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# GENERALIZATIONS OF FOX HOMOTOPY GROUPS, WHITEHEAD PRODUCTS AND GOTTLIEB GROUPS*

# УЗАГАЛЬНЕННЯ ГОМОТОПІЧНИХ ГРУП ФОКСА, ДОБУТКИ УАЙТХЕДА І ГРУПИ ГОТТЛІБА

In this paper, we redefine the torus homotopy groups of Fox and give a proof of the split exact sequence of these groups. Evaluation subgroups are defined and are related to the classical Gottlieb subgroups. With our constructions, we recover the Abe groups and prove some results of Gottlieb for the evaluation subgroups of Fox homotopy groups. We further generalize Fox groups and define a group $\tau = [\Sigma(V \times W \cup *), X]$ in which the generalized Whitehead product of Arkowitz is again a commutator. Finally, we show that the generalized Gottlieb group lies in the center of $\tau$, thereby improving a result of Varadarajan.

Уточнено означення торових гомотопічних груп Фокса, доведено розщеплення точної послідовності цих груп. Наведено означення оціночних підгруп і знайдено їх зв'язок із класичними підгрупами Готтліба. На основі цих конструкцій встановлено деякі властивості груп Абе та доведено деякі результати Готтліба для оціночних підгруп гомотопічних груп Фокса. Наведено подальше узагальнення груп Фокса та означення групи $\tau = [\Sigma(V \times W \cup *), X]$, у якій узагальнення Арковича добутку Уайтхеда також є комутатором. Насамкінець показано, що узагальнена група Готтліба міститься у центрі групи $\tau$, що покращує результат Варадараяна.

**Introduction.** In 1941, J. H. C. Whitehead [1] introduced the notion of a product, between elements of the higher homotopy groups of a space, now known as the Whitehead product. It is well known that if $\alpha, \beta \in \pi_1(X)$ then the Whitehead product $\alpha \circ \beta$ of $\alpha$ and $\beta$ is simply the commutator $[\alpha, \beta]$. In an attempt to give a more geometric description of the Whitehead product, R. Fox introduced in [2] the torus homotopy groups. Given a path connected space $X$, the $n$-th torus homotopy group $\tau_n(X)$ is isomorphic to the fundamental group of the function space $X^{T^{n-1}}$, where $T^{n-1}$ denotes the $(n-1)$-dimensional torus. In [2], an elegant and geometric description of the elements of $\tau_n(X)$ was given. Take for example when $n = 2$, a typical element of $\tau_2$ is the homotopy class of maps of the form $f\colon F_2 \to X$ where $F_2$ is the pinched 2-torus, i.e., $F_2$ is the quotient of $S^1 \times S^1$ by $S^1 \times \{s_0\}$ for some basepoint $s_0 \in S^1$. It follows that $F_2$ has the same homotopy type as the reduced suspension of $\left(S^1 \cup *\right)$, the disjoint union of the circle with a distinguished point. With this description of the $F_n$ as the pinched $n$-torus, we are led to extend the definition of $\tau_n$. We reprove the main results of [2] using modern language of homotopy theory. Our approach allows us to generalize many results concerning Gottlieb groups or generalized evaluation subgroups. The insight from [2] sheds new light into the generalized Whitehead product given by M. Arkowitz [3]. In particular, we show, in the same spirit as in [2], another way so that the generalized Whitehead product when embedded in a larger (and different) group is a commutator as well.

Although R. Fox introduced his so-called torus homotopy groups [2] (first announced in 1945) in 1948 and made a connection with the Whitehead products, these

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homotopy groups seem to have been forgotten in the development of algebraic topology. It is the purpose of this paper to show that Fox's concept of the torus homotopy groups can be used to generalize and to unify many other results such as those in [3], and in [4].

This paper is organized as follows. In Section 1, we redefine the torus homotopy groups of Fox and reprove the main results of [2]. In Section 2, we study the evaluation subgroups of the Fox groups, following the work of D. Gottlieb [5]. In Section 3, we extend the definition of the Fox groups and obtain a similar split exact sequence (Theorem 3.1). In Section 4, we relate our generalized Fox groups of Section 3 with the generalized Whitehead product of Arkowitz [3]. In particular, we introduce a group $\tau$ in which Arkowitz's product is again a commutator (Theorem 4.1). Moreover, we show that the generalized Gottlieb group lies in the center of $\tau$ (Theorem 4.2).

Throughout, all spaces are assumed to be compactly generated as in [6]. This assumption is made solely for the fact that the two definitions of the Gottlieb groups, namely, the one defined using associated maps, and the one using the evaluation map, are indeed isomorphic in the category of compactly generated spaces.

**1. Fox torus homotopy groups.** Using modern language of homotopy theory, we redefine in this section Fox's torus homotopy groups and we improve upon Fox's results.

**Definition 1.1.** *Let $X$ be a space and $x_0 \in X$ a basepoint. For $n \geq 1$, the n-th Fox group of $X$ is defined to be*

$$\tau_n(X, x_0) = \left[\Sigma\left(T^{n-1} \cup *\right), X\right],$$

*where $T^k$ denotes the k-dimensional torus, $\Sigma$ denotes the reduced suspension, and $[\ ,\ ]$ denotes the set of homotopy classes of based point preserving maps.*

When defining homotopy groups, one considers basepoint homotopy classes of maps from spheres to a space. In a similar fashion, one can interpret the Fox torus homotopy groups as basepoint homotopy classes of maps from the suspension of tori with an extra basepoint to a given space. Thus, we call an ($n$-dimensional) pinched torus $\Sigma(T^{n-1} \cup *)$ *a Fox space*, denoted by $F_n$.

**Proposition 1.1.** *The suspension of a Fox space has the homotopy type of a bouquet of spheres. More precisely,*

$$\Sigma\left(T^{k-1}/T^{k-2}\right) \approx \bigvee_{\ell=2}^{k}\left(S^{\ell}\right)^{\binom{k-2}{\ell-2}}.$$

***Proof.*** Let us consider the following Barratt – Puppe sequence (see e.g. [7] or [8]

$$T^{k-2} \to T^{k-1} \to T^{k-1}/T^{k-2} \to \Sigma T^{k-2} \to \Sigma T^{k-1} \to \Sigma\left(T^{k-1}/T^{k-2}\right) \to \ldots$$

associated with the cofibration $T^{k-2} \hookrightarrow T^{k-1}$. Using the formula

$$\Sigma(X \times Y) \approx \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y) \tag{1.1}$$

we obtain that $\Sigma\left(T^{k-2} \times S^1\right)$ has the same homotopy type as

$$\Sigma T^{k-2} \vee \Sigma S^1 \vee \Sigma\left(T^{k-2} \wedge S^1\right)$$

where $\Sigma\left(T^{k-2} \wedge S^1\right) \approx \Sigma^2\left(T^{k-2}\right)$. Since the projection $X \times Y \to X$ is a left inverse of the inclusion $X \to X \times Y$, it follows that $\Sigma\left(T^{k-1}/T^{k-2}\right)$ has the same homotopy type as $\Sigma S^1 \vee \Sigma^2\left(T^{k-2}\right)$. On the other hand, the suspension of the torus $T^m = \left(S^1\right)^m$, by using the formula (1.1) for the suspension of a product, has the homotopy type of a wedge of spheres where the number of spheres in dimension $\ell$ is given by the binomial coefficient $\binom{m}{\ell-2}$. By taking the suspension again, it is straightforward to deduce the desired formula.

**Theorem 1.1.** *Let $X$ be a path connected space. Then*

$$0 \to \tau_{n-1}(\Omega X) \to \tau_n(X) \overset{\leftarrow}{\to} \tau_{n-1}(X) \to 1 \tag{1.2}$$

*is split exact. Moreover,*

$$\tau_{n-1}(\Omega X) \cong \prod_{i=2}^{n} \pi_i(X)^{\alpha_i},$$

*where $\alpha_i$ is the binomial coefficient $\binom{n-2}{i-2}$.*

***Proof.*** Consider the Barratt – Puppe sequence

$$T^{n-2} \cup * \to T^{n-1} \cup * \to T^{n-1}/T^{n-2} \to$$
$$\to \Sigma\left(T^{n-2} \cup *\right) \to \Sigma\left(T^{n-1} \cup *\right) \to \Sigma\left(T^{n-1}/T^{n-2}\right) \to \ldots$$

associated with $T^{k-2} \cup * \hookrightarrow T^{k-1} \cup *$ and the long exact sequence

$$\ldots \to \left[\Sigma\left(T^{n-1}/T^{n-2}\right), X\right] \to \left[\Sigma\left(T^{n-1} \cup *\right), X\right] \to \left[\Sigma\left(T^{n-2} \cup *\right), X\right] \to \ldots$$

of groups by taking basepoint homotopy classes of maps into a space $X$. The projection $T^{n-1} \to T^{n-2}$ onto the first $(n-2)$ coordinates induces a basepoint preserving map $\left(T^{n-1} \cup *\right) \to \left(T^{n-2} \cup *\right)$, and hence a homomorphism $\tau_{n-1}(Y) \to \to \tau_n(Y)$ which is a right inverse of the homomorphism $\tau_n(Y) \to \tau_{n-1}(Y)$ for any space $Y$. Consequently, the exactness and the splitting of the short exact sequence (1.2) follow. The second part follows from Proposition 1.1 in a straightforward manner.

Not only does Theorem 1.1 contain the following result of Fox, it also expresses the kernel of the short exact sequence (1.2) in terms of torus homotopy groups. In [2], Fox proved the following theorem.

**Theorem 1.2.** *Let $X$ be a space. Then*

$$0 \to \prod_{i=2}^{n} \pi_i(X)^{\alpha_i} \to \tau_n(X) \overset{\leftarrow}{\to} \tau_{n-1}(X) \to 1 \tag{1.3}$$

*is split exact where $\alpha_i = \binom{n-2}{i-2}$.*

As indicated in [2], Theorem 1.2 asserts, in particular, that we have

$$\tau_1(X) \subseteq \tau_2(X) \subseteq \tau_3(X) \subseteq \ldots, \tag{1.4}$$

where the inclusions are the sections as in (1.3).

To conclude this section, we ask how the torus homotopy groups are related with respect to fibrations. We obtain the following theorem.

**Theorem 1.3.** *Let $F \hookrightarrow E \to B$ be a fibration. For any positive integer $k$, there is a long exact sequence*

$$\ldots \to \tau_k\left(\Omega^n F\right) \to \tau_k\left(\Omega^n E\right) \to \tau_k\left(\Omega^n B\right) \xrightarrow{d_*} \tau_k\left(\Omega^{n-1} F\right) \to \ldots .$$

***Proof.*** Consider the long sequence

$$\ldots \to \Omega^n F \to \Omega^n E \to \Omega^n B \xrightarrow{d_*} \Omega^{n-1} F \to \ldots$$

associated to a fibration (see e.g. [8] $F \hookrightarrow E \to B$. By taking homotopy classes of maps from a space $X$ into the spaces of this sequence, we obtain a long exact sequence. When $X$ is the Fox space $F_k$, we obtain the desired sequence.

**2. Gottlieb – Fox groups.** In [5], D. Gottlieb introduced the so-called (classical) Gottlieb groups $G_n(X)$ of a space $X$ as the set of maps $S^n \vee X \to X$ that can be extended to $S^n \times X \to X$. He showed that, when $X$ has the homotopy type of a CW complex, $G_n(X)$ is the image of the homomorphism induced by the evaluation map $ev : \left(X^X, 1_X\right) \to (X, x_0)$ on $\pi_n(X, x_0)$. In the category of compactly generated spaces, these two notions are equivalent.

**Definition 2.1.** *Let $X$ be a space and $x_0 \in X$. For $n \geq 1$, the $n$-th Gottlieb – Fox group of $X$ is defined to be the evaluation subgroup, denoted by $G\tau_n(X)$, i.e., the image of the evaluation homomorphism. Thus,*

$$G\tau_n(X) \;:=\; \operatorname{Im}\left(ev_* : \tau_n\left(X^X, 1_X\right) \to \tau_n(X, x_0)\right).$$

In general, $\tau_n(X)$ is not abelian so it is not clear whether $G\tau_n(X)$ would be. As it turns out, the Gottlieb – Fox groups are indeed abelian and can be expressed in terms of the classical Gottlieb groups. Moreover, we will show that $G\tau_n(X)$ is central in $\tau_n(X)$.

**Theorem 2.1.** *The Gottlieb – Fox group is a direct product of ordinary Gottlieb groups. In fact, we have*

$$G\tau_n(X) \;=\; \prod_{i=1}^{n} G_i(X)^{\gamma_i},$$

*where $\gamma_i$ is the binomial coefficient* $\binom{n-1}{i-1}$.

***Proof.*** First note that $(\Omega X)^X \approx \Omega\left(X^X\right)$. Consider the commutative diagram

$$\begin{array}{ccccc}
\tau_{n-1}\left(\Omega X^X\right) & \xrightarrow{ev_*} & G\tau_{n-1}(\Omega X) & \xrightarrow{\subseteq} & \tau_{n-1}(\Omega X) \\
\downarrow & & \downarrow & & \downarrow \\
\tau_n\left(X^X\right) & \xrightarrow{ev_*} & G\tau_n(X) & \xrightarrow{\subseteq} & \tau_n(X) \\
\text{proj.}\downarrow & & \text{proj.}\downarrow & & \text{proj.}\downarrow \\
\tau_{n-1}\left(X^X\right) & \xrightarrow{ev_*} & G\tau_{n-1}(X) & \xrightarrow{\subseteq} & \tau_{n-1}(X)
\end{array},$$

where the vertical columns arc short split-exact sequences as in Theorem 1.1. The fact that $G\tau_{n-1}(\Omega X)$ is the kernel of the middle vertical sequence follows from a diagram chasing argument. Moreover, $G\tau_{n-1}(\Omega X) = \prod_{i=2} G_i(X)^{\alpha_i}$ so that

$$G\tau_n(X) \;\cong\; \left(\prod_{i=2} G_i(X)^{\alpha_i}\right) \rtimes \; G\tau_{n-1}(X).$$

By induction and the fact that the action becomes conjugation in $\tau_n(X)$, the semi-direct product is in fact a direct product. Finally, an easy combinatorial argument shows that

$$\sum_{j=2}^{n-k+2} \binom{n-j}{k-2} \;=\; \gamma_k$$

and the formula for $G\tau_n(X)$ follows.

In [5], it was proved that the Whitehead product of the elements of the Gottlieb group with any other element in the higher homotopy groups is zero. This together with the interpretation of the Whitehead product given by Fox in [2] yields the following corollary.

***Corollary* 2.1.** *Denote by* $Z(G)$ *the center of a group* $G$. *Then,*

$$G\tau_n(X) \subset Z(\tau_n(X)),$$

*i.e.,* $G\tau_n(X)$ *is central in* $\tau_n(X)$.

In general, there is no relationship among the classical Gottlieb groups $\{G_n\}$. On the other hand, the nested property of the torus homotopy groups (1.4) also holds for the Gottlieb – Fox groups, i.e., $G\tau_{n-1} \subseteq G\tau_n$.

In [5] the elements of the Gottlieb group were shown to contain the image of the boundary homomorphism in the long exact sequence of homotopy groups of a fibration. We will show a similar result for the torus homotopy groups.

**Theorem 2.2.** *Let* $F \hookrightarrow E \to B$ *be a fibration, then*

$$d_*(\tau_n(\Omega B)) \subset G\tau_n(F),$$

*where* $d_*$ *is induced by the action of* $\Omega B$ *on* $F$. *Furthermore, the group* $G\tau_n(F)$ *is the union of the images of the homomorphism* $d_*\colon \tau_n(\Omega\tilde{B}) \to \tau_n(F)$ *where the union is taken over all fibration* $F \hookrightarrow \tilde{E} \to \tilde{B}$ *having the some fiber* $F$.

***Proof.*** The action of $\Omega B$ on $F$ determines a map $D : \Omega B \to F^F$ up to homotopy. The map $d$ then factors through $F^F$ and $d = ev \circ D$ up to homotopy, where $ev$ is the evaluation map. For the second part, consider the universal fibration over the base $\tilde{B} = B\,\mathrm{Aut}(F)$, the classifying space for the group $\mathrm{Aut}(F)$. The image of the boundary homomorphism is exactly the image of the evaluation given in Definition 2.1 and the result follows.

**3. Generalization of the functor $\tau_n$.** One can define a generalization of the Fox torus group as follows.

**Definition 3.1.** *Let* $X$ *be a space and* $x_0 \in X$ *a basepoint. For any space* $W$, *the* $W$*-Fox group of* $X$ *is defined to be*

$$\tau_W(X, x_0) = [\Sigma(W \cup *), X],$$

*where* $\Sigma$ *denotes the reduced suspension, and* $[\ ,\ ]$ *denotes the set of homotopy classes of basepoint preserving maps.*

First of all, we have the following useful description of $\tau_W(X, x_0)$.

**Lemma 3.1.** *For any* $X$ *and* $W$, *we have*

$$\tau_W(X, x_0) \cong \pi_1\left(X^W, c_0\right) \cong [\Sigma W, X] \rtimes \pi_1(X, x_0),$$

*where* $c_0\colon W \to X$ *is the constant map at* $x_0$.

***Proof.*** For any $X$ and $W$, we have a fibration

$$X_*^W \to X^W \to X,$$

where $X_*^W$ is the function space of basepoint preserving maps. This fibration admits a section and the based point in $X^W$ is the constant map. We then obtain a long exact sequence similar to that of Theorem 1.3. The last three terms of the resulting long exact sequence together with the splitting gives the result.

***Remark* 3.1.** We should point out that several authors have studied the track groups in $W$-topology (see e.g. [9] and subsequent works such as [10 – 12]). In these works, the identification $\left[\Sigma^k W, X\right] \cong \pi_k\left(X^W\right)$ is used, where $X^W$ is understood to be the set of based-point preserving maps from $W$ to $X$ whereas the notation $X^W$ used in this paper denotes the set $\mathrm{Map}(W, X)$ of all maps from $W$ to $X$. For example, $X^{S^1}$ is the free loop space $\Lambda X$ here but $X^{S^1} = \Omega X$ in [9].

As an immediate consequence of Lemma 3.1, we show that the Abe groups of [13] are semi-direct products. More precisely, the $n$-th Abe group $\kappa_n(X)$, which is defined to be the fundamental group of the function space $X^{S^{S^{n-1}}}$, is the split extension of $\pi_n(X)$ by $\pi_1(X)$.

***Corollary* 3.1.** *The Abe groups are semi-direct products, i.e.,*

$$\kappa_n(X) = \tau_{S^{n-1}}(X) = \pi_n(X) \rtimes \pi_1(X).$$

***Remark* 3.2.** Note that there is a canonical map $T^{n-1} \bigcup * \to S^{n-1}$ by collapsing the $(n-2)$-th skeleton of $T^{n-1} \bigcup *$. This induces a monomorphism $\tau_{S^{n-1}}(X) \to \tau_n(X)$. This fact that $\tau_n(X)$ contains an isomorphic copy of $\kappa_n(X)$ was already established in [2].

Next, we give a generalization of Fox's theorem (1.2) as follows.

**Theorem 3.1.** *For any path connected* $X$, $V$ *and* $W$, *the following sequence:*

$$1 \to [(V \times W)/V, \Omega X] \to \tau_{V\times W}(X) \overset{\leftarrow}{\to} \tau_V(X) \to 1 \tag{3.1}$$

*is split exact. If* $W = \Sigma A$ *is a suspension, then* $[(V \times W)/V, \Omega X]$ *is abelian and is isomorphic to* $[V \wedge W, \Omega X] \times [W, \Omega X]$.

***Proof.*** From the following Barratt – Puppe sequence

$$V \bigcup * \to (V \times W) \bigcup * \to (V \times W)/V \to \Sigma(V \bigcup *) \to$$
$$\to \Sigma((V \times W) \bigcup *) \to \Sigma((V \times W)/V) \to \ldots$$

and the fact that the map $V \to (V \times W)$ admits a left inverse, by taking homotopy classes of maps into $X$, we obtain the following split exact sequence of groups:

$$1 \to [\Sigma((V \times W)/V), X] \to [\Sigma((V \times W) \bigcup *), X] \overset{\leftarrow}{\to} [\Sigma(V \bigcup *), X] \to 1.$$

Now we study the group $[\Sigma((V \times W)/V), X]$.

Consider the Barratt – Puppe sequence

$$W \to (V \times W)/V \to V \wedge W \to \Sigma W \to \Sigma((V \times W)/V) \to \Sigma(V \wedge W) \to \ldots$$

and its associated sequence of groups by taking homotopy classes of maps into $X$. Since the map $W \to (V \times W)/V$ admits a left inverse we obtain the following sequence which is split exact:

$$1 \to [V \wedge W, \Omega X] \to [(V \times W)/V, \Omega X] \overset{\leftarrow}{\to} [W, \Omega X] \to 1. \tag{3.2}$$

It follows that

$$[\Sigma((V \times W)/V), X] \cong [V \wedge W, \Omega X] \rtimes [W, \Omega X].$$

Since

$$\Sigma(V \times W) \approx \Sigma V \vee \Sigma W \vee \Sigma(V \wedge W),$$

it follows that

$$\Sigma(W \vee (V \wedge W)) \approx \Sigma(V \times W)/V.$$

Now, if $W = \Sigma A$ for some $A$, then

$$\Sigma(W \vee (V \wedge W)) \approx \Sigma(V \times W)/V$$

is a double suspension and thus $[\Sigma(W \vee (V \wedge W)), X]$ is abelian with the group structure given by the double suspension loop structure. There are canonical projections

$$\alpha\colon (V \times W)/V \to W \quad \text{and} \quad \beta\colon (V \times W)/V \to V \wedge W.$$

The co-multiplication on $W = \Sigma A$ gives rise to a co-multiplication

$$\nu\colon (V \times W)/V \;\to\; ((V \times W)/V) \vee ((V \times W)/V).$$

Together with $\alpha$ and $\beta$, we have a suspension map

$$\Sigma(V \times W)/V \;\to\; \Sigma(W \vee (V \wedge W))$$

which induces an isomorphism on homology and hence on homotopy classes since the spaces are simply connected. Also the suspension map induces a group homomorphism

$$[\Sigma(W \vee (V \wedge W)), X] \;\to\; [\Sigma(V \times W)/V, X].$$

Therefore, we have an isomorphism and this implies that the semi-direct product (3.2) is indeed a direct product.

***Remark* 3.3.** If $W = S^1$, $V = T^{n-2}$, then (3.1) becomes (1.2) since $(V \times W)V = T^{n-1}/T^{n-2}$ is the Fox space $F_{n-1}$ (pinched torus) so that $(V \times W)/V \approx \Sigma\left(T^{n-2} \cup *\right)$.

**4. Generalized Whitehead products and Gottlieb groups.** In [3], a generalized Whitehead product was defined between elements of $[\Sigma A, X]$ and of $[\Sigma B, X]$. Here, we give further information concerning Arkowitz's product with the insight gained from the Fox torus homotopy groups. We also consider the generalized Gottlieb group with respect the functor $\tau_W$.

We start by recalling the definition of the generalized Whitehead product given in [3]. Let $f\colon \Sigma A \to X$, $g\colon \Sigma B \to X$ be two maps,

$$\Sigma p_A\colon \Sigma(A \times B) \;\to\; \Sigma A, \qquad \Sigma p_B\colon \Sigma(A \times B) \;\to\; \Sigma B$$

the suspension of the corresponding projections and

$$f' \;=\; f \circ \Sigma p_A, \qquad g' \;=\; g \circ \Sigma p_B,$$

the composites respectively. Using the co-multiplication of $\Sigma(A \times B)$, we have a well-defined map

$$\left(f'^{-1} \circ g'^{-1}\right) \circ \left(f' \circ g'\right)\colon \Sigma(A \times B) \;\to\; X.$$

This map, when restricted to $\Sigma A \vee \Sigma B$, is homotopic to the constant map. Now we let $K\colon \Sigma(A \times B) \to X$ be a map homotopic to $\left(f'^{-1} \circ g'^{-1}\right) \circ \left(f' \circ g'\right)$ whose restriction to $\Sigma A \vee \Sigma B$ is the constant map.

**Definition 4.1.** *The map* $K\colon \Sigma(A \times B) \to X$, *as above, defines a map* $K'\colon \Sigma(A \times B) \to X$ *and the homotopy class* $\left[K'\right]$ *is a well-defined class called the generalized Whitehead product of* $[f]$ *and* $[g]$, *and is denoted by* $[f] \circ [g]$ (*see* [3]).

In the spirit of [2], we reinterpret Arkowitz's generalized Whitehead product as follows.

**Theorem 4.1.** *Given* $\alpha \in [\Sigma A, X]$ *and* $\beta \in [\Sigma B, X]$, *then the image of* $(\alpha \circ \beta)$ *in* $\tau_{A\times B}(X)$ *is the commutator of the image of* $\alpha^{-1}$ *and the image of* $\beta^{-1}$ *in* $\tau_{A\times B}(X)$.

***Proof.*** The image of $[f] \circ [g]$ in $\tau_{A\times B}(X)$ is the homotopy class of the composite

$$\Sigma((A \times B) \cup *) \;\to\; \Sigma(A \times B) \;\to\; \Sigma(A \wedge B) \;\xrightarrow{\alpha\circ\beta}\; X.$$

Call $\left[\bar{f}\right]$, $[\bar{g}]$ the images of $[f]$, $[g]$ in $\tau_{A\times B}(X)$, respectively, which are the homotopy classes of the composites

$$\Sigma((A \times B) \bigcup *) \to \Sigma(A \times B) \to \Sigma A \xrightarrow{f} X$$

and

$$\Sigma((A \times B) \bigcup *) \to \Sigma(A \times B) \to \Sigma B \xrightarrow{g} X,$$

respectively. The definition of the operation in $\tau_{A \times B}(X)$ uses the fact that the domain $\Sigma((A \times B) \bigcup *)$ is a suspension. The commutator of the images of $\alpha^{-1}$ and $\beta^{-1}$ in $[\Sigma(A \times B), X]$, by the definition of the operation when the domain is a suspension, has the map $\left(f'^{-1} \circ g'^{-1}\right) \circ \left(f' \circ g'\right)$ as a representative. But the homotopy class of this map is also from (see [3]) the image of the generalized Whitehead product in $[\Sigma(A \times B), X]$. Now by composing with the suspension of the projection $\Sigma((A \times B) \bigcup *) \to \Sigma(A \times B)$, the result follows.

Extending the definition of the classical Gottlieb group, we can define the *generalized Gottlieb group* $\mathcal{G}(\Sigma V, X)$ as follows.

**Definition 4.2.** *For any connected spaces $A$ and $X$, with a basepoint $x_0 \in X$, we define*

$$\mathcal{G}(\Sigma A, X) := \operatorname{Im}\left(ev_* : \left[\Sigma A, \left(X^X, 1_X\right)\right] \to [\Sigma A, (X, x_0)]\right)$$

*to be the generalized Gottlieb group.*

Similarly, we let

$$\tilde{\mathcal{G}}(\Sigma A, X) := \{\alpha\colon \Sigma A \to X \mid F \circ j \approx a \vee 1_X \quad \text{for some} \quad F\colon \Sigma A \times X \to X\},$$

where $j\colon \Sigma A \vee X \hookrightarrow \Sigma A \times X$ is the inclusion.

The following result follows from [14].

**Proposition 4.1.** *Suppose $A$ and $X$ are compactly generated*

$$\mathcal{G}(\Sigma A, X) = \tilde{\mathcal{G}}(\Sigma A, X).$$

We now analyze the question "Is $\mathcal{G}$ central in some group?". In [14], it was shown that for a co-$H$-space $A$, the Gottlieb group $\tilde{\mathcal{G}}(A, X)$ is central in $[A, X]$. (See also [15] for related results.) In our case, as a result of the projection $V \times W \to V$, we can regard $[\Sigma V, X]$ as a subgroup of $[\Sigma(V \times W), X]$. Under this identification, we have the following theorem.

**Theorem 4.2.** *The generalized Gottlieb group $\mathcal{G}(\Sigma V, X)$, regarded as a subgroup of $\tau_{V \times W}(X)$, is central in $\tau_{V \times W}(X)$ for any $W$. In particular, it is central in $[\Sigma V, X]$.*

***Proof.*** Let us consider the image of an element of $[\Sigma V, X]$ in $[\Sigma(V \times W), X]$ under the composite

$$\Sigma(V \times W) \xrightarrow{\Sigma p} \Sigma V \xrightarrow{f} X.$$

Since we assume that $f$ belongs to $\mathcal{G}(\Sigma V, X)$, the map

$$f \vee 1_X\colon \Sigma V \vee X \to X$$

has an extension $H\colon \Sigma V \times X \to X$, and hence the map

$$(f \circ \Sigma p) \vee 1_X\colon \Sigma(V \times W) \vee X \to X$$

has an extension to the product $\Sigma(V \times W) \times X$.

Now, consider the composite

$$\Sigma(V \times W) \times \Sigma(V \times W) \xrightarrow{1\_ \times g} \Sigma(V \times W) \times X \xrightarrow{H} X,$$

where $g$ is an arbitrary map from $\Sigma(V \times W)$ to $X$. We use the fact that

$$\tau_{V \times W}(X \times Y) \cong \tau_{V \times W}(X) \times \tau_{V \times W}(Y)$$

as groups so that an arbitrary element from $\tau_{V \times W}(X)$ commutes with an arbitrary element of $\tau_{V \times W}(Y)$, when they are regarded as elements of $\tau_{V \times W}(X \times Y)$. If we apply the above observation for our case, we obtain that the images of $[g]$ and of $[f \circ \Sigma p]$ in $\tau_{V \times W}(X)$ commute and the result follows.

As a consequence of Theorem 4.1, which says that in the group $\tau_{V \times W}(X)$ the generalized Whitehead product becomes a commutator, and of the Theorem above, we obtain that the Whitehead product of an element with any element of the Gottlieb group vanishes.

By combining Theorem 4.1 and Theorem 4.2, we deduce the following result first obtained by K. Varadarajan [4].

**Theorem 4.3.** *Given* $\alpha \in \mathcal{G}(\Sigma A, X)$, *for any* $\beta \in [\Sigma B, X]$, *we have*

$$\alpha \circ \beta = 0.$$

When the target space $X$ is a suspension, an immediate consequence of the result above gives the following description of the generalized Gottlieb groups.

***Corollary* 4.1.** *Let*

$$\mathcal{P}(\Sigma V, \Sigma X) := \operatorname{Ker}(\omega \colon [\Sigma V, \Sigma X] \to [\Sigma(V \wedge X), \Sigma X]),$$

*where* $\omega$ *is given by the generalized Whitehead product. Then,*

$$\mathcal{G}(\Sigma V, \Sigma X) = \mathcal{P}(\Sigma V, \Sigma X).$$

***Remark* 4.1.** The equality of Theorem 4.3 was also shown by C. Hoo [15] and later generalized by H. Marcum [16].

***Remark* 4.2.** Given a fibration $F \hookrightarrow E \to B$, there is an associated Eckmann – Hilton exact sequence with a boundary map $\partial : [A, \Omega B] \to [A, F]$ for any locally finite CW complex $A$. In [4], it was shown that $\partial([A, \Omega B]) \subseteq \mathcal{G}(A, F)$. Thus, when $A = \Sigma\left(T^{n-1} \cup *\right)$, Theorem 2.2 is in fact a special case of Varadarajan's result.


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