stable maps between p-localizations in the sense of Artin – Mazur – Sullivan [9]. For the sake of convenience, we denote the image in $\mathscr{S}_p$ of a polyhedron X by X_p.

Definition 1.1 [11]**.** *We say that two polyhedra X and Y are in the same genus and write $X \sim Y$ if $X_p \simeq Y_p$ for every prime p. By $G(X)$ we denote the genus of the polyhedron X, i.e., the full subcategory of $\mathscr{S}$ consisting of all polyhedra that are in the same genus as X, and by $g(X)$ the number of isomorphism classes in $G(X)$. Note that it is always finite.*

For any abelian group A we denote by $\mathrm{tors}(A)$ its torsion part, i.e., the subgroup of all torsion elements.

Let $\Lambda = \mathrm{End}(X)$ and $\bar{\Lambda} = \Lambda / \mathrm{nil}\,\Lambda$, where $\mathrm{nil}\,\Lambda$ is the nilpotent radical of Λ. Then $\bar{\Lambda}$ is an order in the semisimple algebra $\prod_{i=1}^{k} \mathrm{Mat}(r_i(X), \mathbb{Q})$ for some k, where $r_i(X) = \dim_{\mathbb{Q}} \mathrm{Hos}_{\mathbb{Q}}(S^i, X)$, according to [11] (Corollary 1.8 (3)). Then $g(X) = g(\Lambda) = g(\bar{\Lambda})$ [11], where $g(\Lambda)$ is the number of isomorphism classes of Λ-modules M such that $M_p \simeq \Lambda_p$ for all prime p. In particular, $g(\Lambda) < \infty$ by the Jordan – Zassenhaus theorem [3] (Theorem 24.1).

2. Calculations

For calculation of $g(X)$ for a particular polyhedron X, the following facts are used.

Proposition 2.1 [11]. *Let Λ be an order in $\prod_{i=1}^{k} \mathrm{Mat}(r_i, \mathbb{Q})$ for some k, Γ be a maximal order containing Λ and $\Lambda \supseteq m\Gamma$ for some integer $m > 1$. Then $g(\Lambda)$ equals the number of cosets*

$$\mathrm{Im}\,\gamma\backslash(\Gamma/m\Gamma)^\times/(\Lambda/m\Gamma)^\times,$$

where γ is the natural map $\Gamma^\times \to (\Gamma/m\Gamma)^\times$.

Applied to polyhedra, it gives the following result.

Theorem 2.1 [11]. *Let X be a polyhedron, $B = \bigvee_{i=1}^{k} r_i S^{n_i}$ with different $n_1, n_2, \ldots, n_k$ and some k. Suppose that there are maps $X \xrightarrow{\beta} B \xrightarrow{\alpha} X$ such that $\alpha\beta \equiv m1_X \bmod \mathrm{tors}(X)$ and $\beta\alpha \equiv m1_B \bmod \mathrm{tors}(B)$ for some integer $m > 1$. Then $g(X) = 1$ if $m = 2$ and $g(X) \le \big(\varphi(m)/2\big)^k$ if $m > 2$.*

In the following examples, definitions and results from [6] (Section 3) and [11] (Sections 1, 2) are used. In particular, we denote by a the ath multiple of a generator of the group $\pi_n(S^n) \simeq \mathbb{Z}$, by η the nonzero element of $\pi_n^S(S^{n-1}) \simeq \mathbb{Z}/2$, by η^2 the nonzero element of $\pi_n^S(S^{n-2}) \simeq \mathbb{Z}/2$ and by ν the generator of $\pi_n^S(S^{n-3}) \simeq \mathbb{Z}/24$. We also denote by $\mathbb{Z} \times_m \mathbb{Z}$ the subring of $\mathbb{Z} \times \mathbb{Z}$ consisting of all pairs (α, β) with $\alpha \equiv \beta \pmod m$. Note that $\mathbb{Z} \times_m \mathbb{Z} \supseteq m(\mathbb{Z} \times \mathbb{Z})$, so $g(\mathbb{Z} \times_m \mathbb{Z})$ equals the number of double cosets

$$\{\pm 1\} \times \{\pm 1\}\backslash \mathbb{Z}_m^\times \times \mathbb{Z}_m^\times / \mathbb{Z}_m^\times$$

under the diagonal embedding of $\mathbb{Z}_m^\times$ into $\mathbb{Z}_m^\times \times \mathbb{Z}_m^\times$. It easily gives $g(\mathbb{Z} \times_m \mathbb{Z}) = \varphi(m)/2$.

We consider the torsion free polyhedra from subcategories $\mathscr{S}^4$ and $\mathscr{S}^5$ (see Section 1), i.e., the polyhedra X with torsion free homology groups $H_i(X)$ for all i. Recall that these are just the cases when there are only finitely many isomorphisms classes of torsion free polyhedra. From [12], it is known that the torsion free polyhedra from subcategory $\mathscr{S}^4$ arise from the following cofibration sequences:

$$S^8 \xrightarrow{f} S^5 \to A(v) \to S^9,$$

where $f = v$;

$$S^7 \vee S^8 \xrightarrow{f} S^5 \to A(\eta^2 v) \to S^8 \vee S^9,$$

where $f = (\eta^2, v\nu)$;

$$S^8 \xrightarrow{f} S^5 \vee S^7 \to A(v\eta) \to S^9,$$

with $f = \begin{pmatrix} v\nu \\ \eta \end{pmatrix}$;

$$S^6 \vee S^8 \xrightarrow{f} S^5 \to A(\eta v) \to S^7 \vee S^9,$$

with $f = (\eta, \upsilon\nu)$;

$$S^8 \xrightarrow{f} S^5 \vee S^6 \to A(\upsilon\eta^2) \to S^9,$$

where $f = \begin{pmatrix} \upsilon\nu \\ \eta^2 \end{pmatrix}$;

$$S^7 \vee S^8 \xrightarrow{f} S^5 \vee S^7 \to A(\eta^2\upsilon\eta) \to S^8 \vee S^9,$$

where $f = \begin{pmatrix} \eta^2 & \upsilon\nu \\ 0 & \eta \end{pmatrix}$;

$$S^7 \vee S^8 \to S^5 \vee S^6 \to A(\eta^2\upsilon\eta^2) \to S^8 \vee S^9,$$

where $f = \begin{pmatrix} \eta^2 & \upsilon\nu \\ 0 & \eta^2 \end{pmatrix}$;

$$S^6 \vee S^8 \to S^5 \vee S^6 \to A(\eta\upsilon\eta^2) \to S^7 \vee S^9,$$

with $f = \begin{pmatrix} \eta & \nu\upsilon \\ 0 & \eta^2 \end{pmatrix}$;

$$S^6 \vee S^8 \to S^5 \vee S^7 \to A(\eta\upsilon\eta) \to S^7 \vee S^9,$$

where $f = \begin{pmatrix} \eta & \nu\upsilon \\ 0 & \eta \end{pmatrix}$.

Here $1 \le \upsilon \le 6$, except of the cases $A(\upsilon)$, when $1 \le \upsilon \le 12$, and $A(\eta\upsilon\eta)$, when $1 \le \upsilon \le 3$.

We provide calculations only for $A(\upsilon)$, $A(\eta^2\upsilon)$ and $A(\eta^2\upsilon\eta)$. The rest of the cases is treated in the analogous way with the similar results.

Consider the polyhedron $A(\upsilon)$. It follows from [6] (Theorem 2.4) that modulo the nilpotent radical, End $\big(A(\upsilon)\big)$ is isomorphic to the ring of pairs (α, β), where $\alpha, \beta \in \mathbb{Z}$ and $\alpha\upsilon\nu = \beta\upsilon\nu$, that is $\alpha \equiv \beta \pmod{m}$, where $m = 24/d$ and $d = gcd(\upsilon, 24)$. It is the ring $\mathbb{Z} \times_m \mathbb{Z}$. Therefore

$$g\big(A(\upsilon)\big) = \begin{cases} 4 & \text{if } d = 1, \\ 2 & \text{if } d = 2 \text{ or } d = 3, \\ 1 & \text{if } d > 3. \end{cases}$$

Actually, all atoms $A(\upsilon)$ with fixed $gcd(\upsilon, 24)$ are in the same genus [11].

For the polyhedron $A(\eta^2\upsilon\eta)$ the cofibration sequence is

$$S^7 \vee S^8 \xrightarrow{f} S^5 \vee S^7 \to A(\eta^2\upsilon\eta) \to S^8 \vee S^9$$

where

$$f = \begin{pmatrix} \eta^2 & \upsilon\nu \\ 0 & \eta \end{pmatrix} \qquad \text{and} \qquad \nu \quad \text{is the generator of} \quad \pi_8^S(S^5) \simeq \mathbb{Z}/24.$$

To simplify the calculations of $g(A)$, we can use the Theorem 2.2 from [6] (Section 3). It implies that modulo the nilpotent radical, End $\big(A(\eta^2\upsilon\eta)\big)$ is isomorphic to the ring of pairs (α, β), where

$\alpha\colon S^7 \vee S^8 \to S^7 \vee S^8$ and $\beta\colon S^5 \vee S^7 \to S^5 \vee S^7$ such that $f\alpha = \beta f$, modulo the pairs $(\gamma f, f\gamma')$, where both γ and γ' are the maps $S^5 \vee S^7 \to S^7 \vee S^8$. Here

$$\alpha = \begin{pmatrix} a & b\eta \\ 0 & c \end{pmatrix}, \qquad \beta = \begin{pmatrix} x & y\eta \\ 0 & z \end{pmatrix}$$

for some integers a, b, c, x, y, z. The equation $f\alpha = \beta f$ turns into

$$x\eta^2 = a\eta^2, \qquad xv\nu + 12y\nu = 12b\nu + vc\nu, \qquad z\eta = c\eta,$$

that is

$$x \equiv a (\operatorname{mod} 2), \qquad xv \equiv cv (\operatorname{mod} 12), \qquad z \equiv c (\operatorname{mod} 2),$$

because $\eta^3 = 12\nu$. Both γ and γ' are of the form $\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$, so the products γf, $f\gamma'$ are of the form $\begin{pmatrix} 0 & t\eta \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & t\eta^2 \\ 0 & 0 \end{pmatrix}$ for some $t \in \mathbb{Z}$. It kills $b\eta$ in α and $y\eta^2$ in β, and we obtain the subring Λ of $\Gamma = \mathbb{Z}^4$ consisting of the quadruples (a, c, x, z) such that

$$a \equiv c \equiv x \equiv z (\operatorname{mod} 2), \qquad c \equiv x (\operatorname{mod} m),$$

where $m = 12/gcd(v, 12)$, if $m \neq 3$. If $m = 3$, then we have

$$a \equiv x (\operatorname{mod} 2), \qquad c \equiv z (\operatorname{mod} 2), \qquad c \equiv x (\operatorname{mod} 3).$$

If $m \neq 3$, then the cosets from Proposition 2.1

$$\operatorname{Im} \gamma \backslash (\Gamma/m\Gamma)^\times / (\Lambda/m\Gamma)^\times$$

which describe $g(A)$ become

$$U \backslash C_2^\times \times C_m^\times \times C_m^\times \times C_2^\times / V,$$

where $U = \{\pm 1\} \times \{\pm 1\} \times \{\pm 1\} \times \{\pm 1\}$, $C_m = (\mathbb{Z}/m)^\times$ and

$$V = \big\{ (a, c, x, z) \quad | \quad a \equiv c \equiv x \equiv z (\operatorname{mod} 2),\ c \equiv x (\operatorname{mod} m) \big\}.$$

In case of $m = 3$ we have $\Lambda \supset 6\Gamma$, therefore $g(A) = 1$. So, we always set $g(A) = \varphi(m)/2$, which equals 1 if $m \leq 6$ and 2 if $m = 12$. Hence

$$g(A(\eta^2 v\eta)) = \begin{cases} 2 & \text{if } v = 1 \quad \text{or} \quad v = 5, \\ 1 & \text{otherwise.} \end{cases}$$

For the polyhedron $A(\eta^2 v)$ the cofibration sequence is

$$S^7 \vee S^8 \xrightarrow{f} S^5 \to A(\eta^2 v) \to S^8 \vee S^9,$$

where $f = \begin{pmatrix} \eta^2 & v\nu \end{pmatrix}$.

As it was done in the previous case, we shall use Theorem 2.2 from [6] (Section 3). Here modulo the nilpotent radical, $\mathrm{End}(A(\eta^2 v))$ is isomorphic to the ring of pairs (α, β) with $\alpha\colon S^7 \vee S^8 \to S^7 \vee S^8$ and $\beta\colon S^5 \to S^5$ such that $f\alpha = \beta f$ since there are no nonzero maps $S^5 \to S^7 \vee S^8$. Here

$$\alpha = \begin{pmatrix} a & b\eta \\ 0 & c \end{pmatrix}, \qquad \beta = x$$

for some integers a, b, c, x. The equation $f\alpha = \beta f$ becomes

$$x\eta^2 = a\eta^2,$$

$$xv\nu = 12b\nu + cv\nu,$$

that is

$$x \equiv a (\mathrm{mod}\, 2),$$

$$xv \equiv cv (\mathrm{mod}\, 12).$$

Modulo the nilpotent radical, we obtain the subring Λ of $\Gamma = \mathbb{Z}^3$ consisting of the triples (a, c, x) such that

$$a \equiv c \equiv x (\mathrm{mod}\, 2), \qquad c \equiv x (\mathrm{mod} m),$$

where $m = 12/gcd(v, 12)$ (case $m = 3$ is treated analogously to the previous example). Thus

$$\Gamma^\times \backslash \prod_{p|m} \Gamma_p^\times / \prod_{p|m} \Lambda_p^\times \simeq U \backslash C_2^\times \times C_m^\times \times C_m^\times / V,$$

where $U = \{\pm 1\} \times \{\pm 1\} \times \{\pm 1\}$, $C_m = (\mathbb{Z}/m)^\times$ and

$$V = \{(a, c, x) \mid a \equiv c \equiv x (\mathrm{mod}\, 2),\ c \equiv x (\mathrm{mod}\, m)\}.$$

It gives $g(A) = \varphi(m)/2$, which equals 1 if $m \le 6$ and 2 if $m = 12$. Hence

$$g(A(\eta^2 v)) = \begin{cases} 2 & \text{if } v = 1 \quad \text{or} \quad v = 5, \\ 1 & \text{otherwise.} \end{cases}$$

As the result we obtain

$$g(A(v)) = \begin{cases} 4 & \text{if} \quad d = 1, \\ 2 & \text{if} \quad d = 2 \quad \text{or} \quad d = 3, \\ 1 & \text{if} \quad d > 3, \end{cases}$$

where $d = gcd(v, 24)$, and

$$g(A) = \begin{cases} 2 & \text{if} \quad v = 1 \quad \text{or} \quad v = 5, \\ 1 & \text{otherwise,} \end{cases}$$

in all other cases of the torsion free polyhedra A from $\mathscr{S}^4$.

Now we treat the Baues – Drozd A-atoms from $\mathscr{S}^5$ [6] (Section 5). In all cases $v, w \in \{1,2,3,4,5,6\}$. These atoms are defined by the following cofibration sequences:

$$S^9 \vee S^{10} \xrightarrow{f} S^6 \vee S^7 \to A(v\eta^2 w) \to S^{10} \vee S^{11},$$

where $f = \begin{pmatrix} v\nu_1 & 0 \\ \eta^2 & w\nu_2 \end{pmatrix}$ and ν_1, ν_2 are the generators of $\pi_9^S(S^6) \simeq \mathbb{Z}/24$ and $\pi_{10}^S(S^7) \simeq \mathbb{Z}/24$ respectively;

$$S^8 \vee S^9 \vee S^{10} \xrightarrow{f} S^6 \vee S^7 \vee S^9 \to A(\eta^2 v\eta^2 w\eta) \to S^9 \vee S^{10} \vee S^{11},$$

with $f = \begin{pmatrix} \eta^2 & v\nu_1 & 0 \\ 0 & \eta^2 & w\nu_2 \\ 0 & 0 & \eta \end{pmatrix}$;

$$S^9 \vee S^{10} \xrightarrow{f} S^6 \vee S^7 \vee S^8 \to A(v\eta^2 w\eta^2) \to S^{10} \vee S^{11},$$

where $f = \begin{pmatrix} v\nu_1 & 0 \\ \eta^2 & w\nu_2 \\ 0 & \eta^2 \end{pmatrix}$;

$$S^9 \vee S^{10} \xrightarrow{f} S^6 \vee S^7 \vee S^9 \to A(v\eta^2 w\eta) \to S^{10} \vee S^{11},$$

where $f = \begin{pmatrix} v\nu_1 & 0 \\ \eta^2 & w\nu_2 \\ 0 & \eta \end{pmatrix}$;

$$S^8 \vee S^9 \vee S^{10} \to S^6 \vee S^7 \vee S^8 \to A(\eta^2 v\eta^2 w\eta^2) \to S^9 \vee S^{10} \vee S^{11},$$

with $f = \begin{pmatrix} \eta^2 & v\nu_1 & 0 \\ 0 & \eta^2 & w\nu_2 \\ 0 & 0 & \eta^2 \end{pmatrix}$;

$$S^7 \vee S^9 \vee S^{10} \to S^6 \vee S^7 \vee S^9 \to A(\eta v\eta^2 w\eta) \to S^8 \vee S^{10} \vee S^{11},$$

with $f = \begin{pmatrix} \eta & v\nu_1 & 0 \\ 0 & \eta^2 & w\nu_2 \\ 0 & 0 & \eta \end{pmatrix}$;

$$S^7 \vee S^9 \vee S^{10} \to S^6 \vee S^7 \to A(\eta v\eta^2 w) \to S^8 \vee S^{10} \vee S^{11},$$

where $f = \begin{pmatrix} \eta & v\nu_1 & 0 \\ 0 & \eta^2 & w\nu_2 \end{pmatrix}$;

$$S^8 \vee S^9 \vee S^{10} \to S^6 \vee S^7 \to A(\eta^2 v\eta^2 w) \to S^9 \vee S^{10} \vee S^{11},$$

where $f = \begin{pmatrix} \eta^2 & v\nu_1 & 0 \\ 0 & \eta^2 & w\nu_2 \end{pmatrix}$;

$$S^7 \vee S^9 \vee S^{10} \to S^6 \vee S^7 \vee S^8 \to A(\eta v \eta^2 w \eta^2) \to S^8 \vee S^{10} \vee S^{11},$$

with $f = \begin{pmatrix} \eta & v\nu_1 & 0 \\ 0 & \eta^2 & w\nu_2 \\ 0 & 0 & \eta^2 \end{pmatrix}$.

We provide calculations only for $A(v\eta^2 w)$, $A(\eta v \eta^2 w)$ and $A(\eta^2 v \eta^2 w \eta)$. Other cases are treated in the analogous way giving the similar results.

For $A(v\eta^2 w)$ the cofibration sequence is

$$S^9 \vee S^{10} \xrightarrow{f} S^6 \vee S^7 \to A(v\eta^2 w) \to S^{10} \vee S^{11},$$

where $f = \begin{pmatrix} v\nu_1 & 0 \\ \eta^2 & w\nu_2 \end{pmatrix}$. Analogously to the previous cases, we apply the Theorem 2.2 from [6] (Section 3). Here $\alpha\colon S^9 \vee S^{10} \to S^9 \vee S^{10}$ and $\beta\colon S^6 \vee S^7 \to S^6 \vee S^7$. γ and γ' are zero maps and

$$\alpha = \begin{pmatrix} a & b\eta \\ 0 & c \end{pmatrix}, \qquad \beta = \begin{pmatrix} x & y\eta \\ 0 & z \end{pmatrix}$$

for some integers a, b, c, x, y, z. The equation $f\alpha = \beta f$ implies

$$cw \equiv zw (\operatorname{mod} 12),$$

$$xv \equiv av (\operatorname{mod} 12),$$

$$z \equiv a (\operatorname{mod} 2),$$

and we obtain the subring Λ of $\Gamma = \mathbb{Z}^4$ consisting of the quadruples (a, c, x, z) such that

$$a \equiv z (\operatorname{mod} 2), \qquad a \equiv x (\operatorname{mod} m), \qquad c \equiv z (\operatorname{mod} m'),$$

where $m = gcd(v, 12)$ and $m' = gcd(w, 12)$, if $m, m' \neq 3$. Suppose that both m, m' are even. Then we have

$$a \equiv x (\operatorname{mod} m), \qquad c \equiv z (\operatorname{mod} m'),$$

so that the cosets are defined by the subring $\Lambda^\times \supset \big(a \equiv (m), c \equiv \mu(m'), x \equiv (m), z \equiv \mu(m')\big)$ and by $(\mathbb{Z}/m)^\times \times (\mathbb{Z}/m)^\times$. Therefore $g(A) = \varphi(m)/2 \times \varphi(m')/2$.

Assume now that $m' = 3$ and m is even. Then

$$a \equiv x (\operatorname{mod} m), \qquad c \equiv z (\operatorname{mod} 3), \qquad a \equiv z (\operatorname{mod} 2),$$

so that the cosets are defined by $(a \equiv 1(m),\ c \equiv 1(6),\ x \equiv 1(m),\ z \equiv 1(3))$, wherefrom $g(A) = \varphi(m)/2 \times \varphi(m')/2$. Hence

$$g(A(v\eta^2 w)) = \begin{cases} 1 & \text{if} \quad v, w \in \{2, 3, 4, 6\}, \\ 2 & \text{if} \quad v = 1 \quad \text{or} \quad v = 5, w \in \{2, 3, 4, 6\}, \\ 2 & \text{if} \quad w = 1 \quad \text{or} \quad w = 5, v \in \{2, 3, 4, 6\}, \\ 4 & \text{in all other cases.} \end{cases}$$

The polyhedron $A(\eta v\eta^2 w)$ comes from the cofibration sequence

$$S^7 \vee S^9 \vee S^{10} \xrightarrow{f} S^6 \vee S^7 \to A(\eta v\eta^2 w) \to S^8 \vee S^{10} \vee S^{11},$$

where $f = \begin{pmatrix} \eta & v\nu_1 & 0 \\ 0 & \eta^2 & w\nu_2 \end{pmatrix}$.

In this case $\alpha\colon S^7 \vee S^9 \vee S^{10} \to S^7 \vee S^9 \vee S^{10}$ and $\beta\colon S^6 \vee S^7 \to S^6 \vee S^7$ so that

$$\alpha = \begin{pmatrix} a & b\eta^2 & w_\alpha\nu_2 \\ 0 & c & d\eta \\ 0 & 0 & g \end{pmatrix}, \qquad \beta = \begin{pmatrix} x & y\eta \\ 0 & z \end{pmatrix}$$

for some integers $a, b, c, d, g, w_\alpha, x, y, z$. The equation $f\alpha = \beta f$ gives

$$cv \equiv xv (\mathrm{mod}\, 12),$$

$$gw \equiv zw (\mathrm{mod}\, 12),$$

$$z \equiv c (\mathrm{mod}\, 2),$$

$$a \equiv x (\mathrm{mod}\, 2).$$

Here γ and γ' are of the form $\begin{pmatrix} 0 & t \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, so the products γf, $f\gamma'$ are of the form $\begin{pmatrix} 0 & t\eta^2 & tw\nu_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & t\eta \\ 0 & 0 \end{pmatrix}$ for some $t \in \mathbb{Z}$. It kills $b\eta^2$ and $w_\alpha\nu_2$ in α and $y\eta$ in β and we get the subring Λ of $\Gamma = \mathbb{Z}^6$ consisting of the elements (a, c, d, g, x, z) such that

$$a \equiv c \equiv x \equiv z \equiv g (\mathrm{mod}\, 2),$$

$$c \equiv x (\mathrm{mod}\, m), \qquad g \equiv z (\mathrm{mod}\, m'),$$

where $m = gcd(v, 12)$ and $m' = gcd(w, 12)$.

Applying the same considerations as in the previous case, we obtain that $g(A) = \varphi(m)/2 \times \times \varphi(m')/2$. Hence

$$g(A(\eta v\eta^2 w)) = \begin{cases} 1 & \text{if} \quad v, w \in \{2, 3, 4, 6\}, \\ 2 & \text{if} \quad v = 1 \quad \text{or} \quad v = 5, \quad w \in \{2, 3, 4, 6\}, \\ 2 & \text{if} \quad w = 1 \quad \text{or} \quad w = 5, \quad v \in \{2, 3, 4, 6\}, \\ 4 & \text{in all other cases}. \end{cases}$$

The polyhedron $A(\eta^2 v\eta^2 w\eta)$ appears in the cofibration sequence

$$S^8 \vee S^9 \vee S^{10} \xrightarrow{f} S^6 \vee S^7 \vee S^9 \to A(\eta^2 v\eta^2 w\eta) \to S^9 \vee S^{10} \vee S^{11},$$

where $f = \begin{pmatrix} \eta^2 & v\nu_1 & 0 \\ 0 & \eta^2 & w\nu_2 \\ 0 & 0 & \eta \end{pmatrix}$.

In this case $\alpha\colon S^8 \vee S^9 \vee S^{10} \to S^8 \vee S^9 \vee S^{10}$ and $\beta\colon S^6 \vee S^7 \vee S^9 \to S^6 \vee S^7 \vee S^9$ so that

$$\alpha = \begin{pmatrix} a & b\eta & c\eta^2 \\ 0 & d & f\eta \\ 0 & 0 & g \end{pmatrix}, \qquad \beta = \begin{pmatrix} x & y\eta & v_\beta\nu_1 \\ 0 & z & p\eta^2 \\ 0 & 0 & q \end{pmatrix}$$

for some integers a, b, c, d, f, g, v_β, x, y, z, p, q. The equation $f\alpha = \beta f$ implies

$$dv \equiv xv (\operatorname{mod} 12), \qquad gw \equiv zw (\operatorname{mod} 12), \qquad z \equiv d (\operatorname{mod} 2),$$

$$a \equiv x (\operatorname{mod} 2), \qquad g \equiv q (\operatorname{mod} 2),$$

and γ and γ' are of the form $\begin{pmatrix} 0 & 0 & t\eta \\ 0 & 0 & u \\ 0 & 0 & 0 \end{pmatrix}$, so the products $\gamma f, f\gamma'$ are of the form $\begin{pmatrix} 0 & 0 & t\eta^2 \\ 0 & 0 & u\eta \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & t\eta^3 + uv\nu_1 \\ 0 & 0 & u\eta^2 \\ 0 & 0 & 0 \end{pmatrix}$ for some $t, u \in \mathbb{Z}$.

It kills $c\eta^2$ and $f\eta$ in α and $p\eta^2$ and $v_\beta\nu_1$ in β and we get the subring Λ of $\Gamma = \mathbb{Z}^8$ consisting of the elements (a, b, d, g, x, y, z, q) such that

$$a \equiv x \equiv d \equiv z \equiv g \equiv q (\operatorname{mod} 2),$$

$$d \equiv x (\operatorname{mod} m), \qquad g \equiv z (\operatorname{mod} m'),$$

where $m = gcd(v, 12)$ and $m' = gcd(w, 12)$.

Again, as in the previous case, we obtain that $g(A) = \varphi(m)/2 \times \varphi(m')/2$ and therefore

$$g\big(A(\eta^2 v\eta^2 w\eta)\big) = \begin{cases} 1 & \text{if} \quad v, w \in \{2, 3, 4, 6\}, \\ 2 & \text{if} \quad v = 1 \quad \text{or} \quad v = 5, \quad w \in \{2, 3, 4, 6\}, \\ 2 & \text{if} \quad w = 1 \quad \text{or} \quad w = 5, \quad v \in \{2, 3, 4, 6\}, \\ 4 & \text{in all other cases.} \end{cases}$$

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