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VERTEX OPERATOR REPRESENTATIONS OF TYPE $C_l^{(1)}$ AND PRODUCT-SUM IDENTITIES

ВЕРШИННІ ОПЕРАТОРНІ ЗОБРАЖЕННЯ ТИПУ $C_l^{(1)}$ ТА ТОТОЖНОСТІ ТИПУ СУМ І ДОБУТКІВ

The purposes of this work are to construct a class of homogeneous vertex representations of $C_l^{(1)}$, $l \geq 2$, and to deduce a series of product-sum identities. These identities have fine interpretation in the number theory.

Побудовано клас рівномірних вершинних зображень $C_l^{(1)}$, $l \geq 2$. Отримано низку тотожностей типу сум і добутків. Ці тотожності мають змістовну інтерпретацію теорії чисел.

1. Introduction. It is well known that there is a close relationship between representations of affine Lie algebras and combinatorics. For example, the Jacobi triple product identity can be obtained as the Weyl–Kac denominator formula for the affine Lie algebra $\widehat{sl}_2$ [7]. The famous Rogers–Ramanujan identities can be realized from the character formula of certain level three representations [8]. Like the Jacobi triple product identity, the quintuple product identity is also equivalent to the Weyl–Kac denominator formula for the affine Lie algebra $A_2^{(2)}$. In [6], the following infinite product:

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{6n-1})(1-q^{6n-5})} \tag{1.1}$$

is expressed by a sum of two other infinite products in four different ways.

I. Schur [12] (see also [1]) was probably the first person who studied the partitions described by (1.1). He showed that the number of partitions of n into parts congruent to $\pm 1 (\operatorname{mod} 6)$ is equal to the number of partitions of n into distinct parts congruent to $\pm 1 (\operatorname{mod} 3)$ and is also equal to the number of partitions of n into parts that differ at least 3 with added condition that difference between multiples of 3 is at least 6. His first result can be briefly described by

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{6n-1})(1-q^{6n-5})} = \prod_{n=1}^{\infty} \frac{1+q^n}{1+q^{3n}}. \tag{1.2}$$

Motivated by product-sum identity provided by [6], we study a generalized product-sum relations of some special partitions. Our method uses the vertex representations of affine Lie algebras of type $C_l^{(1)}$. For the related topics, one can refer [4, 5, 10, 11, 13] and references therein.

Theorem 1.1. *For any odd $l \geq 3$, the following product-sum identity holds*:

$$\prod_{n=1}^{\infty} \frac{1+q^n}{1+q^{ln}} = \sum_{s=0}^{\frac{l-1}{2}} q^{\frac{(l-2s)^2-1}{8}} \prod_{n \not\equiv \pm(s+1), 0 (\operatorname{mod} l+2)} \frac{1}{(1-q^{2n})(1-q^{ln})},$$

particularly, it covers the first result of [6] *when* $l = 3$.

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Our result in Theorem 1.1 implies the following partition theorem.

Theorem 1.2. *Suppose that $l = 2r+1 \geq 3$ is an odd number, $A_l(n)$ is the number of partitions of n into distinct parts without multiples of l, and $B_{l,s}(n)$ is the number of partitions of n into*

$$2k_1 + \ldots + 2k_i + lr_1 + \ldots + lr_j + \frac{(l-2s)^2-1}{8}$$

with constraints $k_p, r_p \not\equiv \pm(s+1),\, 0(\operatorname{mod} l+2)$. Then for any positive integer n, we have

$$A_l(n) = B_{l,0}(n) + B_{l,1}(n) + \ldots + B_{l,r}(n).$$

Proof. Let $1 + \sum_{n=1}^{\infty} A_n a^n$ be the power series of $\prod_{n=1}^{\infty} \dfrac{1+q^n}{1+q^{ln}}$. Because

$$\prod_{n=1}^{\infty} \frac{1+q^n}{1+q^{ln}} = \prod_{n\geq 1 \text{ is not a multiple of } l} (1+q^n) = \sum_{\substack{n_1>n_2>\ldots>n_k\geq 1 \\ n_i \text{ is not a multiple of } l,\, k\geq 0}} q^{n_1+\ldots+n_k}.$$

Then A_n is the number of partitions of n into distinct parts without multiples of l and $A_n = A(n)$.

A similar argument on $B_{l,s}(n)$ shows that Theorem 1.1 is equivalent to the relation

$$A_l(n) = B_{l,0}(n) + B_{l,1}(n) + \ldots + B_{l,r}(n), \quad \text{for all positive integer } n.$$

Theorem 1.2 is proved.

For example,

$A_5(15) = 16$		$B_{5,0}(15) = 3$	$B_{5,1}(15) = 7$	$B_{5,2}(15) = 6$
$1+14$	$1+6+8$	$2(2\times 3)+3$	$2(1\times 7)+1$	$2(1\times 5)+5(1)$
$2+13$	$2+4+9$	$2(2+4)+3$	$2(1\times 4+3)+1$	$2(1\times 3+2)+5(1)$
$3+12$	$2+6+7$	$2(3+3)+3$	$2(1+3\times 2)+1$	$2(1+2\times 2)+5(1)$
$4+11$	$3+4+8$		$2(1\times 3+4)+1$	$2(5)+5(1)$
$6+9$	$1+2+3+9$		$2(1+6)+1$	$5(1\times 3)$
$7+8$	$1+2+4+8$		$2(3+4)+1$	$5(1+2)$
$1+2+12$	$1+3+4+7$		$2(1\times 2)+5(1\times 2)$	
$1+3+11$	$2+3+4+6$			

Table 1.1 lists the values of $A_5(n)$, $B_{5,0}(n)$, $B_{5,1}(n)$, $B_{5,2}(n)$ for $n \leq 15$.

The above results will be proved by the irreducible decompositions of vertex module $V(P) = S(\widehat{H}^-) \otimes \mathbb{C}[P]$ of $C_l^{(1)}$, where $1\otimes 1$ has weight Λ_0. If we assume that $1\otimes 1$ has weight Λ_1, then our method also gives the following result.

***Table* 1.1**

n	$A_5(n)$	$B_{5,0}(n)$	$B_{5,1}(n)$	$B_{5,2}(n)$
1	1	0	1	0
2	1	0	0	1
3	2	1	1	0
4	2	0	0	2
5	2	0	1	1
6	3	0	1	2
7	4	1	2	1
8	4	0	1	3
9	6	1	3	2
10	7	0	1	6
11	8	2	4	2
12	10	0	2	8
13	12	3	6	3
14	14	0	3	11
15	16	3	7	6

Theorem 1.3. *For any even $l \geq 2$, the following product-sum identity holds*:

$$\prod_{n=1}^{\infty} \frac{(1+q^{n-1/2})^2}{(1+q^n)(1+q^{ln})} = \frac{\prod_{n\geq 1}(1-q^{\left(l\left(\frac{l+2}{2}\right)\right)(2n-1)})(1-q^{(l+2)(2n-1)})}{\prod_{\left(\frac{l}{2}+1\right)\nmid n}(1-q^{2n})(1-q^{ln})} +$$

$$+2\sum_{s=0}^{l/2-1} q^{\frac{(l-2s)^2}{8}} \prod_{n\not\equiv\pm(s+1),0(\operatorname{mod} l+2)} \frac{1}{(1-q^{2n})(1-q^{ln})},$$

or equivalently,

$$\prod_{n=1}^{\infty} \frac{(1+q^{2n-1})^2}{(1+q^{2n})(1+q^{2ln})} = \frac{\prod_{n\geq 1}(1-q^{(l(l+2)(2n-1)})(1-q^{2(l+2)(2n-1)})}{\prod_{(\frac{l}{2}+1)\nmid n}(1-q^{4n})(1-q^{2ln})} +$$

$$+2\sum_{s=0}^{l/2-1} q^{\frac{(l-2s)^2}{4}} \prod_{n\not\equiv\pm(s+1),0(\operatorname{mod} l+2)} \frac{1}{(1-q^{4n})(1-q^{2ln})}.$$

Throughout the paper, we let $\mathbb{C}$, $\mathbb{Z}$ present the set of complex numbers and the set of integers, respectively.

2. Affine Lie algebra of type $C_l^{(1)}$. ***2.1.*** Let $\dot{\mathcal{G}}$ be a finite-dimensional simple Lie algebra of type C_l, $A = \mathbb{C}[t^{\pm 1}]$ the ring of Laurent polynomials in variable t. Then the affine Lie algebra of type $C_l^{(1)}$ is the vector space

$$\widetilde{\mathcal{G}} = \dot{\mathcal{G}} \otimes A \ \oplus \ \mathbb{C}c \oplus \mathbb{C}d,$$

with Lie bracket:

$$[\, x \otimes t^m, y \otimes t^n \,] = [\, x, y \,] \otimes t^{m+n} + m(x \mid y)\delta_{m+n,0}c,$$

$$[\, c, \mathcal{G} \,] = 0,$$

$$[\, d, x \otimes t^m \,] = mx \otimes t^m,$$

where x, $y \in \dot{\mathcal{G}}$, $m, n \in \mathbb{Z}$ and $(\cdot \mid \cdot)$ is a nondegenerate invariant normalized symmetric bilinear form on $\dot{\mathcal{G}}$.

2.2. Suppose that $\dot{H}$ is a Cartan subalgebra of $\dot{\mathcal{G}}$, and $\dot{H}^*$ the dual space of $\dot{H}$. Then there exists an inner product $(\cdot \mid \cdot)|_{\dot{H}^*_{\mathbf{R}}}$ and an orthogonal normal basis $\{e_1, e_2, \ldots, e_l\}$ in Euclidean space $\dot{H}^*_{\mathbf{R}}$ such that the simple root system

$$\Pi = \left\{ \alpha_1 = \frac{1}{\sqrt{2}}(e_1 - e_2),\ \ldots,\ \alpha_{l-1} = \frac{1}{\sqrt{2}}(e_{l-1} - e_l),\ \alpha_l = \sqrt{2}\, e_l \right\},$$

the short root system

$$\dot{\Delta}_S = \left\{ \pm\frac{1}{\sqrt{2}}(e_i - e_j),\ \pm\frac{1}{\sqrt{2}}(e_i + e_j) \,\middle|\, 1 \le i < j \le l \right\},$$

where $\frac{1}{\sqrt{2}}(e_i - e_j) = \alpha_i + \ldots + \alpha_{j-1}$ for $1 \le i < j \le l$, $\frac{1}{\sqrt{2}}(e_i + e_l) = \alpha_i + \ldots + \alpha_l$ for $1 \le i < l$, $\frac{1}{\sqrt{2}}(e_i + e_j) = \alpha_i + \ldots + \alpha_{j-1} + 2\alpha_j + \ldots + 2\alpha_{l-1} + \alpha_l$ for $1 \le i < j < l$; and the long root system

$$\dot{\Delta}_L = \left\{ \pm\sqrt{2}\, e_i \,\middle|\, 1 \le i \le l \right\},$$

where $\sqrt{2}\, e_i = 2\alpha_i + \ldots + 2\alpha_{l-1} + \alpha_l$ for $1 \le i < l$.

Then the root lattice is

$$Q = \bigoplus_{i=1}^{l} \mathbb{Z}\, \alpha_i,$$

$(\alpha_i|\alpha_i) = 1$, $1 \le i \le l-1$, and $(\alpha_l \mid \alpha_l) = 2$.

Let $\gamma : \dot{H} \longrightarrow \dot{H}^*$ be the linear isomorphism such that

$$\alpha_i(\gamma^{-1}(\alpha_j)) = (\alpha_i \mid \alpha_j), \quad i, j = 1, \ldots, l,$$

and

$$\gamma(\alpha_i^\vee) = 2\alpha_i, \qquad i = 1, \ldots, l-1, \quad \gamma(\alpha_l^\vee) = \alpha_l.$$

Then we have $(\alpha_i^\vee|\alpha_j^\vee) = (\gamma(\alpha_i^\vee)|\gamma(\alpha_j^\vee))$. As usual, we identify $\dot{H}$ with $\dot{H}^*$ via γ, i.e., $\alpha^\vee = \frac{2\alpha}{(\alpha \mid \alpha)}$.

For any weight $\Lambda \in (\dot{H} \oplus \mathbb{C}c \oplus \mathbb{C}d)^*$, let $L(\Lambda)$ denote the irreducible highest weight $\widetilde{\mathcal{G}}$-module with highest weight Λ.

2.3. Define a 2-cocycle $\epsilon_0 \colon Q \times Q \longrightarrow \{\pm 1\}$ by

$$\epsilon_0(a+b,c) = \epsilon_0(a,c)\,\epsilon_0(b,c), \qquad \epsilon_0(a,b+c) = \epsilon_0(a,b)\,\epsilon_0(a,c), \quad a,\, b,\, c \in L,$$

and

$$\epsilon_0(\alpha_i,\alpha_j) = \begin{cases} -1, & i = j+1, \\ 1, & \text{other pairs} \ \ (i,j). \end{cases}$$

Let $P = \bigoplus_{i=1-1}^{l} \mathbb{Z}\,\alpha_i \oplus \dfrac{1}{2}\mathbb{Z}\,\alpha_l$. Extend ϵ_0 to $Q \times P$ with

$$\epsilon_0\left(\alpha_i, \frac{1}{2}\alpha_l\right) = 1.$$

2.4. For $\alpha = \sum_{i=1}^{l} k_i\alpha_i \in \dot{\Delta} \cup \{0\}$, define maps $p : \dot{\Delta} \cup \{0\} \to \dot{\Delta}_S \cup \{0\}$ and $s : \dot{\Delta} \cup \{0\} \to \dot{Q}_L$ by

$$p\left(\sum_{i=1}^{l} k_i\alpha_i\right) = \sum_{i=1}^{l-1} \rho(k_i)\alpha_i, \qquad s\left(\sum_{i=1}^{l} k_i\alpha_i\right) = \sum_{i=1}^{l-1} \big(k_i - \rho(k_i)\big)\alpha_i,$$

where $\dot{Q}_L = \operatorname{Span}_{\mathbb{Z}} \dot{\Delta}_L$ and $\rho(k_i) \in \{0,\, 1\}$ such that $\rho(k_i) \equiv k_i \pmod 2$. It is straightforward to check the following statements.

Lemma 2.1. (i) $p\,(\dot{\Delta}_L \cup \{\,0\,\}) = 0$, *and* $p(-\alpha) = p(\alpha)$ *for any* $\alpha \in \dot{\Delta}_S$.

(ii) *Suppose that* $\alpha,\, \beta,\, \alpha+\beta \in \dot{\Delta}$, *then we have*:

(1) *if* $\alpha \in \dot{\Delta}_L$, *then* $(\alpha \mid \beta) = -1$, $p(\alpha+\beta) = p(\beta)$, $s(\alpha+\beta) = s(\alpha) + s(\beta)$;

(2) *if* $\alpha,\, \beta \in \dot{\Delta}_S$, $\alpha+\beta \in \dot{\Delta}_L$, *then* $(\alpha \mid \beta) = 0$, $p(\alpha) = p(\beta)$, $s(\alpha+\beta) - s(\alpha) - s(\beta) = 2p(\alpha)$;

(3) *if* $\alpha,\, \beta,\, \alpha+\beta \in \dot{\Delta}_S$, *then* $(\alpha \mid \beta) = -\dfrac{1}{2}$ *and* $|(p(\alpha) \mid p(\beta))| = \dfrac{1}{2}$; *moreover,*

(a) $(p(\alpha) \mid p(\beta)) = \dfrac{1}{2}$, *then* $p(\alpha+\beta) = p(\alpha) - p(\beta)$, $s(\alpha+\beta) - s(\alpha) - s(\beta) = 2p(\beta)$, *or* $p(\alpha+\beta) = -p(\alpha) + p(\beta)$, $s(\alpha+\beta) - s(\alpha) - s(\beta) = 2p(\alpha)$;

(b) *if* $(p(\alpha) \mid p(\beta)) = -\dfrac{1}{2}$, *then* $p(\alpha+\beta) = p(\alpha) + p(\beta)$, $s(\alpha+\beta) = s(\alpha) + s(\beta)$.

(iii) *For any* $\alpha \in \dot{\Delta}$, *we have*:

(1) $s(\alpha) \in \big\{\pm\sqrt{2}(e_i - e_l) \mid 1 \le i \le l\big\} \subset \dot{Q}_L$;

(2) $p(\alpha) \in \left\{\dfrac{1}{\sqrt{2}}(e_i - e_j) \;\middle|\; 1 \le i \le j \le l\right\} \subset \dot{\Delta}_S \cup \{0\}$;

(3) $s(\alpha) + s(-\alpha) = -2p(\alpha) \in \dot{Q}_L$;

(4) $\alpha \pm p(\alpha) \in \dot{Q}_L$.

2.5. Define a map $f \colon Q \times Q \to \{\pm 1\}$ by

$$f(\alpha,\beta) = (-1)^{(s(\alpha)|\beta) + (p(\alpha)|p(\beta) + p(\alpha+\beta))}.$$

Set $\epsilon = \epsilon_0 \circ f$, then $\epsilon \colon Q \times Q \longrightarrow \{\pm 1\}$ is still a 2-cocycle, which has the property (ii) in the following lemma.

Lemma 2.2. (i) *For* $\alpha, \beta \in \dot{\Delta}$, *we have*

$$\epsilon_0(\alpha,\beta) = (-1)^{(\alpha|\beta) + (p(\alpha)|p(\beta)) + (s(\alpha)|\beta) + (s(\beta)|\alpha)} \cdot \epsilon_0(\beta,\alpha).$$

(ii) *For* $\alpha,\ \beta,\ \alpha+\beta \in \dot{\Delta}$, *we have* $\epsilon(\alpha,\beta) = -\epsilon(\beta,\alpha)$.

2.6. We have the following proposition.

Proposition 2.1. *The affine Lie algebra* $\widetilde{\mathcal{G}}$ *of type* $C_l^{(1)}$ *has a system of generators*

$$\left\{\alpha_i^\vee \otimes t^n, e_\alpha \otimes t^n \mid 1 \le i \le l,\ n \in \mathbb{Z}\right\}$$

and c, d *with relations*

$$\left[\,\alpha_i^\vee \otimes t^m,\ \alpha_j^\vee \otimes t^n\,\right] = m(\alpha_i^\vee \mid \alpha_j^\vee)\delta_{m+n,0}c,$$

$$\left[\,\alpha_i^\vee \otimes t^m,\ e_\alpha \otimes t^n\,\right] = \alpha(\alpha_i^\vee)\, e_\alpha \otimes t^{m+n},$$

$$\left[\,e_\alpha \otimes t^m,\ e_{-\alpha} \otimes t^n\,\right] = \epsilon(\alpha,-\alpha)\,\frac{2}{(\alpha \mid \alpha)}\left[\,\gamma^{-1}(\alpha)\otimes t^{m+n} + m\delta_{m+n,0}c\right],$$

$$\left[\,e_\alpha \otimes t^m,\ e_\beta \otimes t^n\,\right] = \epsilon(\alpha,\beta)\bigl(1+\delta_{1,\,(p(\alpha)|p(\beta))}\bigr)\, e_{\alpha+\beta}\otimes t^{m+n} \quad \forall \alpha,\beta, \quad \alpha+\beta \in \dot{\Delta},$$

$$\left[\,e_\alpha \otimes t^m,\ e_\beta \otimes t^n\,\right] = 0 \quad \forall \alpha,\beta \in \dot{\Delta}, \quad \alpha+\beta \notin \dot{\Delta}\cup\{\,0\,\},$$

where γ *is the canonical linear space isomorphism from* $\dot{H}$ *to* $\dot{H}^*$.

3. Vertex construction of Lie algebra of type $C_l^{(1)}$. ***3.1.*** Let $H(m)$, $m \in \mathbb{Z}$, be an isomorphic copy of $\dot{H}$. Set $\dot{H}_S := \mathrm{Span}_{\mathbb{C}}\{\,\alpha_i \mid 1 \le i \le l-1\,\}$ and $H_S\left(n-\dfrac{1}{2}\right)$, $n \in \mathbb{Z}$, is an isomorphic copy of $\dot{H}_S$.

Define a Lie algebra

$$\widehat{H} = \bigoplus_{m\in\mathbb{Z}} H(m) \oplus \bigoplus_{n\in\mathbb{Z}} H_S\left(n-\frac{1}{2}\right) \oplus \mathbb{C}c,$$

with Lie bracket

$$\left[\,\widetilde{H}, c\,\right] = 0,$$

$$\left[\,a(m), b(n)\,\right] = m\,(a \mid b)\,\delta_{m,-n}c.$$

Let

$$\widehat{H}^- = \bigoplus_{m\in\mathbb{Z}_-} H(m) \oplus \bigoplus_{n\in\mathbb{Z}_-} H_S\left(n+\frac{1}{2}\right),$$

and let $S(\widehat{H}^-)$ be the symmetric algebra generated by $\widehat{H}^-$. Then $S(\widehat{H}^-)$ is an $\widehat{H}$-module with the action

$$c\cdot v = v, \quad a(m)\cdot v = a(m)v \quad \forall m<0,$$

and

$$a(m)\cdot b(n) = m\,(a,b)\,\delta_{m+n,0} \quad \forall m \ge 0, \quad n<0,$$

where $a,b \in H$, $m,n \in \dfrac{1}{2}\mathbb{Z}$.

3.2. We form a group algebra $\mathbb{C}[P]$ with base elements e^h, $h \in P$, and the multiplication

$$e^{h_1} e^{h_2} = e^{h_1 + h_2} \quad \forall h_1, h_2 \in P.$$

Set

$$V(P) := S(\widehat{H}^-) \otimes \mathbb{C}[P]$$

and extend the action of $\widehat{H}$ to space $V(P)$ by

$$a(m) \cdot (v \otimes e^r) = (a(m) \cdot v) \otimes e^r \quad \forall m \in \frac{1}{2}\mathbb{Z}^*;$$

and define

$$a(0) \cdot (v \otimes e^r) = (a \mid r)\, v \otimes e^r,$$

which makes $V(P)$ into a $\widehat{H}$-module.

3.3. For $r \in P$, $\alpha \in Q$, define $\mathbb{C}$-linear operators as

$$e^{\alpha} \cdot (v \otimes e^r) = v \otimes e^{\alpha + r},$$

$$z^{\alpha} \cdot (v \otimes e^r) = z^{(\alpha | r)}\, v \otimes e^r,$$

$$\epsilon_{\alpha} \cdot (v \otimes e^r) = (-1)^{(s(\alpha)|r)}\, \epsilon_0(\alpha, r)\, v \otimes e^r,$$

$$a(z) = \sum_{j \in \mathbb{Z}} a(j)\, z^{-2j},$$

$$E^{\pm}(\alpha, z) \cdot (v \otimes e^r) = \left(\exp\left(\mp \sum_{n=1}^{\infty} \frac{1}{n} z^{\mp 2n} \alpha(\pm n) \right) \cdot v \right) \otimes e^r,$$

$$F^{\pm}(\alpha, z) \cdot (v \otimes e^r) = \left(\exp\left(\mp \sum_{n=0}^{\infty} \frac{2}{2n+1} z^{\mp(2n+1)} \alpha\left(\pm \frac{2n+1}{2} \right) \right) \cdot v \right) \otimes e^r.$$

Then $a(z)$, $E^{\pm}(\alpha, z)$, $F^{\pm}(\alpha, z) \in (\mathrm{End} V(P))[[z, z^{-1}]]$.

As usual, we shall adopt the notation of normal ordering product

$$: a(i) b(j) := \begin{cases} a(i) b(j), & \text{if } \ i \le j, \\ b(j) a(i), & \text{if } \ j < i, \end{cases}$$

where $a, b \in L$ and $i, j \in \frac{1}{2}\mathbb{Z}$.

3.4. Let $\widetilde{V}(P)$ be the formal completion of $V(P) = S\big(\widehat{H}^-\big) \otimes \mathbb{C}[P]$. We give some vertex operators on $\widetilde{V}(P)$:

(1) For $\alpha \in \dot{\Delta} \cup \{0\}$, set

$$Y(\alpha, z) = E^-(\alpha, z) E^+(\alpha, z) F^-(p(\alpha), z) F^+(p(\alpha), z),$$

$$Z^{\epsilon}(\alpha, z) = z^{(\alpha|\alpha)} e^{\alpha} z^{2\alpha} \epsilon_{\alpha},$$

$$X^{\epsilon}(\alpha, z) := Y(\alpha, z) \otimes Z^{\epsilon}(\alpha, z).$$

(2) For $\alpha, \beta \in \dot{\Delta}$, define

$$X^\epsilon(\alpha, \beta, z, w) =: Y(\alpha, z)Y(\beta, w) : \otimes\, Z^\epsilon(\alpha + \beta, w).$$

3.5. The Laurent series of operators $X^\epsilon(\alpha, z)$ is denoted by

$$X^\epsilon(\alpha, z) = \sum_{k=-\infty}^{\infty} X^\epsilon_{k/2}(\alpha)\, z^{-k}.$$

Then for all $k \in \mathbb{Z}$, $X^\epsilon_{k/2}(\alpha)$ is an operator on $V(P)$. Note that $X^\epsilon_n(\alpha)$ acts as an operator on $V(P)$ in the following way:

$$X^\epsilon_n(\alpha) \cdot (v \otimes e^r) = \epsilon(\alpha, r)\, Y_{n+\frac{1}{2}(\alpha|\alpha)+(\alpha|r)}(\alpha)\,(v) \otimes e^{\alpha+r} \quad \forall v \otimes e^r \in V(P).$$

3.6. For $v = a_1(-n_1)a_2(-n_2)\ldots a_p(-n_p) \otimes e^r \in V(L)$, define the degree action of d on $V(P)$ by

$$d \cdot (v \otimes e^r) = \left(\deg(v) - \frac{1}{2}(r \mid r)\right) v \otimes e^r,$$

where $\deg(v) = -\sum_{i=1}^{p} n_i$.

The number $\deg(v) - \frac{1}{2}(r \mid r)$ is called the degree of $v \otimes e^r$ and denoted by $\deg(v \otimes e^r)$.

3.7. We have the following proposition.

Proposition 3.1. *The affine Lie algebra $\widetilde{\mathcal{G}}$ of type $C_l^{(1)}$ is homomorphic to the Lie algebra J generated by operators $\alpha^\vee(n), X^\epsilon_n(\alpha), c, d$ ($\alpha \in \dot{\Delta}, n \in \mathbb{Z}$) on $V(P) = S(\widehat{H}^-) \otimes \mathbb{C}[P]$, i.e., there exists a unique Lie algebra homomorphism π from $\widetilde{\mathcal{G}}$ to the Lie subalgebra J of* $\mathrm{End}(V(P))$ *such that*

$$\pi(\gamma^{-1}(\alpha_i) \otimes t^n) = \frac{2}{(\alpha_i \mid \alpha_i)}\alpha_i(n),$$

$$\pi(e_\alpha \otimes t^n) = X^\epsilon_n(\alpha),$$

$$\pi(c) = \mathrm{id},$$

$$\pi(d) = d,$$

that is, $V(P)$ is a $\widetilde{\mathcal{G}}$-module.

4. Some computations needed.

Lemma 4.1. *For any l, if Λ_s is the basic weight of $C_l^{(1)}$, then we have*

$$\dim_q(L(\Lambda_s)) = \dim_q(L(\Lambda_{l-s})),$$

$$\dim_q(L(\Lambda_s)) = \prod_{n=1}^{\infty} \frac{(1 - q^{2(l+2)n})(1 - q^{2(l+2)n-2-2s})(1 - q^{2(l+2)n-2l-2+2s})}{1 - q^n}.$$

For the definition of $\dim_q$, one can refer [7, p. 183] (Proposition 10.10).
Define q-series

$$\kappa_q(l,r)=\sum_{n\in\mathbb{Z}} q^{ln^2-rn}, \tag{4.1}$$

for $0<r\le l$. If $r=l$, then

$$\kappa_q(l,l)=2\prod_{n=1}^{\infty}\frac{(1-q^{4n})^2}{1-q^{2n}}, \tag{4.2}$$

by Gauss identity

$$\sum_{n\in\mathbb{Z}} q^{2n^2-n}=\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2}{1-q^n}.$$

Suppose that $V=S(\alpha(-1),\alpha(-2),\ldots)\otimes\mathbb{C}[\mathbb{Z}\alpha]$ with $(\alpha|\alpha)=2$, then V is an irreducible $A_1^{(1)}$-module isomorphic to $L(\Lambda_0)$ (one can see [4] for details). The degree of $v=\alpha(-n_1)\ldots\alpha(-n_k)\otimes \otimes e^{n\alpha}\in V$ is defined as $-n_1-\ldots-n_k-n^2$ and weight of v is $-(n_1+\ldots+n_k+n^2)\delta+n\alpha$. Hence

$$\mathrm{ch}V=e^{\Lambda_0}\frac{1}{\prod_{n=1}^{\infty}(1-e^{-n\delta})}\sum_{n\in\mathbb{Z}} e^{-n^2\delta+n\alpha}.$$

Moreover,

$$\mathrm{ch}L(\Lambda_0)=e^{\Lambda_0}\frac{\sum_{n\in\mathbb{Z}} e^{-(3n^2+n)\delta+3n\alpha}-\sum_{n\in\mathbb{Z}} e^{-(3n^2+n)\delta-(3n+1)\alpha}}{\prod_{n=1}^{\infty}(1-e^{-n\delta})(1-e^{-n\delta+\alpha})(1-e^{-(n-1)\delta-\alpha})}.$$

If $e^{-\delta}$, $e^{-\alpha}$ are specialized as q^l, q^r, respectively, then $V\cong L(\Lambda_0)$ implies the following lemma.

Lemma 4.2. *If* $0<r<l$, *then*

$$\kappa_q(l,r)=\prod_{n=1}^{\infty}\frac{(1-q^{2ln})(1-q^{4ln-2(l-r)})(1-q^{4l(n-1)+2(l-r)})}{(1-q^{2ln-l-r})(1-q^{2l(n-1)+l+r})}.$$

Proof. This lemma can easily be proved using the quintuple product identity (see [3]).

5. The module structure. ***5.1.*** Let $\alpha_0\in H^*$ such that $\{\alpha_0,\alpha_1,\ldots,\alpha_l\}$ is the simple root system of affine Lie algebra $\widetilde{\mathcal{G}}$ and $\alpha_0(\alpha_0^\vee)=2$, $\alpha_0(\alpha_1^\vee)=-2$, $\alpha_0(d)=1$ and $\alpha_0(\alpha_j^\vee)=\alpha_0(c)=0$, $2\le j\le l$. Then $\delta=\alpha_0+2\alpha_1+2\alpha_2+\ldots+2\alpha_{l-1}+\alpha_l$ is the primitive imaginary root of $\widetilde{\mathcal{G}}$. Let $\Lambda_i\in H^*$ be such that

$$\Lambda_i(\alpha_j^\vee)=\delta_{ij},\qquad \Lambda_i(d)=0,\quad 0\le j\le l.$$

Lemma 5.1. *With respect to the Cartan subalgebra* H *of* $\widetilde{\mathcal{G}}$, $V(P)$ *has the weight space decomposition*

$$V(P)=\sum_{\lambda\in\mathrm{weight}(V(P))} V(P)_\lambda,$$

and the weight space $V(P)_\lambda$ *has a basis* $v\otimes e^r$, *where* $r\in P$, $v\in S(\dot{\mathcal{H}}^-)$, *and*

$$\lambda = \Lambda_0 + \left(\deg(v) - \frac{1}{2}(r \mid r)\right)\delta + r,$$

so $\deg(v)$ *and* r *are uniquely determined by* λ.

5.2. The following describes the possible distribution of the maximal weights of $\widetilde{\mathcal{G}}$-module $V(\dot{Q})$.

Lemma 5.2. *For any* $\lambda \in P(V(\dot{Q}))$, *we have*

$$\lambda \leq \Lambda_j - \frac{j}{4}\delta,$$

for some $j \in \mathbb{Z}$, *where* $0 \leq j \leq l$.

Proof. By Lemma 5.1, $\lambda = \Lambda_0 - \left(k + \frac{1}{2}(r \mid r)\right)\delta + r$, where $r = \sum_{i=1}^{l-1} k_i\alpha_i + \frac{k_l}{2}\alpha_l \in P$ and $k \in \frac{1}{2}\mathbb{N}$. At first, we have

$$\frac{1}{2}(r \mid r)\delta - r =$$

$$= \frac{1}{4}\big(k_1^2 + (k_2 - k_1)^2 + \ldots + (k_{l-1} - k_{l-2})^2 + (2k_l - k_{l-1})^2\big)\delta - \sum_{i=1}^{l-1} k_i\alpha_i - \frac{k_l}{2}\alpha_l =$$

$$= \frac{1}{4}\Big[\big(k_1^2\delta - 2k_1(2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{l-1} + \alpha_l)\big) +$$

$$+ \big((k_2 - k_1)^2\delta - 2(k_2 - k_1)(2\alpha_2 + \ldots + 2\alpha_{l-1} + \alpha_l)\big) + \ldots$$

$$\ldots + \big((k_{l-1} - k_{l-2})^2\delta - 2(k_{l-1} - k_{l-2})(2\alpha_{l-1} + \alpha_l)\big) + (k_l - k_{l-1})^2\delta - 2(k_l - k_{l-1})\alpha_l\Big].$$

Suppose that

$$\alpha = 2\alpha_i + \ldots + 2\alpha_{l-1} + \alpha_l < \delta.$$

If $n < 0$, then $n^2\delta - 2n\alpha > 0$. If $n > 1$, then

$$(n^2 - 1)\delta - 2(n - 1)\alpha = (n - 1)((n + 1)\delta - 2\alpha) > 0.$$

Hence we get

$$n^2\delta - 2n\alpha \geq 0$$

or

$$n^2\delta - 2n\alpha \geq \delta - 2\alpha.$$

So

$$\frac{1}{2}(r \mid r)\delta - r \geq$$

$$\geq \frac{s}{4}\delta - \frac{1}{2}\big[(2\alpha_{p_1} + \ldots + 2\alpha_{l-1} + \alpha_l) + \ldots + (2\alpha_{p_s} + \ldots + 2\alpha_{l-1} + \alpha_l)\big] \geq$$

$$\geq \frac{s}{4}\delta - \frac{1}{2}\big[(2\alpha_1 + \ldots + 2\alpha_{l-1} + \alpha_l) + \ldots + (2\alpha_s + \ldots + 2\alpha_{l-1} + \alpha_l)\big] =$$

$$= \frac{1}{2}(\gamma_s \mid \gamma_s)\delta - \gamma_s,$$

for some s, where

$$\gamma_s = \alpha_1 + 2\alpha_2 + \ldots + (s-1)\alpha_{s-1} + s(\alpha_s + \ldots + \alpha_{l-1}) + \frac{s}{2}\alpha_l \in P,$$

and it clear that $\Lambda_s = \Lambda_0 + \gamma_s, (\gamma_s \mid \gamma_s) = \dfrac{s}{2}$. Then we have

$$\lambda = \Lambda_0 - \left(k + \frac{1}{2}(r \mid r)\right)\delta + r \leq \Lambda_0 - \frac{1}{2}(r|r)\delta + r \leq$$

$$\leq \Lambda_0 - \frac{1}{2}(\gamma_s \mid \gamma_s)\delta + \gamma_s = \Lambda_s - \frac{s}{4}\delta$$

for some s, $0 \leq s \leq l$.

***Remark* 5.1.** By the result above, we know that any highest weight of $V(P)$ belongs to the set

$$\bigcup_{s=0}^{l} \left\{ \Lambda_s - \frac{s}{4} - \frac{p}{2}\delta \;\middle|\; p \geq 0,\ s = 0, 1, \ldots, l \right\}.$$

More precisely, any highest weight vector has the form $v \otimes e^{\gamma_s}$ for some s.

Theorem 5.1. *$V(P)$ has the decomposition*

$$V(P) = \bigoplus_{s=0}^{l} V(P)^{[s]},$$

where $V(P)^{[s]}$ is the sum of those irreducible submodules whose highest weights $\lambda \leq \Lambda_s - \dfrac{s}{4}$.

6. Highest weight vectors. ***6.1.*** Define operators

$$S(\alpha, z) = \exp\left(\sum_{n>0} \frac{\alpha(-n+1/2)}{n-1/2} z^{2n-1}\right) \exp\left(-\sum_{n>0} \frac{\alpha(n-1/2)}{n-1/2} z^{-2n+1}\right),$$

with series expansion

$$S(\alpha, z) = \sum_{n \in \frac{1}{2}\mathbb{Z}} S_n(\alpha) z^{-2n}.$$

Lemma 6.1. *For $i = 1, \ldots, l-1$, we have*

$$\{S_n(\alpha_i), S_m(\alpha_i)\} = S_n(\alpha_i)S_m(\alpha_i) + S_m(\alpha_i)S_n(\alpha_i) = -2\delta_{m+n,0}$$

and

$$S_n(\alpha) = (-1)^{2n} S_n(-\alpha), \quad n \in \frac{1}{2}\mathbb{Z}.$$

6.2. Define $\beta_i = \alpha_i$ for $i = 1, \ldots, l-1$ and

$$\beta_l = -\sum_{i=1}^{l-1} \frac{i}{l}\alpha_i,$$

also let

$$y_i = \sum_{j=i}^{l} 2\beta_j, \quad i = 1, \ldots, l.$$

Define

$$Z^{[s]}(z) = \sum_{i\in\mathbb{Z}} Z^{[s]}_{i/2} z^{-i} = \sum_{j=1}^{s} S(y_j, z) - \sum_{j=s+1}^{l} S(y_j, z),$$

for even s. Particularly, $Z^{[l]}(z) = -Z^{[0]}(z)$.

***Remark* 6.1.** The operators $Z^{[s]}$ are the same as (or isomorphic to) those defined by Lepowsky and Wilson in [8, 9], where they are generating operators of vacuum spaces of standard $A_1^{(1)}$-modules of level l. For more details, one can refer to those two papers.

Lemma 6.2. *For any $n \in \dfrac{1}{2}\mathbb{Z}$, if $v \otimes e^{\gamma_s}$ is a highest weight vector and $Z_n^{[s]} v \otimes e^{\gamma_s}$ is not zero, then $Z_n^{[s]} v \otimes e^{\gamma_s}$ is also a highest weight vector.*

Proof. At first, we give the proof for $s = 0$. For $i < l$, we have

$$S_n(y_i) + S_n(y_{i+1}) = \sum_{j\in\mathbb{Z}} S_j(\beta_i) S_{n-j}(\beta_i + 2\beta_{i+1} + \ldots + 2\beta_l),$$

and $(y_j|\alpha_i) = 0$, $j \neq i, i+1$. Hence

$$-X_0^{\epsilon}(\alpha_i) Z_n^{[0]}(v \otimes 1) =$$

$$= X_0^{\epsilon}(\alpha_i)\left\{\sum_{r\neq i,i+1} S_n(y_r) + \sum_{j\in\mathbb{Z}} S_j(\beta_i) S_{n-j}(\beta_i + 2\beta_{i+1} + \ldots + 2\beta_l)\right\} v \otimes 1 =$$

$$= Y_{\frac{1}{2}}(\alpha_i)\left\{\sum_{r\neq i,i+1} S_n(y_r) + \sum_{j\in\mathbb{Z}} S_j(\beta_i) S_{n-j}(\beta_i + 2\beta_{i+1} + \ldots + 2\beta_l)\right\} v \otimes e^{\alpha_i} =$$

$$= Y_{\frac{1}{2}}(\alpha_i) \sum_{r\neq i,i+1} S_n(y_r) v \otimes e^{\alpha_i} +$$

$$+ Y_{\frac{1}{2}}(\alpha_i) \sum_{j\in\mathbb{Z}} S_j(\alpha_i) S_{n-j}(\beta_i + 2\beta_{i+1} + \ldots + 2\beta_l) v \otimes e^{\alpha_i} =$$

$$= \sum_{r\neq i,i+1} S_n(y_r) X_0^{\epsilon}(\alpha_i) v \otimes 1 +$$

$$+\sum_{k\in\mathbb{Z}} E_k(\alpha_i)S_{\frac{1}{2}-k}(\alpha_i)\sum_{j\in\mathbb{Z}} S_j(\alpha_i)S_{n-j}(\beta_i+2\beta_{i+1}+\ldots+2\beta_l)v\otimes e^{\alpha_i}=$$

$$=-\sum_{j\in\mathbb{Z}} S_j(\alpha_i)S_{n-j}(\beta_i+2\beta_{i+1}+\ldots+2\beta_l)X_0^{\epsilon}(\alpha_i)v\otimes 1=0.$$

Moreover, operators $X_0^{\epsilon}(\alpha_l)$ and $X_1^{\epsilon}(-(2\alpha_1+\ldots+2\alpha_{l-1}+\alpha_l))$ commute with $Z_n^{[0]}$, so $Z_n^{[0]}v\otimes 1$ is still a highest weight vector.

The proof for $s=l$ is the same as above.

For $Z_n^{[s]}$ with $0<s<l$,

$$S_n(y_i)-S_n(y_{i+1})=\sum_{j\in\mathbb{Z}+1/2} S_j(\beta_i)S_{n-j}(\beta_i+2\beta_{i+1}+\ldots+2\beta_l),$$

then

$$X_0^{\epsilon}(\alpha_s)Z_n^{[s]}(v\otimes e^{\gamma_s})=$$

$$=X_0^{\epsilon}(\alpha_s)\left\{\left(\sum_{r<s}-\sum_{r>s+1}\right)S_n(y_r)+\sum_{j\in\mathbb{Z}+1/2} S_j(\beta_s)S_{n-j}(\beta_s+2\beta_{s+1}+\ldots+2\beta_l)\right\}v\otimes e^{\gamma_s}=$$

$$=Y_1(\alpha_s)\left\{\left(\sum_{r<s}-\sum_{r>s+1}\right)S_n(y_r)+\sum_{j\in\mathbb{Z}} S_j(\beta_s)S_{n-j}(\beta_s+2\beta_{s+1}+\ldots+2\beta_l)\right\}v\otimes e^{\gamma_s+\alpha_i}=$$

$$=Y_1(\alpha_i)\left(\sum_{r<s}-\sum_{r>s+1}\right)S_n(y_r)v\otimes e^{\gamma_s+\alpha_i}+$$

$$+Y_1(\alpha_i)\sum_{j\in\mathbb{Z}} S_j(\alpha_s)S_{n-j}(\beta_s+2\beta_{s+1}+\ldots+2\beta_l)v\otimes e^{\gamma_s+\alpha_s}=$$

$$=\sum_{r\neq i,i+1} S_n(y_r)X_0^{\epsilon}(\alpha_s)v\otimes e^{\gamma_s}+$$

$$+\sum_{k\in\mathbb{Z}} E_k(\alpha_i)S_{1-k}(\alpha_i)\sum_{j\in\mathbb{Z}+1/2} S_j(\alpha_s)S_{n-j}(\beta_s+2\beta_{s+1}+\ldots+2\beta_l)v\otimes e^{\gamma_s+\alpha_s}=$$

$$=-\sum_{j\in\mathbb{Z}} S_j(\alpha_s)S_{n-j}(\beta_s+2\beta_{s+1}+\ldots+2\beta_l)X_0^{\epsilon}(\alpha_s)v\otimes e^{\gamma_s}=0.$$

For other $X^{\epsilon}(\alpha_i)$ and $X_0^{\epsilon}(\alpha_l)$, $X_1^{\epsilon}\big(-(2\alpha_1+\ldots+2\alpha_{l-1}+\alpha_l)\big)$, the proof is similar to the first case. Then $Z_n^{[s]}v\otimes e^{\gamma_s}$ is also a highest weight vector.

Lemma 6.2 is proved.

For $\lambda=\Lambda_0-\sum_{i=0}^{l}k_i\alpha_i$, define

$$\deg\lambda = \sum_{i=0}^{l} k_i,$$

and

$$V(P)_i = \sum_{\lambda\colon \deg\lambda = i} V(P)_\lambda,$$

then

$$V(P) = \sum V(P)_i.$$

The q-character ch_q is a map from $V(P)$ to $\mathbb{Z}[q^{\pm 1}]$ $\big($to $\mathbb{Z}\big[q^{\pm 1/2}\big]$ if l is even$\big)$ defined by

$$\mathrm{ch}_q V(P) = \sum \dim V(P)_i q^i.$$

Define the highest weight vector space of $V(P)^{[s]}$ as $\Omega_s \otimes e^{\gamma_s}$. Then we have the following theorem.

Theorem 6.1. *Ω_s is generated by operators $Z_i^{[s]}$, $i \in \dfrac{1}{2}\mathbb{Z}_-$. Moreover,*

$$\mathrm{ch}_q\Omega_s = \prod_{n=1}^{\infty} \frac{(1-q^{l(l+2)n})(1-q^{l[(l+2)n-s-1]})(1-q^{l[(l+2)n-l+s-1]})}{1-q^{ln}}.$$

7. Proof of Theorem 6.1. Let

$$\widehat{H}^- = \bigoplus_{n\in\mathbb{Z}_-} H_S\left(n+\frac{1}{2}\right).$$

Theorem 6.1 will be proved by the following lemmas.

Lemma 7.1. *$S(\widehat{H_S}^-)\otimes 1$ can be generated by operators $Z_n^{[s]}$, $n\in\dfrac{1}{2}\mathbb{Z}$, $s=0,\ldots,l$, on $1\otimes 1$.*

Proof. At first, by the definition of operators $Z^{[s]}(z)$,

$$Z_n^{[1]} - Z_n^{[0]} = 2S(y_1),$$

$$Z_n^{[2]} - Z_n^{[1]} = 2S(y_2),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$Z_n^{[l-1]} - Z_n^{[l-2]} = 2S(y_{l-1}),$$

moreover, for $0<s<l$ and $m\in\mathbb{Z}$, $y_s\left(m+\dfrac{1}{2}\right)$ can be generated by operators $S_n(y_s)$, $n\in\dfrac{1}{2}\mathbb{Z}$. So $S(H_S^-)\otimes 1$ can be generated by the $Z_n^{[s]}$'s.

Lemma 7.2. *Suppose that $v\in S(\widehat{H}_S-)$, then $v\otimes e^{\gamma_s}$ is a highest weight vector if and only if for all positive integers m,*

$$S_{m-1/2}(\alpha_i)v\otimes 1 = 0, \quad 0<i<l, \quad i\neq s, \qquad S_m(\alpha_s)v\otimes 1 = 0 \quad (\textit{when}\ \ (\alpha_s|\alpha_s)=1).$$

Proof. As we know that $v \otimes e^{\gamma_s}$ is a highest weight vector if and only if

$$X_0^{\epsilon}(\alpha_i)v \otimes e^{\gamma_s} = X_1^{\epsilon}(-2\alpha_1 - \ldots - 2\alpha_{l-1} - \alpha_l)v \otimes e^{\gamma_s} = 0, \quad i = 1, \ldots, l.$$

For any $v \in S(\widehat{H_S}^-)$, it always holds that

$$X_0^{\epsilon}(\alpha_l)v \otimes e^{\gamma_s} = X_1^{\epsilon}(-2\alpha_1 - \ldots - 2\alpha_{l-1} - \alpha_l)v \otimes e^{\gamma_s} = 0.$$

Let

$$E(\alpha, z) = E^-(\alpha, z)E^+(\alpha, z) = \sum_{j \in \mathbb{Z}} E_j(\alpha) z^{-j},$$

then for $0 < i < l$,

$$X_0^{\epsilon}(\alpha_i)v \otimes e^{\gamma_s} = \epsilon_{\alpha_i} Y_{1/2}(\alpha_i)v \otimes e^{\gamma_s + \alpha_i} = \epsilon_{\alpha_i} \sum_{j \in \mathbb{Z}} E_j(\alpha_i) S_{1/2-j}(\alpha_i) v \otimes e^{\gamma_s + \alpha_i}$$

for $i \neq s$ and

$$X_0^{\epsilon}(\alpha_i)v \otimes e^{\gamma_s} = \epsilon_{\alpha_i} Y_1(\alpha_i)v \otimes e^{\gamma_s + \alpha_i} = \epsilon_{\alpha_i} \sum_{j \in \mathbb{Z}} E_j(\alpha_i) S_{1-j}(\alpha_i) v \otimes e^{\gamma_s + \alpha_i}$$

for $i = s$. Thus this lemma holds.

Lemma 7.3. *If $v \in S(\widehat{H_S}^-)$ and for all positive integer m,*

$$S_{m-1/2}(\alpha_1)v \otimes 1 = 0,$$

then v belongs to the subspace W_1 generated by $Z_{n/2}^{[0]},\ Z_{n/2}^{[2]}, \ldots, Z_{n/2}^{[l-1]},\ Z_{n/2}^{[l]} = -Z_{n/2}^{[0]}$, $n \in \mathbb{Z}$.

Proof. Notice that

$$Z_{n/2}^{[0]} = -\sum_{r \neq 1,2} S_{n/2}(y_r) - \sum_{j \in \mathbb{Z}} S_j(\beta_1) S_{n/2-j}(\beta_1 + 2\beta_2 + \ldots + 2\beta_l) =$$

$$= -\sum_{j \in \mathbb{Z}} S_j(\alpha_1) S_{n/2-j}(\beta_1 + 2\beta_2 + \ldots + 2\beta_l) + \text{terms commuting with } S(\alpha_1),$$

$$Z_{n/2}^{[1]} = \sum_{j \in \mathbb{Z}} S_{j+1/2}(\alpha_1) S_{n/2-j-1/2}(\beta_1 + 2\beta_2 + \ldots + 2\beta_l) + \text{terms commuting with } S(\alpha_1),$$

and

$$Z_{n/2}^{[s]} = \sum_{j \in \mathbb{Z}} S_j(\alpha_1) S_{n/2-j}(\beta_1 + 2\beta_2 + \ldots + 2\beta_l) + \text{terms commuting with } S(\alpha_1),$$

for $s \geq 2$. Since $(\alpha_1 | \beta_1 + 2\beta_2 + \ldots + 2\beta_l) = 0$, a homogeneous non-zero vector

$$v = v \otimes 1 = \sum a_{j_1, \ldots, j_k} Z_{j_1}^{[s_1]} \ldots Z_{j_k}^{[s_k]} \otimes 1$$

can be written as

$$\sum b_{i_1, \ldots, i_r} S_{i_1}(\alpha_1) \ldots S_{i_r}(\alpha_1) \otimes 1, \quad i_1 < \ldots < i_r \leq 0,$$

where $b_{i_1,\ldots,i_r}$ is a non-zero polynomial commuting with $S(\alpha_1)$. Then $v \in W_1$ if and only if $i_1,\ldots,i_r \in \mathbb{Z}$ for any $b_{i_1,\ldots,i_r}$. It is easy to show that if $b_{i_1,\ldots,i_r} \otimes 1 \neq 0$, then

$$S_{-j_1}(\alpha_1)\ldots S_{-j_r}(\alpha_1)v = \text{a scalar of } \ b_{j_1,\ldots,j_r} \otimes 1 \neq 0.$$

Condition $S_{m-1/2}(\alpha_1)v \otimes 1 = 0$ implies all $i_1,\ldots,i_r \in \mathbb{Z}$, so $v \in W_1$.

Lemma 7.3 is proved.

A similar argument shows the following two lemmas.

Lemma 7.4. *If $v \in S\big(\widehat{H_S}^-\big)$ and for all positive integer m,*

$$S_{m-1/2}(\alpha_1)v \otimes 1 = 0, S_{m-1/2}(\alpha_2)v \otimes 1 = 0,$$

then v belongs to the subspace generated by $Z^{[0]}_{n/2},\ Z^{[3]}_{n/2},\ldots,Z^{[l-1]}_{n/2},\ Z^{[l]}_{n/2} = -Z^{[0]}_{n/2}$.

Lemma 7.5. *If $v \in S\big(\widehat{H_S}^-\big)$ and for all positive integer m and $1 < i < l$,*

$$S_{m-1/2}(\alpha_i)v \otimes 1 = 0,$$

then v belongs to the subspace generated by $Z^{[0]}_n$.

Similarly to the proof for $s = 0$ above, for general s, we have the following lemma.

Lemma 7.6. *Suppose that $v \in S\big(\widehat{H_S}^-\big)$ and $0 < s < l$. If*

$$S_{m-1/2}(\alpha_i)v \otimes 1 = 0, \quad 0 < i < l, \quad i \neq s, \qquad S_m(\alpha_s)v \otimes 1 = 0 \quad (\textit{when } (\alpha_s|\alpha_s) = 1),$$

for all positive integer m, then v belongs to the subspace generated by $Z^{[s]}_n$.

Lemma 7.7. *For any $0 \leq s \leq l$, the element $1 \otimes e^{\gamma_s}$ is a highest weight vector.*

Lemma 7.8. *For odd $l \geq 3$, Ω_s has basis*

$$\left\{ Z^{[s]}_{n_1}\ldots Z^{[s]}_{n_k} \otimes 1 \;\Big|\; n_p \in \frac{1}{2}\mathbb{Z}_-,\ n_p \leq n_{p+1},\ n_p \leq n_{p+r} - 1,\ n_{k-\sigma(s)} \leq -1 \right\}.$$

For even $l \geq 2$, Ω_s has basis

$$\Bigg\{ Z^{[s]}_{n_1}\ldots Z^{[s]}_{n_k} \otimes 1 \;\Big|\; n_p \in \frac{1}{2}\mathbb{Z}_-,\ n_p - n_{p+r} < -1 \Rightarrow$$

$$\Rightarrow \sum_{i=0}^{r} n_{p+i} \in \mathbb{Z},\ n_p \leq n_{p+r} - 1,\ n_{k-\sigma(i)} \leq -1 \Bigg\},$$

here $r = \dfrac{l-1}{2}$ if l odd and $r = \dfrac{l}{2}$ if l even, $\sigma(s) = s$ for $s \leq r$, otherwise, $\sigma(s) = r + 1 - s$.

Lemma 7.9. *For any $0 \leq s \leq l$,*

$$\mathrm{ch}_q\Omega_s = \prod_{n=1}^{\infty} \frac{(1 - q^{l(l+2)n})(1 - q^{l[(l+2)n-s-1]})(1 - q^{l[(l+2)n-l+s-1]})}{1 - q^{ln}}.$$

For Lemmas 7.8 and 7.9, one can refer [8] (Theorem 10.4), [9] (Section 14) and [2] (Section 3).

Lemmas 7.1 – 7.9 prove Theorem 6.1.

8. Product-sum identities. Since

$$V(P)=\sum_{s=0}^{l}\Omega_s\otimes L\left(\Lambda_s-\frac{s}{4}\delta\right),$$

we have the specialized character

$$\mathrm{ch}_qV(P)=\sum_{s=0}^{l}\mathrm{ch}_q\Omega_s\ \mathrm{ch}_qL\left(\Lambda_s-\frac{s}{4}\delta\right),$$

the left-hand side is

$$\frac{\sum_{n_1,\ldots,n_l\in\mathbb{Z}}q^{\frac{1}{2}(ln_1^2-n_1+ln_2^2-3n_2+\ldots+ln_l^2-(2l-1)n_l)}}{\prod_{n=1}^{\infty}(1-q^{ln})^{l-1}(1-q^{2ln})}$$

which equals

$$\frac{q^{-l^2/8}[\kappa_{q^{1/2}}(l,1)\kappa_{q^{1/2}}(l,3),\ldots,\kappa_{q^{1/2}}(l,l-1)]^2}{\prod_{n=1}^{\infty}(1-q^{ln})^{l-1}(1-q^{2ln})}=q^{-l^2/8}\prod_{n=1}^{\infty}\frac{(1+q^{n-1/2})^2}{1-q^{ln}}$$

for even l, and equals

$$\frac{q^{-\frac{l^2-1}{8}}[\kappa_{q^{1/2}}(l,1)\kappa_{q^{1/2}}(l,3),\ldots,\kappa_{q^{1/2}}(l,l-2)]^2\kappa_{q^{1/2}}(l,l)}{\prod_{n=1}^{\infty}(1-q^{ln})^{l-1}(1-q^{2ln})}=2q^{-\frac{l^2-1}{8}}\prod_{n=1}^{\infty}\frac{(1-q^{2ln-l})}{(1-q^{2n-1})^2}$$

for odd l. Where κ_q is defined by Eqs. (4.1) and (4.2).

The right-hand side is

$$\sum_{s=0}^{l}\mathrm{ch}_q\Omega_s\ \mathrm{ch}_qL\left(\Lambda_s-\frac{s}{4}\delta\right)=\sum_{s=0}^{l}q^{\frac{(l-s)s}{2}}\mathrm{ch}_q\Omega_s\ \dim_q L(\Lambda_s).$$

Then by the computation of Ω_s and $\dim_q L(\Lambda_s)$ before, the proof for our main theorems is finished.

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