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NEW SHARP INEQUALITIES OF OSTROWSKI TYPE AND GENERALIZED TRAPEZOID TYPE FOR RIEMANN–STIELTJES INTEGRALS AND APPLICATIONS

НОВІ ТОЧНІ НЕРІВНОСТІ ТИПУ ОСТРОВСЬКОГО ТА ТИПУ УЗАГАЛЬНЕНОГО ТРАПЕЦОЇДА ДЛЯ ІНТЕГРАЛІВ РІМАНА – СТІЛЬТЬЄСА ТА ЇХ ЗАСТОСУВАННЯ

We prove new sharp weighted generalizations of Ostrowski type and generalized trapezoid type inequalities for Riemann – Stieltjes integrals. Several related inequalities are deduced and investigated. New Simpson-type inequalities for the $\mathcal{RS}$-integral obtained. Finally, as an application, an error estimate is given for a general quadrature rule for the $\mathcal{RS}$-integral via the Ostrowski — generalized trapezoid quadrature formula.

Доведено нові точні зважені узагальнення нерівностей типу Островського та типу узагальненого трапецоїда для інтегралів Рімана – Стільтьєса. Отримано та досліджено кілька близьких нерівностей. Отримано нові нерівності типу Сімпсона для $\mathcal{RS}$-інтеграла. Як застосування наведено оцінку похибки загального правила квадратур для $\mathcal{RS}$-інтеграла із використанням квадратурної формули Островського — узагальненого трапецоїда.

1. Introduction. In order to approximate the Riemann – Stieltjes integral $\int_a^b f(t)du(t)$, Dragomir [12] has introduced the following (general) quadrature rule:

$$\mathcal{D}\left(f,u;x\right):=f(x)\left[u\left(b\right)-u(a)\right]-\int\limits_a^b f(t)du(t).$$

After that, many authors have studied this quadrature rule under various assumptions of integrands and integrators. In the following, we give a summary of these results: let $f,u\colon[a,b]\to\mathbb{R}$ be as follow:

(1) f is of r-H_f-Hölder type on $[a,b]$, where $H_f>0$ and $r\in(0,1]$ are given,
(1′) u is of s-H_u-Hölder type on $[a,b]$, where $H_u>0$ and $s\in(0,1]$ are given,
(2) f is of bounded variation on $[a,b]$,
(2′) u is of bounded variation on $[a,b]$,
(3) f is L_f-Lipschitz on $[a,b]$,
(3′) u is L_u-Lipschitz on $[a,b]$,
(4) f is monotonic nondecreasing on $[a,b]$,
(4′) u is monotonic nondecreasing on $[a,b]$,
(5) f is $L_{1,f}$-Lipschitz on $[a,x]$ and $L_{2,f}$-Lipschitz on $[x,b]$,
(5′) u is $L_{1,u}$-Lipschitz on $[a,x]$ and $L_{2,u}$-Lipschitz on $[x,b]$,
(6) f is monotonic nondecreasing on $[a,x]$ and $[x,b]$,
(6′) u is monotonic nondecreasing on $[a,x]$ and $[x,b]$,
(7) f is absolutely continuous on $[a,b]$,
(8) $|f'|$ is convex on $[a,b]$.

Then, the following inequalities hold under the corresponding assumptions:

$$|\mathcal{D}(f,u;x)| \leq$$

$$\leq \begin{cases} H_f \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^r \bigvee_a^b(u), & (1),(2')\ [13] \\ H_u \begin{cases} [(x-a)^s + (b-x)^s]\left[\frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^x(f) - \bigvee_x^b(f)\right|\right], \\ [(x-a)^{qs} + (b-x)^{qs}]^{1/q}\left[(\bigvee_a^x(f))^p + \left(\bigvee_x^b(f)\right)^p\right]^{1/p}, \\ \qquad p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^s \bigvee_a^b(f), \end{cases} & (1'),(2)\ [14] \\ \frac{L_u H_f}{r+1}\left[(x-a)^{r+1} + (b-x)^{r+1}\right], & (1),(3')\ [6] \\ \frac{L_f H_u}{s+1}\left[(x-a)^{s+1} + (b-x)^{s+1}\right], & (1'),(3)\ [6] \\ L_u L_f \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2, & (3),(3')\ [6] \\ \max\{L_{1,u}, L_{2,u}\} \times \begin{cases} \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right][f(b)-f(a)], \\ \left[\frac{f(b)-f(a)}{2} + \frac{1}{2}\left|f(x) - \frac{f(a)+f(b)}{2}\right|\right](b-a), \end{cases} & (5'),(6)\ [6] \\ \max\{L_{1,f}, L_{2,f}\} \times \begin{cases} \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right][u(b)-u(a)], \\ \left[\frac{u(b)-u(a)}{2} + \frac{1}{2}\left|u(x) - \frac{u(a)+u(b)}{2}\right|\right](b-a), \end{cases} & (5),(6')\ [6] \\ H_f \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^r [u(b)-u(a)], & (1),(4')\ [11] \\ H_u \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^s [f(b)-f(a)], & (1'),(4)\ [11] \\ \sup_{t\in[a,x]}\{(x-t)\,\mu(f;x,t)\}\bigvee_a^x(u) + \sup_{t\in[x,b]}\{(t-x)\,\mu(f;x,t)\}\bigvee_x^b(u), & (2'),(7)\ [7] \\ \frac{1}{2}\left[(x-a)\bigvee_a^x(u)\,\|f'\|_{\infty,[a,x]} + (b-x)\bigvee_x^b(u)\,\|f'\|_{\infty,[x,b]}\right] + \\ \qquad + \frac{1}{2}|f'(x)|\left[(x-a)\bigvee_a^x(u) + (b-x)\bigvee_x^b(u)\right]. & (2'),(7),(8)\ [7] \end{cases} \tag{1.1}$$

More details about each inequality of the above, the reader may refer to the corresponding mentioned references and the references therein.

From a different view point, the authors of [14] considered the problem of approximating the Stieltjes integral $\int_a^b f(t)du(t)$ via the generalized trapezoid rule

$$\mathcal{T}(f,u;x) := [u(x)-u(a)]\,f(a) + [(b)-u(x)]\,f(b) - \int\limits_a^b f(t)du(t),$$

$$|\mathcal{T}(f,u;x)| \le$$

$$\le \begin{cases} H_u\left[\dfrac{b-a}{2}+\left|x-\dfrac{a+b}{2}\right|\right]^r \bigvee_a^b(f), & (1'),(2)\ [15] \\ H_f\begin{cases} [(x-a)^s+(b-x)^s]\left[\dfrac{1}{2}\bigvee_a^b(u)+\dfrac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right], \\ [(x-a)^{qs}+(b-x)^{qs}]^{1/q}\left[(\bigvee_a^x(u))^p+\left(\bigvee_x^b(u)\right)^p\right]^{1/p}, & (1),(2')\ [8] \\ \qquad p>1, \qquad \dfrac{1}{p}+\dfrac{1}{q}=1, \\ \left[\dfrac{b-a}{2}+\left|x-\dfrac{a+b}{2}\right|\right]^s \bigvee_a^b(u). \end{cases} \end{cases} \tag{1.2}$$

For new quadrature rules involving $\mathcal{RS}$-integral see the recent works [1, 2]. For other results concerning various approximation for $\mathcal{RS}$-integral under various assumptions on f and u, see [3, 4, 8, 9, 15–18] and the references therein.

In the recent work [19], Z. Liu has proved sharp generalization of weighted Ostrowski type inequality for mappings of bounded variation, as follows (see also [20]):

Theorem 1.1. *Let $f\colon [a,b]\to\mathbb{R}$ be a mapping of bounded variation, $g\colon [a,b]\to[0,\infty)$ continuous and positive on (a,b). Then for any $x\in[a,b]$ and $\alpha\in[0,1]$, we have*

$$\left|\int\limits_a^b f(t)g(t)dt - \left[(1-\alpha)\,f(x)\int\limits_a^b g(t)dt + \alpha\left(f(a)\int\limits_a^x g(t)\,dt + f(b)\int\limits_x^b g(t)dt\right)\right]\right| \le$$

$$\le \left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right]\left[\frac{1}{2}\int\limits_a^b g(t)dt + \left|\int\limits_a^x g(t)dt - \frac{1}{2}\int\limits_a^b g(t)dt\right|\right]\bigvee_a^b(f), \tag{1.3}$$

where $\bigvee_a^b(f)$ denotes to the total variation of f over $[a,b]$. The constant $\left[\dfrac{1}{2}+\left|\dfrac{1}{2}-\alpha\right|\right]$ is the best possible.

For recent results concerning Ostrowski inequality for mappings of bounded variation see [11, 19–23].

The main aim in this paper, is to introduce and discuss new weighted generalizations of the Ostrowski and the generalized trapezoid inequalities for the Riemann–Stieltjes integrals.

2. Main results. We begin with the following result:

Theorem 2.1. *Let $g,u\colon [a,b]\to[0,\infty)$ be such that g is continuous and positive on $[a,b]$ and u is monotonic increasing on $[a,b]$. If $f\colon [a,b]\to\mathbb{R}$ is a mapping of bounded variation on $[a,b]$, then for any $x\in[a,b]$ and $\alpha\in[0,1]$, we have*

$$\left|(1-\alpha)\left[f(x)\int\limits_a^{(a+b)/2} g(s)du(s)+f(a+b-x)\int\limits_{(a+b)/2}^{b} g(s)du(s)\right]+\right.$$

$$\left.+\alpha\left[f(a)\int\limits_a^x g(s)du(s)+f(b)\int\limits_x^b g(s)du(s)\right]-\int\limits_a^b f(t)g(t)du(t)\right|\leq$$

$$\leq\left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right]\left[\frac{1}{2}\int\limits_a^b g(t)du(t)+\left|\int\limits_a^x g(t)du(t)-\frac{1}{2}\int\limits_a^b g(t)du(t)\right|\right]\bigvee_a^b(f), \tag{2.1}$$

where $\bigvee_a^b(f)$ denotes to the total variation of f over $[a,b]$. The constant $\left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right]$ is the best possible.

Proof. Define the mapping

$$K_{g,u}(t;x):=\begin{cases}(1-\alpha)\displaystyle\int_a^t g(s)du(s)+\alpha\int\limits_x^t g(s)du(s), & t\in[a,x],\\ (1-\alpha)\displaystyle\int_{(a+b)/2}^t g(s)du(s)+\alpha\int\limits_x^t g(s)du(s), & t\in(x,a+b-x],\\ (1-\alpha)\displaystyle\int_b^t g(s)du(s)+\alpha\int\limits_x^t g(s)du(s), & t\in(a+b-x,b].\end{cases}$$

Using integration by parts, we have the following identity:

$$\int\limits_a^b K_{g,u}(t;x)\,df(t)=\int\limits_a^x\left[(1-\alpha)\int\limits_a^t g(s)du(s)+\alpha\int\limits_x^t g(s)du(s)\right]df(t)+$$

$$+\int\limits_x^{a+b-x}\left[(1-\alpha)\int\limits_{(a+b)/2}^t g(s)du(s)+\alpha\int\limits_x^t g(s)du(s)\right]df(t)+$$

$$+\int\limits_{a+b-x}^{b}\left[(1-\alpha)\int\limits_{b}^{t}g(s)du(s)+\alpha\int\limits_{x}^{t}g(s)du(s)\right]df(t)=$$

$$=(1-\alpha)\left[f(x)\int\limits_{a}^{(a+b)/2}g(s)du(s)+f\left(a+b-x\right)\int\limits_{(a+b)/2}^{b}g(s)du(s)\right]+$$

$$+\alpha\left[f(a)\int\limits_{a}^{x}g(s)du(s)+f(b)\int\limits_{x}^{b}g(s)du(s)\right]-\int\limits_{a}^{b}f(t)g(t)du(t).$$

Using the fact that for a continuous function $p\colon[a,b]\to\mathbb{R}$ and a function $\nu\colon[a,b]\to\mathbb{R}$ of bounded variation, then the Riemann – Stieltjes integral $\int_a^b p(t)d\nu(t)$ exists and one has the inequality

$$\left|\int\limits_{a}^{b}p(t)d\nu(t)\right|\leq\sup_{t\in[a,b]}|p(t)|\bigvee_{a}^{b}(\nu)\,. \tag{2.2}$$

As f is of bounded variation on $[a,b]$, by (2.2) we have

$$\left|(1-\alpha)\left[f(x)\int\limits_{a}^{(a+b)/2}g(s)du(s)+f\left(a+b-x\right)\int\limits_{(a+b)/2}^{b}g(s)du(s)\right]+\right.$$

$$\left.+\alpha\left[f(a)\int\limits_{a}^{x}g(s)du(s)+f(b)\int\limits_{x}^{b}g(s)du(s)\right]-\int\limits_{a}^{b}f(t)g(t)du(t)\right|\leq$$

$$\leq\sup_{t\in[a,b]}|K_{g,u}\left(t;x\right)|\bigvee_{a}^{b}(f). \tag{2.3}$$

Now, define the mappings $p,q\colon[a,b]\to\mathbb{R}$ given by

$$p_1(t):=(1-\alpha)\int\limits_{a}^{t}g(s)du(s)+\alpha\int\limits_{x}^{t}g\left(s\right)du(s)\qquad t\in\left[a,x\right],$$

$$p_2(t):=(1-\alpha)\int\limits_{(a+b)/2}^{t}g(s)du(s)+\alpha\int\limits_{x}^{t}g(s)du(s),\qquad t\in\left(x,a+b-x\right],$$

$$p_3(t):=(1-\alpha)\int\limits_{b}^{t}g(s)du(s)+\alpha\int\limits_{x}^{t}g\left(s\right)du(s),\qquad t\in\left(a+b-x,b\right],$$

for all $\alpha \in [0,1]$, and $x \in [a,b]$. Since g is *positive* continuous and u is monotonic increasing on $[a,b]$ then the Riemann – Stieltjes integral $\int_a^b g(s)du(s)$ exists and *positive*. Also, since the derivative of the monotonic increasing function u is always positive, so that $(gu')(t) > 0$ a.e., it follows that, $p_1'(t),\ p_2'(t),\ p_3'(t) > 0$, almost everywhere on their corresponding domains. Therefore, we have

$$\sup_{t\in[a,x]} |K_{g,u}(t;x)| = \max\left\{(1-\alpha)\int_a^x g(s)du(s), \alpha\int_a^x g(s)du(s)\right\} =$$

$$= \left[\frac{1}{2} + \left|\frac{1}{2}-\alpha\right|\right]\int_a^x g(s)du(s),$$

$$\sup_{t\in(x,a+b-x]} |K_{g,u}(t;x)| =$$

$$= \max\left\{(1-\alpha)\int_x^{(a+b)/2} g(s)du(s), \alpha\int_x^{(a+b)/2} g(s)du(s) + \int_{(a+b)/2}^{a+b-x} g(s)du(s)\right\} =$$

$$= \frac{1}{2}\left[\int_x^{a+b-x} g(s)du(s) + (1-\alpha)\left|\int_{(a+b)/2}^{a+b-x} g(s)du(s)\right|\right],$$

and

$$\sup_{t\in(a+b-x,b]} |K_{g,u}(t;x)| = \max\left\{(1-\alpha)\int_x^b g(s)du(s), \alpha\int_x^b g(s)du(s)\right\} =$$

$$= \left[\frac{1}{2} + \left|\frac{1}{2}-\alpha\right|\right]\int_x^b g(s)du(s).$$

Thus

$$\sup_{t\in[a,b]} |K_{g,u}(t;x)| = \left[\frac{1}{2} + \left|\frac{1}{2}-\alpha\right|\right]\max\left\{\int_a^x g(s)du(s), \int_x^b g(s)du(s)\right\} =$$

$$= \left[\frac{1}{2} + \left|\frac{1}{2}-\alpha\right|\right]\left[\frac{1}{2}\int_a^b g(s)du(s) + \left|\int_a^x g(s)du(s) - \frac{1}{2}\int_a^b g(s)du(s)\right|\right]. \tag{2.4}$$

Therefore, by (2.3) and (2.4) we get (2.1). To prove that the constant $\frac{1}{2} + \left|\frac{1}{2}-\alpha\right|$ is best possible for all $\alpha \in [0,1]$, take $u(t) = t$ for all $t \in [a,b]$ and therefore, we refer to (1.1). Thus, the sharpness follows from (1.1) (consider f and g to be defined as in [19]). Hence, the proof is established and we shall omit the details.

***Corollary* 2.1.** *In Theorem* 2.1, *choose* $\alpha = 0$, *then we get*

$$\left| f(x) \int\limits_a^b g(t)du(t) - \int\limits_a^b f(t)g(t)du(t) \right| \le$$

$$\le \left[\frac{1}{2} \int\limits_a^b g(t)du(t) + \left| \int\limits_a^x g(t)du(t) - \frac{1}{2} \int\limits_a^b g(t)du(t) \right| \right] \bigvee_a^b (f). \tag{2.5}$$

A general weighted version of the above Ostrowski inequality for $\mathcal{RS}$-integrals, may be deduced as follows:

$$\left| f(x) - \frac{\int_a^b f(t)g(t)du(t)}{\int_a^b g(t)du(t)} \right| \le \left[\frac{1}{2} + \left| \frac{\int_a^x g(t)du(t)}{\int_a^b g(t)du(t)} - \frac{1}{2} \right| \right] \bigvee_a^b (f) \tag{2.6}$$

provided that $g(t) \ge 0$, *for almost every* $t \in [a,b]$ *and* $\int_a^b g(t)du(t) \ne 0$.

***Remark* 2.1.** Choosing $\alpha = 1$ in (2.1), then we get

$$\left| f(a) \int\limits_a^x g(s)du(s) + f(b) \int\limits_x^b g(s)du(s) - \int\limits_a^b f(t)g(t)du(t) \right| \le$$

$$\le \left[\frac{1}{2} \int\limits_a^b g(t)du(t) + \left| \int\limits_a^x g(t)du(t) - \frac{1}{2} \int\limits_a^b g(t)du(t) \right| \right] \bigvee_a^b (f), \tag{2.7}$$

which is 'the generalized trapezoid inequality for $\mathcal{RS}$-integrals'.

***Corollary* 2.2.** *In Theorem* 2.1, *let* $g(t) = 1$ *for all* $t \in [a,b]$. *Then we have the inequality*

$$\left| \alpha \left[(u(x) - u(a)) f(a) + ((b) - u(x)) f(b) \right] + (1-\alpha) \left[u(b) - u(a) \right] f(x) - \int\limits_a^b f(t)du(t) \right| \le$$

$$\le \left[\frac{1}{2} + \left| \frac{1}{2} - \alpha \right| \right] \left[\frac{u(b) - u(a)}{2} + \left| u(x) - \frac{u(a) + u(b)}{2} \right| \right] \bigvee_a^b (f). \tag{2.8}$$

The constant $\left[\frac{1}{2} + \left| \frac{1}{2} - \alpha \right| \right]$ *is the best possible.*

For instance,

If $\alpha = 0$, *then we get*

$$\left| [u(b) - u(a)] f(x) - \int\limits_a^b f(t)du(t) \right| \le \left[\frac{u(b) - u(a)}{2} + \left| u(x) - \frac{u(a) + (b)}{2} \right| \right] \bigvee_a^b (f). \tag{2.9}$$

If $\alpha = \frac{1}{3}$, *then we have*

$$\left| \frac{1}{3} \left\{ [u(x) - u(a)] \, f(a) + 2 \, [u(b) - u(a)] \, f(x) + [u(b) - u(x)] \, f(b) \right\} - \int\limits_a^b f(t) du(t) \right| \leq$$

$$\leq \frac{2}{3} \left[\frac{u(b) - u(a)}{2} + \left| u(x) - \frac{u(a) + u(b)}{2} \right| \right] \bigvee_a^b (f). \tag{2.10}$$

If $\alpha = \frac{1}{2}$, *then we obtain*

$$\left| \frac{1}{2} \left\{ [u(x) - u(a)] \, f(a) + [u(b) - u(a)] \, f(x) + [u(b) - u(x)] \, f(b) \right\} - \int\limits_a^b f(t) du(t) \right| \leq$$

$$\leq \frac{1}{2} \left[\frac{u(b) - u(a)}{2} + \left| u(x) - \frac{u(a) + u(b)}{2} \right| \right] \bigvee_a^b (f). \tag{2.11}$$

If $\alpha = 1$, *then we get*

$$\left| [u(x) - u(a)] \, f(a) + [u(b) - u(x)] \, f(b) - \int\limits_a^b f(t) du(t) \right| \leq$$

$$\leq \left[\frac{u(b) - u(a)}{2} + \left| u(x) - \frac{u(a) + (b)}{2} \right| \right] \bigvee_a^b (f). \tag{2.12}$$

Proof. The results follow by Theorem 2.1. It remains to prove the sharpness of (2.8). Suppose $0 \leq \alpha \leq \frac{1}{2}$, assume that (2.8) holds with constant $C_1 > 0$, i.e.,

$$\left| \alpha \left[(u(x) - u(a)) \, f(a) + ((b) - u(x)) \, f(b) \right] + (1 - \alpha) \, [u(b) - u(a)] \, f(x) - \int\limits_a^b f(t) du(t) \right| \leq$$

$$\leq C_1 \left[\frac{u(b) - u(a)}{2} + \left| u(x) - \frac{u(a) + u(b)}{2} \right| \right] \bigvee_a^b (f). \tag{2.13}$$

Let $f, u \colon [a, b] \to \mathbb{R}$ be defined as follows $u(t) = t$ and

$$f(t) = \begin{cases} 0, & t \in [a, b] \setminus \left\{ \frac{a+b}{2} \right\}, \\ \frac{1}{2}, & t = \frac{a+b}{2}, \end{cases}$$

which follows that $\bigvee_a^b(f) = 1$ and $\int_a^b f(t)du(t) = 0$, setting $x = \dfrac{a+b}{2}$ it gives by (2.13)

$$(1-\alpha)\,\frac{b-a}{2} \le C_1\frac{b-a}{2},$$

which proves that $C_1 \ge 1-\alpha$, and therefore $1-\alpha$ is the best possible for all $0 \le \alpha \le \dfrac{1}{2}$.

Now, suppose $\dfrac{1}{2} \le \alpha \le 1$ and assume that (2.8) holds with constant $C_2 > 0$, i.e.,

$$\left|\alpha\left[(u(x)-u(a))\,f(a) + ((b)-u(x))\,f(b)\right] + (1-\alpha)\,[u(b)-u(a)]\,f(x) - \int\limits_a^b f(t)du(t)\right| \le$$

$$\le C_2\left[\frac{u(b)-u(a)}{2} + \left|u(x) - \frac{u(a)+u(b)}{2}\right|\right]\bigvee_a^b(f). \tag{2.14}$$

Let $f, u\colon [a,b] \to \mathbb{R}$ be defined as follows $u(t) = t$ and

$$f(t) = \begin{cases} 0, & t \in (a,b]\,, \\ 1, & t = a, \end{cases}$$

which follows that $\bigvee_a^b(f) = 1$ and $\int_a^b f(t)du(t) = 0$, setting $x = \dfrac{a+b}{2}$ it gives by (2.14)

$$\alpha\frac{b-a}{2} \le C_2\frac{b-a}{2},$$

which proves that $C_2 \ge \alpha$, and therefore α is the best possible for all $\dfrac{1}{2} \le \alpha \le 1$. Consequently, we can conclude that the constant $\left[\dfrac{1}{2} + \left|\dfrac{1}{2} - \alpha\right|\right]$ is the best possible, for all $\alpha \in [0,1]$.

***Corollary* 2.3.** *In* (2.10), *setting* $x = \dfrac{a+b}{2}$ *then we have the following Simpson-type inequality for Riemann – Stieltjes integral*:

$$\left|\frac{1}{3}\left\{\left[u\left(\frac{a+b}{2}\right) - u(a)\right] f(a) + 2\,[u(b)-u(a)]\,f\left(\frac{a+b}{2}\right) + \right.\right.$$

$$\left.\left. + \left[u(b) - u\left(\frac{a+b}{2}\right)\right] f(b)\right\} - \int\limits_a^b f(t)du(t)\right| \le$$

$$\le \frac{2}{3}\left[\frac{u(b)-u(a)}{2} + \left|u\left(\frac{a+b}{2}\right) - \frac{u(a)+u(b)}{2}\right|\right]\bigvee_a^b(f). \tag{2.15}$$

The constant $\dfrac{2}{3}$ *is the best possible.*

***Remark* 2.2.** For recent three-point quadrature rules and related inequalities regarding Riemann – Stieltjes integrals, the reader may refer to the work [2].

***Corollary* 2.4.** *In* (2.8), *let* $u(t) = t$ *for all* $t \in [a, b]$, *then we get*

$$\left|\alpha\left((x-a)\,f(a)+(b-x)\,f(b)\right)+(1-\alpha)\,(b-a)\,f(x)-\int\limits_a^b f(t)dt\right|\le$$

$$\le\left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right]\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f). \tag{2.16}$$

For $x = \dfrac{a+b}{2}$, *we have*

$$\left|(b-a)\left[\alpha\frac{f(a)+f(b)}{2}+(1-\alpha)\,f\left(\frac{a+b}{2}\right)\right]-\int\limits_a^b f\,(t)\,dt\right|\le$$

$$\le\left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right]\frac{b-a}{2}\bigvee_a^b(f). \tag{2.17}$$

***Remark* 2.3.** Under the assumptions of Theorem 2.1, a weighted generalization of Montgomery's type identity for Riemann – Stieltjes integrals may be deduced as follows:

$$f(x)=\frac{1}{\int_a^b g(s)du(s)}\int\limits_a^b K_{g,u}\,(t;x)\,df(t)+\frac{1}{\int_a^b g(s)du(s)}\int\limits_a^b f(t)g(t)du(t),$$

for all $x \in [a, b]$, where

$$K_{g,u}\,(t;x):=\begin{cases}\displaystyle\int_a^t g(s)du(s), & t\in[a,x],\\ \displaystyle\int_b^t g(s)du(s), & t\in(x,b].\end{cases}$$

Provided that $\int_a^b g(s)du(s) \neq 0$.

3. On L-Lipschitz integrators.

Theorem 3.1. *Let* g *be as in Theorem* 2.1. *Let* $u\colon [a,b] \to [0,\infty)$ *be of bounded variation on* $[a, b]$. *If* $f\colon [a,b] \to \mathbb{R}$ *is L-Lipschitzian on* $[a, b]$, *then for any* $x \in [a, b]$ *and* $\alpha \in [0, 1]$, *we have*

$$\left|\alpha\left[f(a)\int\limits_a^x g(s)du(s)+f(b)\int\limits_x^b g(s)du(s)\right]+\right.$$

$$\left.+\,(1-\alpha)\,f(x)\int\limits_a^b g(s)du(s)-\int\limits_a^b f(t)g(t)du(t)\right|\le$$

$$\leq L \max\left\{(x-a)\sup_{t\in[a,x]}\{M(t)\},\ (b-x)\sup_{t\in[x,b]}\{N(t)\}\right\}\bigvee_a^b(u), \tag{3.1}$$

where

$$M(t) := \max\left\{(1-\alpha)\sup_{s\in[a,t]}|g(s)|,\ \alpha\sup_{s\in[t,x]}|g(s)|\right\}$$

and

$$N(t) := \max\left\{(1-\alpha)\sup_{s\in[t,b]}|g(s)|,\ \alpha\sup_{s\in[t,x]}|g(s)|\right\}.$$

Proof. By Theorem 2.1, we have the identity

$$\int_a^b K_{g,u}(t;x)\,df(t) =$$

$$= \alpha\left[f(a)\int_a^x g(s)du(s) + f(b)\int_x^b g(s)du(s)\right] +$$

$$+ (1-\alpha)\,f(x)\int_a^b g(s)du(s) - \int_a^b f(t)g(t)du(t).$$

Using the fact that for a Riemann integrable function $p\colon [c,d]\to\mathbb{R}$ and L-Lipschitzian function $\nu\colon [c,d]\to\mathbb{R}$, the inequality one has the inequality

$$\left|\int_c^d p(t)d\nu(t)\right| \leq L\int_c^d |p(t)|\,dt. \tag{3.2}$$

As f is L-Lipschitz mapping on $[a,b]$, by (3.2) we have

$$\left|\int_a^b K_{g,u}(t;x)\,df(t)\right| \leq L\int_a^b |K_{g,u}(t;x)|\,dt = L\left[\int_a^x |p(t)|\,dt + \int_x^b |q(t)|\,dt\right]. \tag{3.3}$$

However, as u is of bounded variation on $[a,b]$ and g is continuous, by (2.2) we obtain

$$|p(t)| \leq (1-\alpha)\left|\int_a^t g(s)du(s)\right| + \alpha\left|\int_x^t g(s)du(s)\right| \leq$$

$$\leq (1-\alpha)\sup_{s\in[a,t]}|g(s)|\bigvee_a^t(u) + \alpha\sup_{s\in[t,x]}|g(s)|\bigvee_t^x(u) \leq$$

$$\leq \max\left\{(1-\alpha)\sup_{s\in[a,t]}|g(s)|\,,\alpha\sup_{s\in[t,x]}|g(s)|\right\}\bigvee_a^x(u):=$$

$$:=M(t)\bigvee_a^x(u). \tag{3.4}$$

Similarly, we get

$$|q(t)|\leq \max\left\{(1-\alpha)\sup_{s\in[t,b]}|g(s)|\,,\alpha\sup_{s\in[t,x]}|g(s)|\right\}\bigvee_x^b(u):=N(t)\bigvee_x^b(u). \tag{3.5}$$

Thus, by (3.3) – (3.5), we have

$$\left|\int_a^b K_{g,u}\left(t;x\right)df(t)\right|\leq L\left[\int_a^x|p(t)|\,dt+\int_x^b|q(t)|\,dt\right]\leq$$

$$\leq L\left[\left(\int_a^x M(t)dt\right)\bigvee_a^x(u)+\left(\int_x^b N(t)dt\right)\bigvee_x^b(u)\right]\leq$$

$$\leq L\left[(x-a)\sup_{t\in[a,x]}\{M(t)\}\bigvee_a^x(u)+(b-x)\sup_{t\in[x,b]}\{N(t)\}\bigvee_x^b(u)\right]\leq$$

$$\leq L\max\left\{(x-a)\sup_{t\in[a,x]}\{M(t)\},\ (b-x)\sup_{t\in[x,b]}\{N(t)\}\right\}\bigvee_a^b(u),$$

which gives the result.

***Remark* 3.1.** In Theorem 3.1, if $g(t)=1$ for all $t\in[a,b]$. Then

$$M(t)=N(t)=\left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right],\quad \text{for all}\quad t\in[a,b].$$

***Corollary* 3.1.** *In Theorem* 3.1, *let* $g(t)=1$ *for all* $t\in[a,b]$. *Then, we have the inequality*

$$\left|\alpha\left[\left(u(x)-u(a)\right)f(a)+\left((b)-u(x)\right)f(b)\right]+(1-\alpha)\left[u(b)-u(a)\right]f(x)-\int_a^b f(t)du(t)\right|\leq$$

$$\leq L\left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right]\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u). \tag{3.6}$$

The constant $\left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right]$ *is the best possible.*

For instance,

If $\alpha = 0$, *then we get*

$$\left|[u(b) - u(a)]\, f(x) - \int\limits_a^b f(t)du(t)\right| \leq L\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b(u). \tag{3.7}$$

If $\alpha = \frac{1}{3}$, *then we obtain*

$$\left|\frac{1}{3}\left\{[u(x) - u(a)]\, f(a) + 2\,[u\,(b) - u(a)]\, f(x) + [u(b) - u(x)]\, f(b)\right\} - \int\limits_a^b f(t)du(t)\right| \leq$$

$$\leq \frac{2}{3}L\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b(u). \tag{3.8}$$

If $\alpha = \frac{1}{2}$, *then we have*

$$\left|\frac{1}{2}\left\{[u(x) - u(a)]\, f(a) + [u(b) - u(a)]\, f(x) + [u(b) - u(x)]\, f(b)\right\} - \int\limits_a^b f(t)du(t)\right| \leq$$

$$\leq \frac{1}{2}L\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b(u). \tag{3.9}$$

If $\alpha = 1$, *then we get*

$$\left|[u(x) - u(a)]\, f(a) + [u(b) - u(x)]\, f(b) - \int\limits_a^b f(t)du(t)\right| \leq$$

$$\leq L\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b(u). \tag{3.10}$$

Proof. The results follow by Theorem 3.1. It remains to prove the sharpness of (3.6). Suppose $0 \leq \alpha \leq \frac{1}{2}$, assume that (3.6) holds with constant $C_1 > 0$, i.e.,

$$\Big|\alpha\left[(u(x) - u(a))\, f(a) + ((b) - u(x))\, f(b)\right] +$$

$$+ (1-\alpha)\,[u(b) - u(a)]\, f(x) - \int\limits_a^b f(t)du(t)\Big| \leq$$

$$\leq LC_1\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u). \tag{3.11}$$

Let $f, u\colon [a,b]\to\mathbb{R}$ be defined as follows $f(t)=t-b$ and

$$u(t)=\begin{cases}0, & t\in[a,b),\\ 1, & t=b.\end{cases}$$

Therefore, f is L-Lipschitz with $L=1$ and $\bigvee_a^b(u)=1$ and $\int_a^b f(t)du(t)=0$, setting $x=\dfrac{a+b}{2}$ it gives by (3.11)

$$(1-\alpha)\,\frac{b-a}{2}\leq C_1\frac{b-a}{2},$$

which proves that $C_1\geq 1-\alpha$, and therefore $1-\alpha$ is the best possible for all $0\leq\alpha\leq\dfrac{1}{2}$.

Now, suppose $\dfrac{1}{2}\leq\alpha\leq 1$ and assume that (3.6) holds with constant $C_2>0$, i.e.,

$$\left|\alpha\left[(u(x)-u(a))\,f(a)+((b)-u(x))\,f(b)\right]+(1-\alpha)\left[u(b)-u(a)\right]f(x)-\int\limits_a^b f(t)du(t)\right|\leq$$

$$\leq LC_2\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u). \tag{3.12}$$

Let $f, u\colon [a,b]\to\mathbb{R}$ be defined as follows $f(t)=t-a$ and

$$u(t)=\begin{cases}0, & t\in[a,b]\setminus\left\{\dfrac{a+b}{2}\right\},\\ \dfrac{1}{2}, & t=\dfrac{a+b}{2}.\end{cases}$$

Therefore, f is L-Lipschitz with $L=1$ and $\bigvee_a^b(u)=1$ and $\int_a^b f(t)du(t)=0$, setting $x=\dfrac{a+b}{2}$ it gives by (3.12)

$$\alpha\frac{b-a}{2}\leq C_2\frac{b-a}{2},$$

which proves that $C_2\geq\alpha$, and therefore α is the best possible for all $\dfrac{1}{2}\leq\alpha\leq 1$. Consequently, we can conclude that the constant $\left[\dfrac{1}{2}+\left|\dfrac{1}{2}-\alpha\right|\right]$ is the best possible, for all $\alpha\in[0,1]$.

***Corollary* 3.2.** *In* (3.8), *choosing* $x=\dfrac{a+b}{2}$, *then we have the following Simpson-type inequality for* $\mathcal{RS}$*-integrals*:

$$\left|\frac{1}{3}\left\{\left[u\left(\frac{a+b}{2}\right)-u(a)\right]f(a)+2\left[u(b)-u(a)\right]f\left(\frac{a+b}{2}\right)+\right.\right.$$

$$\left.\left.+\left[u(b)-u\left(\frac{a+b}{2}\right)\right]f(b)\right\}-\int\limits_a^b f(t)du(t)\right|\leq$$

$$\leq\frac{1}{3}L\left(b-a\right)\bigvee_a^b(u). \tag{3.13}$$

The constant $\frac{1}{3}$ *is the best possible.*

***Corollary* 3.3.** *In* (3.6), *let* $u(t)=t$ *for all* $t\in[a,b]$, *then we get*

$$\left|\alpha\left(\left(x-a\right)f(a)+\left(b-x\right)f(b)\right)+\left(1-\alpha\right)\left(b-a\right)f(x)-\int\limits_a^b f(t)dt\right|\leq$$

$$\leq L\left(b-a\right)\left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right]\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]. \tag{3.14}$$

For $x=\dfrac{a+b}{2}$, *we have*

$$\left|\left(b-a\right)\left[\alpha\frac{f(a)+f(b)}{2}+\left(1-\alpha\right)f\left(\frac{a+b}{2}\right)\right]-\int\limits_a^b f\left(t\right)dt\right|\leq$$

$$\leq L\left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right]\frac{\left(b-a\right)^2}{2}. \tag{3.15}$$

4. On monotonic nondecreasing integrators.

Theorem 4.1. *Let* g,u *be as in Theorem* 3.1. *If* $f\colon[a,b]\to\mathbb{R}$ *is monotonic nondecreasing on* $[a,b]$, *then for any* $x\in[a,b]$ *and* $\alpha\in[0,1]$, *we have*

$$\left|\alpha\left[f(a)\int\limits_a^x g(s)du(s)+f(b)\int\limits_x^b g(s)du(s)\right]+\right.$$

$$\left.+\left(1-\alpha\right)f(x)\int\limits_a^b g(s)du(s)-\int\limits_a^b f(t)g(t)du(t)\right|\leq$$

$$\leq\sup_{t\in[a,x]}\left\{M(t)\right\}\left[f(x)-f(a)\right]\bigvee_a^x(u)+\sup_{t\in[x,b]}\left\{N(t)\right\}\left[f(b)-f(x)\right]\bigvee_x^b(u), \tag{4.1}$$

where $M(t)$ *and* $N(t)$ *are defined in Theorem* 3.1.

Proof. Using the identity

$$\alpha\left[f(a)\int\limits_a^x g(s)du(s)+f(b)\int\limits_x^b g(s)du(s)\right]+$$

$$+\left(1-\alpha\right)f(x)\int\limits_a^b g(s)du(s)-\int\limits_a^b f(t)g(t)du(t)=$$

$$=\int\limits_a^b K_{g,u}\left(t;x\right)df(t).$$

It is well-known that for a monotonic nondecreasing function $\nu\colon[a,b]\to\mathbb{R}$ and continuous function $p\colon[a,b]\to\mathbb{R}$, one has the inequality

$$\left|\int\limits_a^b p(t)d\nu(t)\right|\le\int\limits_a^b\left|p(t)\right|d\nu(t). \tag{4.2}$$

As f is monotonic nondecreasing on $[a,b]$, by (4.2) we have

$$\left|\int\limits_a^b K_{g,u}\left(t;x\right)df(t)\right|\le\int\limits_a^b\left|K_{g,u}\left(t;x\right)\right|df(t)=$$

$$=\int\limits_a^x\left|p(t)\right|df(t)+\int\limits_x^b\left|q(t)\right|df(t). \tag{4.3}$$

Now, as u is of bounded variation on $[a,b]$ and g is continuous, by (3.4), (3.5) we obtain

$$\begin{aligned}|p(t)|&\le M(t)\bigvee_a^x(u),\\ |q(t)|&\le N(t)\bigvee_x^b(u).\end{aligned} \tag{4.4}$$

Thus, by (4.3) and (4.4), we get

$$\left|\int\limits_a^b K_{g,u}\left(t;x\right)df(t)\right|\le\int\limits_a^x\left|p(t)\right|df(t)+\int\limits_x^b\left|q(t)\right|df(t)\le$$

$$\le\left(\int\limits_a^x M(t)df(t)\right)\bigvee_a^x(u)+\left(\int\limits_x^b N(t)df(t)\right)\bigvee_x^b(u)\le$$

$$\leq \sup_{t\in[a,x]}\{M(t)\}\,[f(x)-f(a)]\bigvee_a^x(u)+\sup_{t\in[x,b]}\{N(t)\}\,[f(b)-f(x)]\bigvee_x^b(u),$$

which gives the result.

***Corollary* 4.1.** *In Theorem* 4.1, *let* $g(t)=1$ *for all* $t\in[a,b]$. *Then, we have the inequality*

$$\left|\alpha\left[(u(x)-u(a))\,f(a)+((b)-u(x))\,f(b)\right]+(1-\alpha)\left[u(b)-u(a)\right]f(x)-\int_a^b f(t)du(t)\right|\leq$$

$$\leq\left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right]\left\{[f(x)-f(a)]\bigvee_a^x(u)+[f(b)-f(x)]\bigvee_x^b(u)\right\}\leq$$

$$\leq\left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right]\left[\frac{f(b)-f(a)}{2}+\left|f(x)-\frac{f(a)+f(b)}{2}\right|\right]\bigvee_a^b(u). \qquad (4.5)$$

For the last inequality, the constant $\left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right]$ *is the best possible.*

For instance,

If $\alpha=0$, *then we have*

$$\left|[u(b)-u(a)]\,f(x)-\int_a^b f(t)du(t)\right|\leq$$

$$\leq[f(x)-f(a)]\bigvee_a^x(u)+[f(b)-f\,(x)]\bigvee_x^b(u)\leq$$

$$\leq\left[\frac{f(b)-f(a)}{2}+\left|f(x)-\frac{f(a)+f\,(b)}{2}\right|\right]\bigvee_a^b(u). \qquad (4.6)$$

If $\alpha=\frac{1}{3}$, *then we get*

$$\left|\frac{1}{3}\left\{[u(x)-u(a)]\,f(a)+2\,[u\,(b)-u(a)]\,f(x)+[u(b)-u(x)]\,f(b)\right\}-\int_a^b f(t)du(t)\right|\leq$$

$$\leq\frac{2}{3}\left\{[f(x)-f(a)]\bigvee_a^x(u)+[f(b)-f(x)]\bigvee_x^b(u)\right\}\leq$$

$$\leq\frac{2}{3}\left[\frac{f(b)-f(a)}{2}+\left|f(x)-\frac{f(a)+f(b)}{2}\right|\right]\bigvee_a^b(u). \qquad (4.7)$$

If $\alpha = \frac{1}{2}$, *then we obtain*

$$\left| \frac{1}{2} \left\{ [u(x) - u(a)] \, f(a) + [u(b) - u(a)] \, f(x) + [u(b) - u(x)] \, f(b) \right\} - \int\limits_a^b f(t) du(t) \right| \leq$$

$$\leq \frac{1}{2} \left\{ [f(x) - f(a)] \bigvee_a^x (u) + [f(b) - f(x)] \bigvee_x^b (u) \right\} \leq$$

$$\leq \frac{1}{2} \left[\frac{f(b) - f(a)}{2} + \left| f(x) - \frac{f(a) + f(b)}{2} \right| \right] \bigvee_a^b (u). \tag{4.8}$$

If $\alpha = 1$, *then we have*

$$\left| [u(x) - u(a)] \, f(a) + [u(b) - u(x)] \, f(b) - \int\limits_a^b f(t) du(t) \right| \leq$$

$$\leq [f(x) - f(a)] \bigvee_a^x (u) + [f(b) - f\,(x)] \bigvee_x^b (u) \leq$$

$$\leq \left[\frac{f(b) - f(a)}{2} + \left| f(x) - \frac{f(a) + f\,(b)}{2} \right| \right] \bigvee_a^b (u). \tag{4.9}$$

Proof. The results follow by Theorem 4.1. It remains to prove the sharpness of (4.5). Suppose $0 \leq \alpha \leq \frac{1}{2}$, assume that (4.5) holds with constant $C_1 > 0$, i.e.,

$$\Big| \alpha \left[\left(u(x) - u(a) \right) f(a) + \left((b) - u(x) \right) f(b) \right] +$$

$$+ (1 - \alpha) \, [u(b) - u(a)] \, f(x) - \int\limits_a^b f(t) du(t) \Big| \leq$$

$$\leq C_1 \left[\frac{f(b) - f(a)}{2} + \left| f(x) - \frac{f(a) + f\,(b)}{2} \right| \right] \bigvee_a^b (u). \tag{4.10}$$

Let $f, u \colon [a, b] \to \mathbb{R}$ be defined as follows:

$$f(t) = \begin{cases} -1, & t = a, \\ 0, & t = (a, b], \end{cases}$$

and

$$u(t)=\begin{cases}0, & t\in[a,b)\,,\\ 1, & t=b.\end{cases}$$

Therefore, f is monotonic nondecreasing on $[a,b]$ and $\bigvee_a^b(u)=1$ and $\int_a^b f(t)du(t)=0$, setting $x=a$ it gives by (4.10) that $1-\alpha\le C_1$, and which proves that $1-\alpha$ is the best possible for all $0\le\alpha\le\frac{1}{2}$.

Now, suppose $\frac{1}{2}\le\alpha\le 1$ and assume that (4.5) holds with constant $C_2>0$, i.e.,

$$\Bigg|\alpha\left[(u(x)-u(a))\,f(a)+((b)-u(x))\,f(b)\right]+$$

$$+(1-\alpha)\left[u(b)-u(a)\right]f(x)-\int\limits_a^b f(t)du(t)\Bigg|\le$$

$$\le C_2\left[\frac{f(b)-f(a)}{2}+\left|f(x)-\frac{f(a)+f\,(b)}{2}\right|\right]\bigvee_a^b(u). \tag{4.11}$$

Let $f,u\colon[a,b]\to\mathbb{R}$ be defined as $f(t)$ as above, and $u(t)=t$, which follows that $\bigvee_a^b(u)=b-a$, and $\int_a^b f(t)du(t)=0$, setting $x=b$ it gives by (4.11) $\alpha\le C_2$, and therefore α is the best possible for all $\frac{1}{2}\le\alpha\le 1$. Consequently, we can conclude that the constant $\left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right]$ is the best possible, for all $\alpha\in[0,1]$.

***Corollary* 4.2.** *In* (4.7), *choosing* $x=\frac{a+b}{2}$, *then we have the following Simpson-type inequality for* $\mathcal{RS}$*-integrals*:

$$\Bigg|\frac{1}{3}\left\{\left[u\left(\frac{a+b}{2}\right)-u(a)\right]f(a)+2\left[u(b)-u(a)\right]f\left(\frac{a+b}{2}\right)+\right.$$

$$\left.+\left[u(b)-u\left(\frac{a+b}{2}\right)\right]f(b)\right\}-\int\limits_a^b f(t)du(t)\Bigg|\le$$

$$\le\frac{2}{3}\left\{\left[f\left(\frac{a+b}{2}\right)-f(a)\right]\bigvee_a^{(a+b)/2}(u)+\left[f(b)-f\left(\frac{a+b}{2}\right)\right]\bigvee_{(a+b)/2}^b(u)\right\}\le$$

$$\le\frac{2}{3}\left[\frac{f(b)-f(a)}{2}+\left|f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{2}\right|\right]\bigvee_a^b(u). \tag{4.12}$$

For the last inequality, the constant $\dfrac{2}{3}$ *is the best possible.*

***Corollary* 4.3.** *In* (4.5), *let* $u(t) = t$ *for all* $t \in [a, b]$, *then we get*

$$\left|\alpha\left((x-a)f(a)+(b-x)f(b)\right)+(1-\alpha)(b-a)f(x)-\int_a^b f(t)dt\right|\leq$$

$$\leq\left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right]\left\{(x-a)\left[f(x)-f(a)\right]+(b-x)\left[f(b)-f(x)\right]\right\}\leq$$

$$\leq\left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right]\left[\frac{f(b)-f(a)}{2}+\left|f(x)-\frac{f(a)+f(b)}{2}\right|\right](b-a). \tag{4.13}$$

For $x = \dfrac{a+b}{2}$, *we have*

$$\left|(b-a)\left[\alpha\frac{f(a)+f(b)}{2}+(1-\alpha)f\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,dt\right|\leq$$

$$\leq\frac{1}{2}(b-a)\left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right]\left[f(b)-f(a)\right]\leq$$

$$\leq\left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right]\left[\frac{f(b)-f(a)}{2}+\left|f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{2}\right|\right](b-a). \tag{4.14}$$

***Remark* 4.1.** We give an attention to the interested reader, is that, in Theorems 2.1, 3.1, 4.1, one may observe various new inequalities by replacing the assumptions on u, e.g. to be of bounded variation, L_u-Lipschitz or monotonic nondecreasing on $[a, b]$, which therefore gives in some cases the ‘dual’ of the above obtained inequalities.

It remains to mention that, in Theorem 3.1, and according to the assumptions on u one may observe several estimations for the functions $p(t)$ and $q(t)$ which therefore gives different functions $M(t)$ and $N(t)$.

***Remark* 4.2.** In Theorems 2.1, 3.1, 4.1, a different result(s) in terms of L_p norms may be stated by applying the well-known Hölder integral inequality, by noting that

$$\left|\int_c^d g(s)du(s)\right|\leq\sqrt[q]{u(d)-u(c)}\times\sqrt[p]{\int_c^d|g(s)|^p\,du(s)},$$

where $p > 1$, $\dfrac{1}{p}+\dfrac{1}{q}=1$.

***Remark* 4.3.** One can point out some results for the Riemann integral of a product, in terms of L_1-, L_p-, and L_∞-norms by using a similar argument considered in [12] (see also [1, 2]).

5. Applications to Ostrowski generalized trapezoid quadrature formula for $\mathcal{RS}$-integrals. Let $I_n\colon a = x_0 < x_1 < s < x_n = b$ be a division of the interval $[a, b]$. Define the general Riemann–Stieltjes sum

$$S\left(f, u, I_n, \xi\right) = \sum_{i=0}^{n-1} \alpha\left\{\left[u(\xi_i) - u(x_i)\right] f(x_i) + \left[u(x_{i+1}) - u(\xi_i)\right] f(x_{i+1})\right\} +$$

$$+ \left(1 - \alpha\right)\left[u(x_{i+1}) - u(x_i)\right] f\left(\xi_i\right). \tag{5.1}$$

In the following, we establish an upper bound for the error approximation of the Riemann–Stieltjes integral $\int_a^b f(t)du(t)$ by its Riemann–Stieltjes sum $S\left(f, u, I_n, \xi\right)$. As a sample we apply the inequality (2.8).

Theorem 5.1. *Under the assumptions of Corollary* 2.2, *we have*

$$\int\limits_a^b f(t)du(t) = S\left(f, u, I_n, \xi\right) + R\left(f, u, I_n, \xi\right),$$

where $S\left(f, u, I_n, \xi\right)$ *is given in* (5.1) *and the remainder* $R\left(f, u, I_n, \xi\right)$ *satisfies the bound*

$$\left|R\left(f, u, I_n, \xi\right)\right| \le \left[\frac{1}{2} + \left|\frac{1}{2} - \alpha\right|\right]\left[u(b) - u(a)\right]\bigvee_a^b(f). \tag{5.2}$$

Proof. Applying Corollary 2.2 on the intervals $[x_i, x_{i+1}]$, we may state that

$$\Bigg|\alpha\left\{\left[u(\xi_i) - u(x_i)\right] f(x_i) + \left[u(x_{i+1}) - u(\xi_i)\right] f(x_{i+1})\right\} +$$

$$+ \left(1 - \alpha\right)\left[u\left(x_{i+1}\right) - u(x_i)\right] f\left(\xi_i\right) - \int\limits_{x_i}^{x_{i+1}} f(t)du(t)\Bigg| \le$$

$$\le \left[\frac{1}{2} + \left|\frac{1}{2} - \alpha\right|\right]\left[\frac{u(x_{i+1}) - u(x_i)}{2} + \left|u(\xi_i) - \frac{u(x_i) + u(x_{i+1})}{2}\right|\right]\bigvee_{x_i}^{x_{i+1}}(f)$$

for all $i \in \{0, 1, 2, s, n-1\}$.

Summing the above inequality over i from 0 to $n-1$ and using the generalized triangle inequality, we deduce

$$\left|R\left(f, u, I_n, \xi\right)\right| = \sum_{i=0}^{n-1}\Bigg|\alpha\left\{\left[u(\xi_i) - u(x_i)\right] f(x_i) + \left[u\left(x_{i+1}\right) - u(\xi_i)\right] f\left(x_{i+1}\right)\right\} +$$

$$+ \left(1 - \alpha\right)\left[u\left(x_{i+1}\right) - u(x_i)\right] f\left(\xi_i\right) - \int\limits_{x_i}^{x_{i+1}} f(t)du(t)\Bigg| \le$$

$$\le \left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right]\sum_{i=0}^{n-1}\left[\frac{u\left(x_{i+1}\right)-u(x_i)}{2}+\left|u\left(\xi_i\right)-\frac{u(x_i)+u\left(x_{i+1}\right)}{2}\right|\right]\bigvee_{x_i}^{x_{i+1}}(f)\le$$

$$\le \left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right]\left[\sum_{i=0}^{n-1}\frac{u\left(x_{i+1}\right)-u(x_i)}{2}+\sum_{i=0}^{n-1}\left|u(\xi_i)-\frac{u(x_i)+u(x_{i+1})}{2}\right|\right]\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(f)\le$$

$$\le \left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right]\left[\frac{u(b)-u(a)}{2}+\sup_{i=0,1,\ldots,n-1}\left|u(\xi_i)-\frac{u(x_i)+u\left(x_{i+1}\right)}{2}\right|\right]\bigvee_{a}^{b}(f)\le$$

$$\le \left[\frac{1}{2}+\left|\frac{1}{2}-\alpha\right|\right][u(b)-u(a)]\bigvee_{a}^{b}(f).$$

Since

$$\sup_{i=0,1,\ldots,n-1}\left|u(\xi_i)-\frac{u(x_i)+u\left(x_{i+1}\right)}{2}\right|\le\sup_{i=0,1,\ldots,n-1}\frac{u(x_{i+1})-u(x_i)}{2}=\frac{u(b)-u(a)}{2}$$

and

$$\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(f)=\bigvee_{a}^{b}(f),$$

which completes the proof.

***Remark* 5.1.** One may use the remaining inequalities in Section 2, to obtain other bounds for $R\left(f,u,I_n,\xi\right)$. We shall omit the details.

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