UDC 517.91

S. Uğuz (Harran Univ., Turkey)

SPECIAL WARPED-LIKE PRODUCT MANIFOLDS WITH (WEAK) G_2 HOLONOMY

СПЕЦІАЛЬНИЙ СПОТВОРЕНИЙ ДОБУТОК МНОГОВИДІВ ЗІ (СЛАБКОЮ) G_2 ГОЛОНОМІЄЮ

By using fiber-base decomposition of the manifolds, the definition of warped-like product is considered as a generalization of multiply-warped product manifolds, by allowing the fiber metric to be not block diagonal. We consider $(3+3+1)$ decomposition of 7-dimensional warped-like product manifolds, which is called a special warped-like product of the form $M = F \times B$, where the base B is a one-dimensional Riemannian manifold and the fibre F is of the form $F = F_1 \times F_2$ where F_i, $i = 1, 2$, are Riemannian 3-manifolds. If all fibers are complete, connected, and simply connected, then the fibers are isometric to S^3 with constant curvature $k > 0$ in the class of special warped-like product metrics admitting the (weak) G_2 holonomy determined by the fundamental 3-form.

З використанням волоконних розкладів многовидів розглянуто визначення спотвореного добутку як узагальнення багаторазово спотворених добутків многовидів, при цьому волоконна метрика може не бути блочно-діагональною. Вивчено $(3+3+1)$ розклади 7-вимірних спотворених добутків многовидів, що називаються спеціальними спотвореними виду $M = F \times B$, де база B — одновимірний ріманів многовид, а волокно F має фому $F = F_1 \times F_2$, де F_i, $i = 1, 2,$ — ріманові 3-многовиди. Якщо всі волокна є повними і однозв'язними, то вони є ізометричними до S^3 зі сталою кривиною $k > 0$ у класі спеціальних спотворених метрик добутку, що допускають (слабку) G_2 голономію, визначену фундаментальною 3-формою.

1. Introduction. The notion of holonomy group was introduced by Elie Cartan in 1923 [3, 4] and proved to be an efficient tool for the classification of Riemannian manifolds. The list of possible restricted holonomy groups of irreducible, simply-connected nonsymmetric spaces was given by M. Berger in 1955 [5]. Berger's list (refined later by the work of [6, 7]) includes the groups $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$, $Sp(n)Sp(1)$ that could occur in dimensions n, $2n$ and $4n$ respectively and two special cases, G_2 holonomy in 7 dimensions and Spin (7) holonomy in 8 dimensions. Manifolds with holonomy $SO(n)$ constitute the generic case, all others are denoted as manifolds with "special holonomy"and the last two cases are described as manifolds with "exceptional holonomy".

The existence of manifolds with exceptional holonomy was first demonstrated by R. Bryant [9], then complete examples were given by R. Bryant and S. Salamon [10] and the first compact examples were found by D. Joyce in 1996 [11]. The study of manifolds with exceptional holonomy and the construction of explicit examples is still an active research area both for mathematics and physics [12 – 20] (see also references of papers). The concept of weak holonomy was introduced by A. Gray in 1971 as an extension of holonomy [21]. Manifolds with weak holonomy groups are also investigated as given in [22, 23].

The motivation for our work was firstly the explicit Spin (7) metric on $S^3 \times S^3 \times R^2$ given by Yasui and Ootsuka [24] and secondly the explicit G_2 metric on $SU(2) \times SU(2) \times R$ given by Konishi and Naka [2]. We investigated whether one could obtain other solutions by relaxing some of their assumptions, in particular without requiring the three dimensional submanifolds to be S^3. We noticed that their metric ansatzs were a generalization of warped products and we called "warped-like product"as a general framework for multiply warped product manifolds. In this paper we study special warped-like product manifolds with (weak) G_2 holonomy.

The outline of paper is given as follows: we set up the basic definitions of our study in Section 2. We defined “warped-like product metrics”as a general framework for our metrical ansatz and for a special case we present $(3+3+1)$ warped-like product manifolds in Section 3. As studied Spin (7) case in [17, 18], we show that the requirement that the fundamental 3-form φ and the Hodge dual of φ be closed forms, determines the connection and with suitable global assumptions, hence the three manifolds (fibers) we started with are three spheres and we recover the Konishi and Naka solution after gauge transformations in Section 4. In Section 5 weak holonomy in 7-dimensional case is investigated for the special warped-like product metrics. Using the fundamental 3-form φ and its relation with weak holonomy, we prove also that the fibers are isometric to S^3 with constant curvature $k > 0$ as obtained in Section 4. Conclusions of the study with further remarks are summarized in Section 6.

2. Technical preliminaries. ***2.1. G_2 manifolds and the fundamental 3-form.*** As G_2 is a subgroup of $SO(7)$, a manifold M with G_2 holonomy is a real orientable 7-dimensional manifold, called a G_2 *manifold* which is classified by the existence of a certain 3-form φ which is called fundamental form, denoted by φ [9].

2.1.1. G_2-structure. A 7-dimensional manifold M admits a G_2-structure if the structure group of the frame bundle reduces to the exceptional Lie group $G_2 \subset SO(7) \subset GL(7)$ [27]. The existence of a G_2-structure on M is equivalent to the existence of a positive nondegenerate 3-form φ defined on the whole manifold and using this 3-form it is possible to define a Riemannian metric g_φ on M [8]

$$g_\varphi(X,Y)vol = \frac{1}{6} i_X\varphi \wedge i_Y\varphi \wedge \varphi. \tag{2.1}$$

If φ is parallel with respect to the Levi–Civita connection, i.e., $\nabla\varphi = 0$, then the holonomy group is contained in G_2, the G_2-structure is called parallel and the corresponding manifolds are called G_2-manifolds [27]. In this case the induced metric g_φ is Ricci-flat [25].

2.1.2. Manifolds with G_2 holonomy. The definition of G_2 manifold by using holonomy is presented in the following definition.

Definition 2.1. *Let (M, g) be a Riemannian manifold. If the holonomy group of g is contained in G_2, then M is called a G_2 manifold.*

In the present paper we are interested in 7-dimensional real oriented manifolds whose holonomy group is a subgroup of G_2. These manifolds are characterized by the existence of a closed, and G_2 invariant 3-form called the “*fundamental 3-form φ*” [9]. Conversely, if the fundamental form and its Hodge dual are closed, then the manifold has G_2 holonomy, as given by the following theorem of Fernandez and Gray.

Proposition 2.1 [27]. *The holonomy group of a Riemannian metric* (*as given in* (2.1)) *defined by the fundamental 3-form φ is contained in G_2 if and only if* $d\varphi = d * \varphi = 0$.

The proposition above implies that, assuming the existence of a globally defined fundamental 3-form (i.e., existence of a positive nondegenerate 3-form), the problem of proving M has G_2 holonomy is reduced to the local problem of checking that φ and $*\varphi$ are closed forms. We shall do this under a simplifying assumption, that we call “warped-like product”metric ansatz. As a special case in seven dimensions, we shall consider product manifolds $M = F_1 \times F_2 \times B$, where F_1 and F_2 are 3-manifolds and B is diffeomorphic to R. Since all 3-manifolds are paralellizable, the first assumption ensures the existence of independent sections of the fiber in the product decomposition $M = F \times B$ and the second assumption is made for convenience.

2.2. Weak holonomy group G_2. The concept of weak holonomy group was introduced by Alfred Gray in [7]. Much of the early work of Gray was concerned with the study of Riemannian manifolds with special holonomy groups. We present Alfred Gray's definition of the weak holonomy group of a seven dimensional Riemannian manifold. Gray shows the following important result about the weak holonomy group G_2.

Theorem 2.1 [7]. *Let (M, g) be a 7-dimensional Riemannian manifold with weak holonomy group G_2. Then M is an Einstein manifold.*

It is well known that a manifold with holonomy G_2 is Ricci-flat [25]. Thus, it follows from this result that the weak holonomy G_2 is indeed a more general notion.

Then the manifold with weak G_2 holonomy can be obtained by the following definition.

Definition 2.2 [21]. *A G_2-structure φ is said to be weak holonomy G_2 if $d\varphi = \lambda * \varphi$ with constant λ.*

From the definition above, it is clear that $d * \varphi = 0$ and thus this may indeed be considered as a generalization of the holonomy equations $d\varphi = 0$, $d * \varphi = 0$. Our notation is given as follows: e_i and e^i, $i = 1, \ldots, n$, denote respectively local orthonormal frames for the tangent and the cotangent bundles. This gives rise to local bases for k-forms denoted by

$$e^{ij} = e^i \wedge e^j, \qquad e^{ijk} = e^i \wedge e^j \wedge e^k, \qquad e^{ijkl} = e^i \wedge e^j \wedge e^k \wedge e^l \quad \ldots . \tag{2.2}$$

In the following we shall omit the wedge symbol in exterior products. The explicit expression of the fundamental 3-form φ is chosen as [2]

$$\varphi \;=\; e^{123} - e^{156} + e^{246} - e^{345} + e^{147} + e^{367} + e^{257}. \tag{2.3}$$

And the Hodge dual of the fundamental 3-form is written as follows:

$$*\varphi \;=\; e^{4567} - e^{2347} + e^{1357} - e^{1267} + e^{2356} + e^{1245} + e^{1346}. \tag{2.4}$$

3. Warped-like product manifolds. Let (F, g_F), (B, g_B) be Riemannian manifolds and $f > 0$ be smooth function on B. A *warped product manifold* is a product manifold $M = F \times B$ equipped with the metric

Let (F, g_F), (B, g_B) be Riemannian manifolds and $f > 0$ be smooth function on B. A *warped product manifold* is a product manifold $M = F \times B$ equipped with the metric

$$g = \pi_2^* g_B + (f \circ \pi_2)^2 \pi_1^* g_F,$$

where $\pi_1 : F \times B \longrightarrow F$ and $\pi_2 : F \times B \longrightarrow B$ are the natural projections [30]. A generalization of the notion of warped product metrics is the "multiply-warped products defined as follows [31]. Let (F_i, g_i), $i = 1, 2, \ldots, k$, and (B, g_B) be Riemannian manifolds and $f_i > 0$ be smooth functions on B. A *multiply-warped product manifold* is the product manifold $F_1 \times F_2 \times \ldots \times F_k \times B$, equipped with the metric

$$g = \pi_B^* g_B + \sum_{i=1}^{k} (f_i \circ \pi_B)^2 \pi_i^* g_i,$$

where $\pi_B : F_1 \times F_2 \times ... \times F_k \times B \longrightarrow B$ and $\pi_i : F_1 \times F_2 \times ... \times F_k \times B \longrightarrow F_i$ are the natural projections on B and F_i respectively. In this scheme, the metric is block diagonal, with the metrics of the F_i's are multiplied by a conformal factor depending on the coordinates of the base. We further generalize this concept by allowing nondiagonal blocks in the fiber space [32].

***Remark* 3.1.** We can define a metric on M by choosing linearly independent local sections of the cotangent bundle T^*M and declaring these to be orthonormal.

By using fiber-base decomposition, we see that warped-like product is considered as a generalization of multiply-warped product manifolds, by allowing the fiber metric to be non block diagonal [32].

Definition 3.1 [32]. *Let M be the topologically product manifold $M = F_1 \times F_2 \times ... \times F_k \times B$, where* $\dim F_a = n_a$, $a = 1, ..., k$, $\dim B = n$. *Assume that these manifolds are equipped with Riemannian metrics g_{F_a} and g_B respectively. Let $U_a \subset F_a$ and $V \subset B$ be coordinate neighborhoods on F_a and B respectively, and let $U_1 \times U_2 \times ... \times U_k \times V$. Denote the local sections of the cotangent bundle of each F_a respectively by $\{\theta_a^i\}_{i=1}^{n_a}$, the local coordinates of each F_a by $\{y_a^i\}_{i=1}^{n_a}$, and the local coordinates on B by $x^1, x^2, ..., x^n$. If the metric on M is defined by the following orthonormal frame*:

$$e_a^i = \sum_{b=1}^{k}\sum_{j=1}^{n_b} A_{aj}^{bi}\theta_b^j, \qquad i = 1, ..., n_a, \quad a = 1, .., k,$$

$$e_B^i = \sum_{j=1}^{n} a_{Bj}^i \, dx^j, \qquad i = 1, ..., n,$$

where

$$A_{aj}^{bi} = A_{aj}^{bi}(x^1, x^2, ..., x^n), \qquad a_{Bj}^i = a_{Bj}^i(x^1, x^2, ..., x^n),$$

then (M, e^i) is called as a "warped-like product"manifold.

3.1. 7-Dimensional special warped-like product manifolds. For a special case, we will define 7-dimensional special warped-like product manifolds in the following section.

Definition 3.2. *Let $M = F_1 \times F_2 \times B$ be an 7-dimensional topologically product manifold where F_1, F_2 are 3-manifolds and B is a one dimensional manifold, each equipped with Riemannian metrics. Let $\theta^i, \theta^{\hat{i}}$ be orthonormal sections of the cotangent bundles of F_1 and F_2 respectively and x be local coordinate on B. If the metric on M is defined by the following orthonormal frame*:

$$e^i = A(x)\theta^i + B(x)\theta^{\hat{i}}, \qquad e^{\hat{i}} = \hat{A}(x)\theta^i + \hat{B}(x)\theta^{\hat{i}}, \qquad e^7 = a(x)dx, \quad i = 1, 2, 3, \tag{3.1}$$

then we call (M, e^i) $i = 1, 2, ..., 7$, a "special warped-like product"on 7-dimensional manifold.

3.2. Fundamental 3-form and 7-dimensional special warped-like product structure. When relabeling the indices $\hat{1} = 4$, $\hat{2} = 5$ and $\hat{3} = 6$, we get the following forms which are more suitable for our purposes:

$$\varphi \;=\; (e^{1\hat{1}} + e^{2\hat{2}} + e^{3\hat{3}})e^7 + e^{123} - e^{1\hat{2}\hat{3}} - e^{\hat{1}2\hat{3}} - e^{\hat{1}\hat{2}3}, \tag{3.2}$$

$$*\varphi \;=\; e^{12\hat{1}\hat{2}} + e^{13\hat{1}\hat{3}} + e^{23\hat{2}\hat{3}} + (e^{\hat{1}\hat{2}\hat{3}} - e^{\hat{1}23} - e^{1\hat{2}3} - e^{12\hat{3}})e^7. \tag{3.3}$$

When we introduce the exterior forms β, μ and ν

$$\beta = e^{1\hat{1}} + e^{2\hat{2}} + e^{3\hat{3}}, \qquad \mu = e^{123} - e^{1\hat{2}\hat{3}} - e^{\hat{1}2\hat{3}} - e^{\hat{1}\hat{2}3}, \qquad \nu = e^{\hat{1}\hat{2}\hat{3}} - e^{\hat{1}23} - e^{1\hat{2}3} - e^{12\hat{3}}, \quad (3.4)$$

we can write φ and $*\varphi$ as

$$\varphi = \beta e^7 + \mu, \qquad *\varphi = \nu e^7 - \frac{1}{2}\beta^2.$$

Proposition 3.1. *Let F be a 6-dimensional Riemannian manifold of the form $F = F_1 \times F_2$ and F_i, $i = 1, 2$, be 3-manifolds. Let $\theta^i, \theta^{\hat{i}}$, $i = 1, 2, 3$, be orthonormal sections of the cotangent bundles of F_1 and F_2 respectively. Let $(M = F \times R,\ e^i)$ be a 7-dimensional special warped-like product manifold given in Definition 3.2. Then the fundamental form and its Hodge dual are written as*

$$\begin{aligned}\varphi &= f\omega e^7 + \phi_1^+ m_1 + \phi_2^+ m_2 + \phi_1^- n_1 + \phi_2^- n_2,\\ *\varphi &= -\frac{1}{2}f^2\omega^2 + \left(\phi_1^+ \tilde{m}_1 + \phi_2^+ \tilde{m}_2 + \phi_1^- \tilde{n}_1 + \phi_2^- \tilde{n}_2\right) e^7,\end{aligned}$$

where

$$\begin{aligned}&\omega = \theta^{1\hat{1}} + \theta^{2\hat{2}} + \theta^{3\hat{3}}, \quad \phi_1^+ = \theta^{123}, \quad \phi_1^- = \theta^{\hat{1}\hat{2}\hat{3}},\\ &\phi_2^+ = \theta^{1\hat{2}\hat{3}} + \theta^{\hat{1}2\hat{3}} + \theta^{\hat{1}\hat{2}3}, \quad \phi_2^- = \theta^{\hat{1}23} + \theta^{1\hat{2}3} + \theta^{12\hat{3}},\end{aligned} \tag{3.5}$$

and $f, m_i, n_i, \tilde{m}_i, \tilde{n}_i$, $i = 1, 2$, are given

$$\begin{aligned}&f = A\hat{B} - B\hat{A}, \quad m_1 = [A^3 - 3A\hat{A}^2], \quad m_2 = [AB^2 - 2B\hat{A}\hat{B} - A\hat{B}^2],\\ &n_1 = [B^3 - 3B\hat{B}^2], \quad n_2 = [A^2B - 2A\hat{A}\hat{B} - B\hat{A}^2],\end{aligned} \tag{3.6}$$

$$\begin{aligned}&\tilde{m}_1 = [\hat{A}^3 - 3A^2\hat{A}], \quad \tilde{m}_2 = [\hat{A}\hat{B}^2 - 2AB\hat{B} - \hat{A}B^2],\\ &\tilde{n}_1 = [\hat{B}^3 - 3B^2\hat{B}], \quad \tilde{n}_2 = [\hat{A}^2\hat{B} - 2AB\hat{A} - A^2\hat{B}].\end{aligned} \tag{3.7}$$

Proof. When we substitute the special warped-like product structure, we get μ and ν as

$$\begin{aligned}\mu &= [A^3 - 3A\hat{A}^2]\theta^{123} + [AB^2 - 2B\hat{A}\hat{B} - A\hat{B}^2](\theta^{1\hat{2}\hat{3}} + \theta^{\hat{1}2\hat{3}} + \theta^{\hat{1}\hat{2}3}) +\\ &+ [B^3 - 3B\hat{B}^2]\theta^{\hat{1}\hat{2}\hat{3}} + [A^2B - 2A\hat{A}\hat{B} - B\hat{A}^2](\theta^{1\hat{2}3} + \theta^{\hat{1}23} + \theta^{12\hat{3}}),\\ \nu &= [\hat{A}^3 - 3A^2\hat{A}]\theta^{123} + [\hat{A}\hat{B}^2 - 2AB\hat{B} - B^2\hat{A}](\theta^{1\hat{2}\hat{3}} + \theta^{\hat{1}2\hat{3}} + \theta^{\hat{1}\hat{2}3}) +\\ &+ [\hat{B}^3 - 3B^2\hat{B}]\theta^{\hat{1}\hat{2}\hat{3}} + [\hat{A}^2\hat{B} - 2AB\hat{A} - A^2\hat{B}](\theta^{1\hat{2}3} + \theta^{\hat{1}23} + \theta^{12\hat{3}}).\end{aligned}$$

We introduce new variables to simplify the notation $\phi_i^\pm$, $i = 1, 2$, as

$$\phi_1^+ = \theta^{123}, \qquad \phi_2^+ = \theta^{1\hat{2}\hat{3}} + \theta^{\hat{1}2\hat{3}} + \theta^{\hat{1}\hat{2}3},$$

$$\phi_1^- = \theta^{\hat{1}\hat{2}\hat{3}}, \qquad \phi_2^- = \theta^{\hat{1}23} + \theta^{1\hat{2}3} + \theta^{12\hat{3}}.$$

Then we can write

$$\mu = \phi_1^+ m_1 + \ \phi_2^+ m_2 + \phi_1^- n_1 + \phi_2^- n_2,$$

$$\nu = \phi_1^+ \tilde{m}_1 + \ \phi_2^+ \tilde{m}_2 + \phi_1^- \tilde{n}_1 + \phi_2^- \tilde{n}_2,$$

where the coefficient functions m_i and $n_i,\ i = 1, 2$, are given by the equations (3.6). Hence we write the fundamental 3-form φ and its dual form $*\varphi$ on M as follows:

$$\varphi = f\omega e^7 + \phi_1^+ m_1 + \phi_2^+ m_2 + \phi_1^- n_1 + \phi_2^- n_2,$$

$$*\varphi = -\frac{1}{2} f^2 \omega^2 + \left(\phi_1^+ \tilde{m}_1 + \ \phi_2^+ \tilde{m}_2 + \phi_1^- \tilde{n}_1 + \phi_2^- \tilde{n}_2\right) e^7.$$

Proposition 3.1 is proved.

3.3. Fibre-base decomposition of 7-dimensional special warped-like product manifolds. We consider the decomposition of the manifold M as “base”and “fiber”, then we decompose the exterior algebra as

$$\Lambda^p(M) = \bigoplus_{a+k=p} \Lambda^{(a,k)}(M),$$

where $a = 1, \ldots, 6$ and $k = 1$. Under the exterior derivative these summands are mapped as

$$d: \quad \Lambda^{(a,k)}(M) \longrightarrow \Lambda^{(a+1,k)} \oplus \Lambda^{(a,k+1)}.$$

We can refine this decomposition by splitting the components for each fiber as

$$\Lambda^p(M) = \bigoplus_{a+b+k=p} \Lambda^{(a,b,k)}(M),$$

where a and b range from 1 to 3 and $k = 1$ as before. The effect of the exterior derivative is given by

$$d: \quad \Lambda^{(a,b,k)}(M) \longrightarrow \Lambda^{(a+1,b,k)} \oplus \Lambda^{(a,b+1,k)} \oplus \Lambda^{(a,b,k+1)}.$$

By using the structure of 7-dimensional special warped-like product manifolds, we investigate G_2 and the weak G_2 holonomy metrics on these type of manifolds and prove a main theorem related to the special warped-like product manifolds with these G_2 structures in the following sections.

4. Special warped-like product manifolds with G_2 holonomy. In this section we consider the case where the seven dimensional manifold has a $(3+3+1)$ decomposition, i.e., the base is one dimensional and the fiber is a product of 3-manifolds. As all 3-manifolds are paralellizable [33] we work with global sections of the cotangent bundles of the fibers and for simplicity we assume that the base is R. Here we will prove that under suitable global assumptions the fibers are isometric to 3-spheres S^3 with constant curvature $k > 0$.

Theorem 4.1. *Let M be diffeomorphic to $F \times B$, where the base B is a one dimensional Riemannian manifold diffeomorphic to R, the fibre F is a 6-manifold of the form $F = F_1 \times F_2$, and $F_i, \ i = 1, 2,$ are complete, connected and simply connected 3-manifolds. Let the metric on M be a special warped-like product with the following orthonormal frame*:

$$e^i = A(x)\theta^i + B(x)\theta^{\hat{i}}, \qquad i = 1, 2, 3,$$

$$e^{\hat{i}} = \hat{A}(x)\theta^i + \hat{B}(x)\theta^{\hat{i}}, \qquad i = 1, 2, 3,$$

$$e^7 = a(x)dx.$$

Let φ be the fundamental 3-form on M given by

$$\varphi = f\omega e^7 + \phi_1^+ m_1 + \phi_2^+ m_2 + \phi_1^- n_1 + \phi_2^- n_2$$

and its dual

$$*\varphi = -\frac{1}{2}f^2\omega^2 + \left(\phi_1^+ \tilde{m}_1 + \ \phi_2^+ \tilde{m}_2 + \phi_1^- \tilde{n}_1 + \phi_2^- \tilde{n}_2\right) e^7.$$

If $d\varphi = d * \varphi = 0$, then F_1 and F_2 are isometric to S^3 with constant curvature $k > 0$.

Before proving the above theorem, we present two propositions which give the closeness properties of φ and $*\varphi$ respectively. The crucial step in the proof of this theorem is to find projections of the 4-form $d\varphi$ into subspaces of $\Lambda^4(M)$ determined by the special warped-like product structure.

Proposition 4.1. *Let (M, e^i) be a 7-dimensional special warped-like product manifold as in Theorem* 4.1. *If $d\varphi = 0$, then the following two conditions must be satisfied*:

$$f d\omega e^7 = \phi_1^+ dm_1 + \phi_2^+ dm_2 + \phi_1^- dn_1 + \phi_2^- dn_2, \tag{4.1}$$

$$d\phi_2^+ m_2 + d\phi_2^- n_2 = 0, \tag{4.2}$$

where $f, \omega, \phi_i^\pm, m_i, n_i, \ i = 1, 2,$ are given in equations (3.5), (3.6).

Proof. We substitute e^i and $e^{\hat{i}}$ given by the equations (3.1) into the expressions of β, μ and ν given in equations (3.4), we obtain

$$\varphi = \left[f\omega e^7\right] + \left[\phi_1^+ m_1 + \ \phi_2^+ m_2 + \phi_1^- n_1 + \phi_2^- n_2\right],$$

as in Proposition 3.1. The terms in the brackets belong to subspaces $\Lambda^{2,1}$, and $\Lambda^{3,0}$ respectively. Note that $dfe^7 = de^7 = 0$ since the base of the multi-warped product is one dimensional. Similarly, as each F_i is three dimensional, their volume forms are closed, i.e.,

$$d\phi_1^+ = d\phi_1^- = 0.$$

Then $d\varphi = 0$ reduces to

$$d\varphi = \left[f d\omega e^7 - \phi_1^+ dm_1 - \phi_2^+ dm_2 - \phi_1^- dn_1 - \phi_2^- dn_2\right] +$$

$$+ \left[d\phi_2^+ m_2 + d\phi_2^- n_2\right], \tag{4.3}$$

where the terms in the brackets belong respectively to $\Lambda^{3,1}(M)$ and $\Lambda^{4,0}(M)$.

Proposition 4.1 is proved.

Proposition 4.2. *Let* (M, e^i) *be a* 7*-dimensional special warped-like product manifold as in Theorem* 4.1. *If* $d * \varphi = 0$, *then the following two conditions must be satisfied*:

$$\omega d\omega = 0, \tag{4.4}$$

$$f df \omega^2 = \left(d\phi_2^+ \tilde{m_2} + d\phi_2^- \tilde{n_2}\right) e^7, \tag{4.5}$$

where $f, \omega, \phi_i^\pm, \tilde{m_i}, \tilde{n_i}$, $i = 1, 2$, *are given in equations* (3.5), (3.6).

Proof. We can write

$$*\varphi = \left[-\frac{1}{2} f^2 \omega^2\right] + \left[\phi_1^+ \tilde{m_1} + \ \phi_2^+ \tilde{m_2} + \phi_1^- \tilde{n_1} + \phi_2^- \tilde{n_2}\right] e^7,$$

as in Proposition 3.1. The terms in the brackets belong to subspaces $\Lambda^{4,0}$ and $\Lambda^{3,1}$ respectively. By using the previous proposition arguments, $d * \varphi = 0$ reduces to

$$d * \varphi = \left[-f^2 \omega d\omega\right] + \left[-f df \omega^2 + \left(d\phi_2^+ \tilde{m_2} + d\phi_2^- \tilde{n_2}\right) e^7\right],$$

where the terms in the brackets belong respectively to $\Lambda^{5,0}(M)$ and $\Lambda^{4,1}(M)$.

Proposition 4.2 is proved.

Here we prove that the equation (4.1) given in Proposition 4.1 fixes the exterior derivatives of the θ^i's and $\theta^{\hat{i}}$'s completely for the manifold M in Theorem 4.1.

Proposition 4.3. *Let* (M, e^i) *be a* 7*-dimensional special warped-like product manifold as in Theorem* 4.1. *If*

$$f d\omega e^7 - \phi_1^+ dm_1 - \phi_2^+ dm_2 - \phi_1^- dn_1 - \phi_2^- dn_2 = 0,$$

then

$$\begin{aligned} d\theta^1 &= \lambda_1 \theta^{23}, \qquad d\theta^2 = -\lambda_1 \theta^{13}, \qquad d\theta^3 = \lambda_1 \theta^{12}, \\ d\theta^{\hat{1}} &= \lambda_2 \theta^{\hat{2}\hat{3}}, \qquad d\theta^{\hat{2}} = -\lambda_2 \theta^{\hat{1}\hat{3}}, \qquad d\theta^{\hat{3}} = \lambda_2 \theta^{\hat{1}\hat{2}}, \end{aligned} \tag{4.6}$$

where λ_i, $i = 1, 2$, *are arbitrary nonzero constants.*

Proof. Let us write the exterior derivative m_i, n_i, $i = 1, 2$, are of the following form:

$$dm_1 = u_1 e^7, \qquad dm_2 = u_2 e^7,$$

$$dn_1 = v_1 e^7, \qquad dn_2 = v_2 e^7,$$

where u_1, u_2, v_1, v_2 are functions on B. Then we can factorize e^7 in the condition and obtain

$$[f d\omega] - \left[\phi_1^+ u_1\right] - \left[\phi_2^+ u_2\right] - \left[\phi_1^- v_1\right] - \left[\phi_2^- v_2\right] = 0. \tag{4.7}$$

In (4.7) the terms in the brackets belong to subspaces $\Lambda^{(2,1,0)} \oplus \Lambda^{(1,2,0)}$, $\Lambda^{(3,0,0)}$, $\Lambda^{(1,2,0)}$, $\Lambda^{(0,3,0)}$ and $\Lambda^{(2,1,0)}$ respectively. This implies that $u_1 = v_1 = 0$, that is,

$$dm_1 = dn_1 = 0.$$

Thus we obtain

$$f d\omega = \phi_2^+ u_2 + \phi_2^- v_2.$$

If we write explicitly ω, ϕ_2^+ and ϕ_2^-, then

$$fd(\theta^{1\hat{1}} + \theta^{2\hat{2}} + \theta^{3\hat{3}}) = (\theta^{1\hat{2}\hat{3}} + \theta^{\hat{1}2\hat{3}} + \theta^{\hat{1}\hat{2}3})u_2 + (\theta^{\hat{1}23} + \theta^{1\hat{2}3} + \theta^{12\hat{3}})v_2.$$

When we rearrange the equality,

$$(fd\theta^1 - v_2\theta^{23})\theta^{\hat{1}} - (fd\theta^{\hat{1}} + u_2\theta^{\hat{2}\hat{3}})\theta^1 +$$

$$+(fd\theta^2 + v_2\theta^{13})\theta^{\hat{2}} - (fd\theta^{\hat{2}} - u_2\theta^{\hat{1}\hat{3}})\theta^2 +$$

$$+(fd\theta^3 - v_2\theta^{12})\theta^{\hat{3}} - (fd\theta^{\hat{3}} + u_2\theta^{\hat{1}\hat{2}})\theta^3 = 0,$$

we obtain

$$d\theta^1 = \frac{v_2}{f}\theta^{23}, \qquad d\theta^2 = -\frac{v_2}{f}\theta^{13}, \qquad d\theta^3 = \frac{v_2}{f}\theta^{12}, \tag{4.8}$$

$$d\theta^{\hat{1}} = -\frac{u_2}{f}\theta^{\hat{2}\hat{3}}, \qquad d\theta^{\hat{2}} = \frac{u_2}{f}\theta^{\hat{1}\hat{3}}, \qquad d\theta^{\hat{3}} = -\frac{u_2}{f}\theta^{\hat{1}\hat{2}}. \tag{4.9}$$

If we take the exterior derivative of $d\theta^1 = \dfrac{v_2}{f}\theta^{23}$, we get

$$d\left(\frac{v_2}{f}\right)\theta^{23} + \frac{v_2}{f}d\theta^2\theta^3 - \frac{v_2}{f}\theta^2 d\theta^3 = 0.$$

Using the equations (4.8), it is seen that $d\left(\dfrac{v_2}{f}\right) = 0$, in similar way $d\left(\dfrac{u_2}{f}\right) = 0$, that is, $\dfrac{v_2}{f}, \dfrac{u_2}{f}$ are constants. This proves the Proposition 4.3 if the nonzero constants are chosen as λ_1 and λ_2.

We complete the proof of Theorem 4.1 by using the following result.

Theorem 4.2 [34]. *Any two connected, simply connected complete Riemannian manifolds of constant curvature k are isometric to each other.*

***Proof of Theorem* 4.1.** One can see that the equations (4.6) describes the Lie algebra $su(2)$, it follows that if the fibers are connected and simply connected, then they are diffeomorphic to S^3 [36, p. 127] (Section 3.65). Using the equations (4.6), it is seen that the sectional curvatures of F_1 and F_2 are positive, i.e., $K(F_i) = \dfrac{\lambda_i^2}{4} > 0$. Then by the Theorem 4.2, it follows that F_1 and F_2 are isometric to S^3 with constant curvature $k > 0$.

Theorem 4.1 is proved.

***Remark* 4.1.** For the existence of the solution, we have to find $A, B, \hat{A}, \hat{B}$ and $a(x)$ such that the equations in Propositions 4.1, 4.2 are satisfied. From the exterior derivatives of the basis 1-forms θ^i and $\theta^{\hat{i}}$, it is seen that the equation (4.4) of Proposition 4.2 holds identically. The other equations are to be solved, but instead of this computation, we will use Konishi – Naka solution in the following section.

4.1. Konishi – Naka solution. The aim of this section is to prove that Konishi – Naka metric ansatz is unique in the class of special warped-like product metrics admitting the G_2 structure determined by the fundamental 3-form given in the equation (2.3).

Now we recall that the Konishi – Naka solution [24] on

$$M = SU(2) \times SU(2) \times R$$

is given by the following (global) orthonormal frame:

$$\begin{aligned} e^i &= A(x)\theta^i, && i = 1,2,3, \\ e^{\hat{i}} &= \widehat{A}\left(\theta^{\hat{i}} - \frac{1}{2}\theta^i\right), && i = 1,2,3, \\ & e^7 = dx, \end{aligned} \tag{4.10}$$

where the local sections of the cotangent bundle of each $SU(2)$ respectively by θ^i, $\theta^{\hat{i}}$ and the functions $A(x)$, $\widehat{A}$ satisfy the differential equations

$$\frac{dA}{dx} = \frac{\widehat{A}}{2A}, \quad \frac{d\widehat{A}}{dx} = 1 - \frac{\widehat{A}^2}{4A^2}. \tag{4.11}$$

Thus the metric is

$$g = A(x)^2 \sum_{i=1}^{3} (\theta^i)^2 + \widehat{A}(x)^2 \left(\sum_{i=1}^{3} \left[\theta^{\hat{i}} - \frac{1}{2}\theta^i \right] \right)^2 + dx^2. \tag{4.12}$$

We can take $e^7 = dx$, as in [2]. We will show that we can also set $B = 0$ in the equation (3.1) by a frame transformation and obtain exactly the Konishi – Naka metrical ansatz. An orthogonal transformation of the cotangent frame $\{e^i, e^{\hat{i}}\}$ is given by

$$\begin{aligned} \tilde{e}^i &= P^i_j e^j + Q^i_j e^{\hat{j}}, \quad i = 1,2,3, \\ \tilde{e}^{\hat{i}} &= \hat{P}^i_j e^j + \hat{Q}^i_j e^{\hat{j}}, \quad i = 1,2,3, \end{aligned}$$

where P, Q, $\hat{P}$, $\hat{Q}$ satisfy

$$PP^t + QQ^t = I, \qquad P\hat{P}^t + Q\hat{Q}^t = 0, \qquad \hat{P}\hat{P}^t + \hat{Q}\hat{Q}^t = I.$$

The new basis elements $\tilde{e}^i$, $\tilde{e}^{\hat{i}}$ can be written now as

$$\tilde{e}^i = \tilde{A}\theta^i + \tilde{B}\theta^{\hat{i}}, \qquad \tilde{e}^{\hat{i}} = \tilde{\hat{A}}\theta^i + \tilde{\hat{B}}\theta^{\hat{i}}, \tag{4.13}$$

where

$$\begin{aligned} \tilde{A} &= AP + \hat{A}Q, \qquad \tilde{B} = BP + \hat{B}Q, \\ \tilde{\hat{A}} &= A\hat{P} + \hat{A}\hat{Q}, \qquad \tilde{\hat{B}} = B\hat{P} + \hat{B}\hat{Q}. \end{aligned} \tag{4.14}$$

We will now show that we can set $\tilde{B} = 0$ by an orthogonal transformation. Note that if B is nonzero, but $\hat{B}$ is zero, then, $\tilde{B} = 0$ gives $BP = 0$, and since B is a scalar, the matrix P is identically zero. From (4.13) it follows that Q is a unitary hence nonsingular matrix and $\hat{Q}$ is identically zero. Finally the last equation in (4.14) implies that $\hat{P}$ is also a unitary matrix. But since in (4.14), the quantities $\tilde{A}$, $\tilde{B}$, $\tilde{\hat{A}}$, $\tilde{\hat{A}}$ are scalars, it follows that the orthogonal matrices $\hat{P}$ and Q are proportional to identity. It follows that the transformation interchanges the roles of the subspaces.

Assuming now that both B and $\hat{B}$ are nonzero, the equation $\tilde{B} = BP + \hat{B}Q = 0$ implies that the matrix P is proportional to the matrix Q, i.e., $P = -\dfrac{\hat{B}}{B}Q$. Substituting this in $\tilde{A}$, we see that $\tilde{A}I = \left(\hat{A} - \dfrac{A\hat{B}}{B}\right) Q$ hence $Q = Q_0(x,y)I$, that is, Q is the proportional to identity. Then from the first equation in (4.13), we can determine Q_0 and obtain P and Q as

$$Q = \pm\frac{B}{\sqrt{B^2 + \hat{B}^2}}I, \quad P = \mp\frac{\hat{B}}{\sqrt{B^2 + \hat{B}^2}}I.$$

As $\hat{P} = \dfrac{B}{\hat{B}}\hat{Q}$ and substituting in $\hat{A}$ we see that $\hat{Q}$ is also proportional to identity and determine $\hat{P}$ and $\hat{Q}$ as

$$\hat{Q} = \epsilon\frac{\hat{B}}{\sqrt{B^2 + \hat{B}^2}}I \quad \text{and} \quad \hat{P} = \epsilon\frac{B}{\sqrt{B^2 + \hat{B}^2}}I,$$

where $\epsilon^2 = 1$. The transformation matrix

$$\begin{pmatrix} P & Q \\ \hat{P} & \hat{Q} \end{pmatrix} = \frac{1}{\sqrt{B^2 + \hat{B}^2}}\begin{pmatrix} \mp\hat{B}I & \pm BI \\ \epsilon BI & \epsilon\hat{B}I \end{pmatrix}$$

is clearly orthogonal and the coefficients of the new frame are

$$\tilde{A} = \mp\frac{f}{\sqrt{B^2 + \hat{B}^2}}, \qquad \tilde{B} = 0,$$

$$\tilde{\hat{A}} = \epsilon\frac{AB + \hat{A}\hat{B}}{\sqrt{B^2 + \hat{B}^2}}, \qquad \tilde{\hat{B}} = \epsilon\sqrt{B^2 + \hat{B}^2}.$$

If we choose the (global) orthonormal frame as in the equation (4.10), then we can see that

$$A = A(x), \qquad B = 0,$$

$$\hat{A} = -\frac{1}{2}\hat{A}(x), \qquad \hat{B} = \hat{A}(x),$$

$$a = a(x) = 1,$$

where $A(x), \hat{A}(x)$ satisfy the condition given in (4.11). By a straight forward computation using the equations (4.11), it can be seen that the conditions given in Propositions 4.1, 4.2 are satisfied, hence we obtain a direct proof that the solution given in [2] is a G_2 metric. Thus we have the following corollary which implies that the Konishi – Naka solution is unique up to gauge transformations.

***Corollary* 4.1.** *Let M be a special warped-like product manifold. Consider the G_2 holonomy structure determined by the fundamental 3-form given in the equations (2.3) on M. Then there exists a unique metric in the class of special warped-like product metrics admitting this special G_2 structure and the metric is obtained as given in the equations (4.12) up to gauge transformation.*

Let us consider the extension of the holonomy concept in 7-dimensional manifolds, that is, if we replace the condition from G_2 holonomy to weak G_2 holonomy on M, then we obtain that the fibers are the same (S^3) for this special warped-like product as in Section 4.

5. Special warped-like product manifolds with weak G_2 holonomy. We now consider the weak holonomy G_2 for (3+3+1) decomposition. It is proved that under the same global assumptions in Section 4, the fibers are also isometric to S^3.

Theorem 5.1. *Let (M, e^i) be 7-dimensional special warped-like product manifold as in Theorem 4.1. If $d\varphi = \lambda * \varphi$ with $\lambda \neq 0$, then F_1 and F_2 are also isometric to S^3 with constant curvature $k > 0$.*

Let us find the projections of the 4-form $d\varphi$ into subspaces of $\Lambda^4(M)$ under the warped-like product structure.

Proposition 5.1. *Let (M, e^i) be a 7-dimensional special warped-like product manifold as in Theorem 4.1. If $d\varphi = \lambda * \varphi$ with $\lambda \neq 0$, then the following two conditions must be satisfied*:

$$f d\omega e^7 - \sum_{i=1}^{2} \left(\phi_i^+ dm_i + \phi_i^- dn_i\right) = \lambda \sum_{i=1}^{2} \left(\phi_i^+ \tilde{m}_i + \phi_i^- \tilde{n}_i\right) e^7, \tag{5.1}$$

$$d\phi_2^+ m_2 + d\phi_2^- n_2 = -\frac{1}{2}\lambda f^2 \omega^2, \tag{5.2}$$

where $f, \omega, \phi_i^\pm, m_i, n_i$, $i = 1, 2$, are given in equations (3.5), (3.6).

Proof. As similarly obtained before, the exterior derivative of φ can be written

$$d\varphi = \left[f d\omega e^7 - \phi_1^+ dm_1 - \phi_2^+ dm_2 - \phi_1^- dn_1 - \phi_2^- dn_2\right] + \\ + \left[d\phi_2^+ m_2 + d\phi_2^- n_2\right],$$

where the terms in the brackets belong respectively to $\Lambda^{3,1}(M)$ and $\Lambda^{4,0}(M)$. Also

$$*\varphi = \left[-\frac{1}{2} f^2 \omega^2\right] + \left[\left(\phi_1^+ \tilde{m}_1 + \phi_2^+ \tilde{m}_2 + \phi_1^- \tilde{n}_1 + \phi_2^- \tilde{n}_2\right) e^7\right],$$

has the terms in the brackets belong respectively to $\Lambda^{4,0}(M)$ and $\Lambda^{3,1}(M)$. If we impose the conditions, this gives us the two equations of Proposition 5.1.

In the following, it is proved that the equation (5.1) given in Proposition 5.1 fixes the exterior derivatives of the θ^i's and $\theta^{\hat{i}}$'s completely for the manifold M in Theorem 5.1.

Proposition 5.2. *Let (M, e^i) be a 7-dimensional special warped-like product manifold as in Theorem 4.1. If*

$$fd\omega e^7 - \sum_{i=1}^{2}\left(\phi_i^+ dm_i + \phi_i^- dn_i\right) = \lambda \sum_{i=1}^{2}\left(\phi_i^+ \tilde{m}_i + \phi_i^- \tilde{n}_i\right) e^7,$$

then

$$d\theta^1 = \lambda_1 \theta^{23}, \quad d\theta^2 = -\lambda_1 \theta^{13}, \quad d\theta^3 = \lambda_1 \theta^{12},$$

$$d\theta^{\hat{1}} = \lambda_2 \theta^{\hat{2}\hat{3}}, \quad d\theta^{\hat{2}} = -\lambda_2 \theta^{\hat{1}\hat{3}}, \quad d\theta^{\hat{3}} = \lambda_2 \theta^{\hat{1}\hat{2}},$$

where λ_1 *and* λ_2 *are arbitrary nonzero constants.*

Proof. Consider the exterior derivative $m_i, n_i, \ i = 1, 2$, as

$$dm_1 = u_1 e^7, \quad dm_2 = u_2 e^7,$$

$$dn_1 = v_1 e^7, \quad dn_2 = v_2 e^7,$$

where u_1, u_2, v_1, v_2 are functions on B. Then we can factorize e^7 in the condition and obtain

$$[fd\omega] - \left[\phi_1^+ (u_1 + \lambda \tilde{m}_1)\right] - \left[\phi_2^+ (u_2 + \lambda \tilde{m}_2)\right] -$$

$$- \left[\phi_1^- (v_1 + \lambda \tilde{n}_1)\right] - \left[\phi_2^- (v_2 + \lambda \tilde{n}_2)\right] = 0. \tag{5.3}$$

In (5.3) the terms in the brackets belong to subspaces $\Lambda^{(2,1,0)} \oplus \Lambda^{(1,2,0)}$, $\Lambda^{(3,0,0)}$, $\Lambda^{(1,2,0)}$, $\Lambda^{(0,3,0)}$ and $\Lambda^{(2,1,0)}$ respectively. This implies that

$$u_1 + \lambda \tilde{m}_1 = v_1 + \lambda \tilde{n}_1 = 0.$$

Thus we obtain

$$fd\omega = \phi_2^+ (u_2 + \lambda \tilde{m}_2) + \phi_2^- (v_2 + \lambda \tilde{n}_2).$$

If we write explicitly ω, ϕ_2^+ and ϕ_2^-, then

$$fd(\theta^{1\hat{1}} + \theta^{2\hat{2}} + \theta^{3\hat{3}}) = (\theta^{1\hat{2}\hat{3}} + \theta^{\hat{1}2\hat{3}} + \theta^{\hat{1}\hat{2}3})(u_2 + \lambda \tilde{m}_2) +$$

$$+ (\theta^{\hat{1}23} + \theta^{1\hat{2}3} + \theta^{12\hat{3}})(v_2 + \lambda \tilde{n}_2).$$

When we rearrange the equality,

$$\left(fd\theta^1 - (v_2 + \lambda \tilde{n}_2)\theta^{23}\right)\theta^{\hat{1}} - \left(fd\theta^{\hat{1}} + (u_2 + \lambda \tilde{m}_2)\theta^{\hat{2}\hat{3}}\right)\theta^1 +$$

$$+ \left(fd\theta^2 + (v_2 + \lambda \tilde{n}_2)\theta^{13}\right)\theta^{\hat{2}} - \left(fd\theta^{\hat{2}} - (u_2 + \lambda \tilde{m}_2)\theta^{\hat{1}\hat{3}}\right)\theta^2 +$$

$$+ \left(fd\theta^3 - (v_2 + \lambda \tilde{n}_2)\theta^{12}\right)\theta^{\hat{3}} - \left(fd\theta^{\hat{3}} + (u_2 + \lambda \tilde{m}_2)\theta^{\hat{1}\hat{2}}\right)\theta^3 = 0,$$

we obtain

$$d\theta^1 = \frac{v_2 + \lambda\tilde{n}_2}{f}\theta^{23}, \qquad d\theta^2 = -\frac{v_2 + \lambda\tilde{n}_2}{f}\theta^{13}, \qquad d\theta^3 = \frac{v_2 + \lambda\tilde{n}_2}{f}\theta^{12}, \tag{5.4}$$

$$d\theta^{\hat{1}} = -\frac{u_2 + \lambda\tilde{m}_2}{f}\theta^{\hat{2}\hat{3}}, \qquad d\theta^{\hat{2}} = \frac{u_2 + \lambda\tilde{m}_2}{f}\theta^{\hat{1}\hat{3}}, \qquad d\theta^{\hat{3}} = -\frac{u_2 + \lambda\tilde{m}_2}{f}\theta^{\hat{1}\hat{2}}. \tag{5.5}$$

If we take the exterior derivative of $d\theta^1 = \left(\frac{v_2 + \lambda\tilde{n}_2}{f}\right)\theta^{23}$, we get

$$d\left(\frac{v_2 + \lambda\tilde{n}_2}{f}\right)\theta^{23} + \left(\frac{v_2 + \lambda\tilde{n}_2}{f}\right)d\theta^2\theta^3 - \left(\frac{v_2 + \lambda\tilde{n}_2}{f}\right)\theta^2 d\theta^3 = 0.$$

Using the equation (5.4), it is seen that $d\left(\frac{v_2 + \lambda\tilde{n}_2}{f}\right) = 0$, in similar way $d\left(\frac{u_2 + \lambda\tilde{m}_2}{f}\right) = 0$, that is, $\frac{v_2 + \lambda\tilde{n}_2}{f}, \frac{u_2 + \lambda\tilde{m}_2}{f}$ are constants. If the nonzero constants are chosen as λ_i, $i = 1, 2$, this proves the Proposition 5.2.

***Proof of Theorem* 5.1.** It can be proved in similar way in the proof of Theorem 4.1.

Finally we obtain the following main result for the 7-dimensional special warped-like product manifolds with (weak) G_2 holonomy.

Theorem 5.2. *Let M be diffeomorphic to $F \times B$, where the base B is a one dimensional Riemannian manifold diffeomorphic to R, the fibre F is a 6-manifold of the form $F = F_1 \times F_2$, and F_i, $i = 1, 2$, are complete, connected and simply connected 3-manifolds. Let the metric on M be a special warped-like product metric* (3.1). *If M is the manifold with the G_2 holonomy or with the weak G_2 holonomy determined by the fundamental 3-form* (3.2), *then the fibers F_i's are isometric to S^3 with constant curvature $k > 0$. Also there exists a unique metric in the class of special warped-like product metrics admitting the G_2 holonomy, and the metric is written as given* (4.12) *up to gauge transformation.*

6. Conclusions. In this paper we define warped-like product metrics as a generalization of multiply warped products and study special type of these metrics for G_2 cases. Different types of fibers-base decompositions will be investigated in the next studies. We believe that our approach of the warped-like product metrics will be an important notion for the manifolds with special holonomies. Some other interesting results and further connections for the other holonomies wait to be explored.

Acknowledgments. We would like to thank Prof. Dr. Ayse H. Bilge for many fruitful discussions. This work was initiated in the Humboldt University (visiting research), Department of Mathematics in Berlin. The author also thanks Prof. Dr. Thomas Friedrich and Prof. Dr. Ilka Agricola for providing support and all facilities.

1. *Agricola, I.* The Srni lectures on non-integrable geometries with torsion // Arch. Math. – 2006. – **42**. – P. 5 – 84.
2. *Konishi Y., Naka M.* Coset construction of Spin (7), G_2 gravitational instantons // Class. Quantum Grav. – 2001. – **18**. – P. 5521 – 5544.
3. *Cartan E.* Sur les varietes a connexion affine et la theorie de la relativite generalisee I - II // Ann. sci. Ecol. norm. supér. – 1923. – **40**. – P. 325 – 412; 1924. – **41**. – P. 1 – 25.
4. *Cartan E.* Sur une classe remarquable d'espaces de Riemann // Bull. Soc. math. France. – 1926. – **54**. – P. 214 – 264; 1927. – **55**. – P. 114 – 134.

5. *Berger M.* Sur les groupes d'holonomie homogиnes de variйtиs a conexion affine et des varits riemanniennes // Bull. Soc. math. France. – 1955. – **283**. – P. 279 – 330.
6. *Alekseevskii D.* Riemannian spaces with unusual holonomy groups // Funct. Anal. and Appl. – 1968. – **2**. – P. 97 – 105.
7. *Gray A., Brown R. B.* Riemannian manifolds with holonomy group Spin(9) // Different. Geom. in honor of K. Yano. – Tokyo: Kinokuniya, 1972. – P. 41 – 59.
8. *Joyce D.* Compact manifolds with special holonomy. – Oxford: Oxford Univ. Press, 2000.
9. *Bryant R. L.* Metrics with exceptional holonomy // Ann. Math. – 1987. – **126**. – P. 525 – 576.
10. *Bryant R. L., Salamon S. M.* On the construction of some complete metrics with exceptional holonomy // Duke Math. J. – 1989. – **58**. – P. 829 – 850.
11. *Joyce D.* Compact 8-manifolds with holonomy Spin (7) // Invent. math. – 1996. – **123**. – P. 507 – 552.
12. *Gibbons G. W., Page D. N., Pope C. N.* Einstein metrics on S^3, R^3 and R^4 bundles // Communs Math. Phys. – 1990. – **127**. – P. 529 – 553.
13. *Bakas I., Floratos E. G., Kehagias A.* Octonionic gravitational instantons // Phys. Lett. B. – 1998. – **445**. – P. 69 – 76.
14. *Floratos E. G., Kehagias A.* Eight-dimensional self-dual spaces // Phys. Lett. B. – 1998. – **427**. – P. 283 – 290.
15. *Cvetic M., Gibbons G. W., Lu H., Pope C. N.* New complete non-compact Spin (7) manifolds // Nucl. Phys. B. – 2002. – **620**. – P. 29.
16. *Kanno H., Yasui Y.* On Spin (7) holonomy metric based on $SU(3)/U(1)$: I-II // J. Geom. and Phys. – 2002. – **43**. – P. 293 – 309; 310 – 326.
17. *Uğuz S., Bilge A. H.* $(3+3+2)$ Warped-like product manifolds with Spin (7) holonomy // J. Geom. and Phys. – 2011. – **61**. – P. 1093 – 1103.
18. *Uğuz S.* Lee form and special warped-like product manifolds with locally conformally parallel Spin(7) structure // Ann. Global Anal. and Geom. – 2013. – **43**, Issue 2. – P. 123 – 141.
19. *Uğuz S.* Conformally parallel Spin(7) structures on solvmanifolds // Turk. J. Math. – 2013.
20. *Bilge A. H., Dereli T., Kocak S.* Maximal linear subspaces of strong self-dual 2-forms and the Bonan 4-form // Linear Algebra and its Appl. – 2011. – **434**. – P. 1200 – 1214.
21. *Gray A.* Weak holonomy groups // Math. Zentr. – 1971. – **123**. – P. 290 – 300.
22. *Friedrich T.* Weak Spin(9)-structures on 16-dimensional Riemannian manifolds // Asian J. Math. – 2001. – **5**. – P. 129 – 160.
23. *Swann A.* Weakening holonomy. – Preprint ESI № 816. – 2000.
24. *Yasui Y., OotsukaT.* Spin(7) holonomy manifold and superconncetion // Class. Quantum Grav. – 2001. – **18**. – P. 807 – 816.
25. *Bonan E.* Sur les varieties riemanniennes a groupe d'holonomie G_2 ou Spin(7) // C. r. Acad. sci. Paris. – 1966. – **262**. – P. 127 – 129.
26. *Cabrera F., Monar M., Swann A.* Classification of G_2-structures // J. London Math. Soc. – 1996. – **53**. – P. 407 – 416.
27. *Fernandez M., Gray A.* Riemannian manifolds with structure group G_2 // Ann. mat. pura ed appl. – 1982. – **132**. – P. 19 – 45.
28. *Friedrich T., Ivanov S.* Killing spinor equations in dimension 7 and geometry of integrable G_2-manifolds // J. Geom. and Phys. – 2003. – **48**. – P. 1 – 11.
29. *Schwachhöfer L. J.* Riemannian, symplectic and weak holonomy // Ann. Global Anal. and Geom. – 2000. – **18**. – P. 291 – 308.
30. *O'Neil B.* Semi Riemannian geometry. – London: Acad. Press, Inc., 1983.
31. *Flores J. L., Sanchez M.* Geodesic connectedness of multiwarped spacetimes // J. Different. Equat. – 2002. – **186**. – P. 1 – 30.
32. *Bilge A. H., Uğuz S.* A generalization of warped product manifolds with Spin (7) holonomy // Geom. and Phys. (XVI Int. Fall Workshop. AIP Conf. Proc., 1023). – 2008. – P. 165 – 171.
33. *Hempel J.* 3-Manifolds. – Princeton Univ. Press, 1976.
34. *Kobayashi S., Nomizu K.* Foundations of differential geometry . – Intersci., 1969. – Vol. I.
35. *Salamon S. M.* Riemannian geometry and holonomy groups // Pitman Res. Notes Math. Ser. – 1989. – **201**.
36. *Warner F. W.* Foundations of differentiable manifolds and Lie groups. – Springer, 1983.

Received 20.11.11,
after revision — 20.05.13