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FIXED-POINT THEOREMS AND COMMON FIXED-POINT THEOREMS ON SPACES EQUIPPED WITH VECTOR-VALUED METRICS

ТЕОРЕМИ ПРО НЕРУХОМУ ТОЧКУ ТА СПІЛЬНУ НЕРУХОМУ ТОЧКУ НА ПРОСТОРАХ ІЗ ВЕКТОРНОЗНАЧНОЮ МЕТРИКОЮ

We show the existence of a fixed point and common fixed point for single-valued generalized contractions on spaces equipped with vector-valued metrics.

Показано існування нерухомої точки та спільної нерухомої точки для однозначних узагальнених стискувальних відображень на просторах із векторнозначною метрикою.

1. Introduction. The classical Banach contraction principle was extended for contraction mappings on spaces endowed with vector-valued metrics by Perov in 1964 [4]. Filip et al. [2] studied fixed point property of a self mapping on generalized metric space (X, d) and generalized the results of Perov. In this paper, Theorem 2.1 of [2] is generalized, and local fixed point property of a self mapping on generalized metric space (X, d) is considered. Finally, common fixed point property of two single-valued self mappings on generalized metric space (X, d) is studied.

Throughout this paper $\mathcal{C}, \mathcal{R}$ and $\mathcal{N}$ are the sets of all complex, real and natural numbers, respectively.

Let $(\mathcal{V}, \preceq)$ be an ordered Banach space. The cone $\mathcal{V}_+ = \{v \in \mathcal{V} \colon \theta \preceq v\}$, where θ is the zero-vector of $\mathcal{V}$, satisfies the usual properties

1) $\mathcal{V}_+ \cap -\mathcal{V}_+ = \{\theta\}$;

2) $\mathcal{V}_+ + \mathcal{V}_+ \subset \mathcal{V}_+$;

3) $\alpha\mathcal{V}_+ \subset \mathcal{V}_+$ for $\alpha \geq 0$.

Let X be a nonempty set. A mapping $d \colon X \times X \longrightarrow \mathcal{V}$ is called a vector-valued metric on X, if the following properties are satisfied:

1) $d(x, y) \geq \theta$ for each $x, y \in X$, if $d(x, y) = \theta$, then $x = y$;

2) $d(x, y) = d(y, x)$ for each $x, y \in X$;

3) $d(x, y) \preceq d(x, z) + d(z, y)$ for each $x, y, z \in X$.

The pair (X, d) is called vector-valued metric space. Agarwal and Khamsi [3] (Theorem 2) show that for lower semi-continuous function F from complete vector-valued metric space (X, d) over an order complete and order continuous Banach lattice $\mathcal{V}$, if function $T \colon X \longrightarrow X$ satisfied in the following condition for every $x \in X$

$$d(x, T(x)) \leq F(x) - F(T(x)),$$

then $\mathrm{Fix}(T) \neq \varnothing$. Now, we replace $\mathcal{V}$ by $\mathcal{R}^m$ and have the following definition for vector-valued metric space.

Let X be a nonempty set. A mapping $d \colon X \times X \longrightarrow \mathcal{R}^m$ is called a vector-valued metric on X, if the following properties are satisfied:

1) $d(x,y) \geq 0$ for each $x, y \in X$, if $d(x,y) = 0$, then $x = y$;

2) $d(x,y) = d(y,x)$ for each $x, y \in X$;

3) $d(x,y) \leq d(x,z) + d(z,y)$ for each $x, y, z \in X$.

A set X equipped with a vector-valued metric d is called a generalized metric space and denoted by (X,d). Let x_1 be an element of generalized metric space X and $r = (r_i)_{i=1}^m \in \mathcal{R}^m$, with $r_i > 0$ for each $1 \leq i \leq n$, then $B(x_1, r) = \{x \in X | d(x_1, x) < r\}$ is the open ball centered in x_1 with radius r, also $\widetilde{B}(x_1, r) = \{x \in X | d(x_1, x) \leq r\}$ is the closed ball centered in x_1 with radius r.

Let $f \colon X \longrightarrow X$ be a single-valued map. $\operatorname{Fix}(f) = \{x \in X | f(x) = x\}$ is the set of all fixed points of f.

$M_{m,m}(\mathcal{R}^+)$ means the set of all $m \times m$ matrices with positive elements, Θ the zero matrix, and I the identity $m \times m$ matrix. Let $A \in M_{m,m}(\mathcal{R}^+)$, A is said to be convergent to zero, if and only if $A^n \longrightarrow 0$ as $n \longrightarrow \infty$ (see [7], for more details).

Let $\alpha, \beta \in \mathcal{R}^m$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ and $c \in \mathcal{R}$. Note that $\alpha \leq \beta$ (resp. $\alpha < \beta$) means $\alpha_i \leq \beta_i$ (resp. $\alpha_i < \beta_i$) for each $1 \leq i \leq m$, and also $\alpha \leq c$ (resp. $\alpha < c$) means $\alpha_i \leq c$ (resp. $\alpha_i < c$) for $1 \leq i \leq m$, respectively. As well as, we can define addition and multiplication on $\mathcal{R}^m$ as follows:

$$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_m + \beta_m),$$

and

$$\alpha \cdot \beta = (\alpha_1 \beta_1, \alpha_2 \beta_2, \ldots, \alpha_m \beta_m),$$

for every $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathcal{R}^m$.

Now, we need the following equivalent statements:

1) A is convergent towards zero;

2) $A^n \longrightarrow 0$ as $n \longrightarrow \infty$;

3) the eigenvalues of A are in the open unit disc, that is, $|\lambda| < 1$, for each $\lambda \in C$ with $\det(A - \lambda I) = 0$;

4) the matrix $I - A$ is nonsingular and

$$(I - A)^{-1} = I + A + \ldots + A^n + \ldots;$$

5) $A^n q \longrightarrow 0$ and $qA^n \longrightarrow 0$ as $n \longrightarrow \infty$, for each $q \in \mathcal{R}^m$.

The proof of the above statements are the classical results in matrix analysis (see [1, 5, 6] for more details). For the sake of simplicity, we make an identification between row and column vectors in $\mathcal{R}^m$.

2. Fixed point property. Let (X,d) be a vector-valued metric space and A be an operator such as [3] (Theorem 1). In [3], $d_A(x,y)$ identified by $(I - A)d(x,y)$, for every $x, y \in X$. Then d_A is a vector-valued metric on X, and the vector-valued metric space (X, d_A) is complete.

Theorem 2.1. *Let (X,d) be a complete generalized metric space, and $f \colon X \longrightarrow X$ be a continuous map with the property that, there exists $A, B, C \in M_{m,m}(\mathcal{R}_+)$ such that*

$$d(f(x), f(y)) \leq Ad(x,y) + Bd(x,y)d(y, f(x))[d(x, f(x)) + d(y, f(y))] +$$

$$+ Cd(x,y)[d(x, f(y))d(y, f(x))] \qquad (2.1)$$

for every $x, y \in X$. Suppose A is a matrix converging to zero. Then $\operatorname{Fix}(f) \neq \varnothing$.

Proof. It is sufficient to provide conditions [3] (Theorem 2). By condition (2.1), we have

$$d(f(x), f^2(x)) \leq Ad(x, f(x)), \tag{2.2}$$

for every $x \in X$. Therefore

$$d_A(x, f(x)) = (I - A)d(x, f(x)) \leq d(x, f(x)) - d(f(x), f^2(x)).$$

Similar to [3] (Corollary 1), we define $F(x) := d(x, f(x))$, for every $x \in X$. Then F is a continuous map, and by [3] (Theorem 2]), $\mathrm{Fix}(f) \neq \varnothing$.

We recall $\widetilde{B}(x_1, r) = \{x \in X | d(x_1, x) \leq r\}$, is the closed ball centered in x_1 with radius r.

Theorem 2.2. *Let (X, d) be a complete generalized metric space, $r := (r_i)_{i=1}^m \in \mathcal{R}_+^m$ with $0 < r_i \leq 1$ for each $i \in \{1, 2, \ldots, m\}$ and $f : \widetilde{B}(x_1, r) \longrightarrow X$ having the property that, there exists $A, B, C \in M_{m,m}(\mathcal{R}_+)$ such that*

$$d(f(x), f(y)) \leq A(d(x, y))d(x, y) + Bd(x, y)d(y, f(x))[d(x, f(x)) + d(y, f(y))]+$$

$$+Cd(x, y)[d(x, f(y))d(y, f(x))]$$

for every $x, y \in X$. Suppose

(i) *A is a matrix converging to zero;*

(ii) *if $u \in \mathcal{R}_+^m$ is such that $u(I - A)^{-1} \leq (I - A)^{-1}r$, then $u \leq r$;*

(iii) *$d(x_1, f(x_1))(I - A)^{-1} \leq r$.*

Then $\mathrm{Fix}(f) \neq \varnothing$.

Proof. Construct the sequence $(x_n)_{n \in \mathcal{N}}$ as follows: for each $n \in \mathcal{N}$, set $x_{n+1} = f(x_n)$. By (iii), $d(x_1, x_2)(I - A)^{-1} = d(x_1, f(x_1))(I - A)^{-1} \leq r \leq (I - A)^{-1}r$. Therefore, by using (ii), $d(x_1, x_2) \leq r$ and

$$d(x_1, x_2)(d(x_1, x_2)(I - A)^{-1}) = d(x_1, x_2)(d(x_1, f(x_1))(I - A)^{-1} \leq$$

$$\leq d(x_1, x_2)r \leq d(x_1, x_2)(I - A)^{-1}r \leq r^2.$$

Therefore, $A(d(x_1, x_2))d(x_1, x_2)(I - A)^{-1} \leq Ar^2$. Similarly,

$$d(x_2, x_3)(I - A)^{-1} = d(f(x_1), f(x_2))(I - A)^{-1} \leq$$

$$\leq A(d(x_1, x_2))d(x_1, x_2)(I - A)^{-1}+$$

$$+Bd(x_1, x_2)d(x_2, f(x_1))[d(x_1, f(x_1)) + d(x_2, f(x_2))](I - A)^{-1}+$$

$$+ Cd(x_1, x_2)[d(x_1, f(x_2)d(x_2, f(x_1))](I - A)^{-1} \leq$$

$$\leq Ar^2 + \Theta = Ar^2.$$

Above inequality hold by this fact since $x_2 = f(x_1)$, thus $d(x_2, f(x_1)) = \Theta$,

$$Bd(x_1, x_2)d(x_2, f(x_1))[d(x_1, f(x_1)) + d(x_2, f(x_2))](I - A)^{-1} = \Theta,$$

and $Cd(x_1, x_2)[d(x_1, f(x_2)d(x_2, f(x_1))](I - A)^{-1} = \Theta$.

As well as, since $d(x_1,x_3) \le d(x_1,x_2) + d(x_2,x_3)$, $0 < r < 1$ and $r^2 < r$, therefore,

$$d(x_1,x_3)(I-A)^{-1} \le d(x_1,x_2)(I-A)^{-1} + d(x_2,x_3)(I-A)^{-1} \le$$
$$\le Ir + Ar^2 \le Ir + Ar \le (I + A + A^2 + \ldots + A^n + \ldots)r =$$
$$= (I-A)^{-1}r.$$

By induction, for all $n \in \mathcal{N}$, the sequence (x_n) is in $\widetilde{B}(x_1,r)$ and satisfying
(1) $x_{n+1} = f(x_n)$;
(2) $d(x_0,x_n)(I-A)^{-1} \le (I-A)^{-1}r$;
(3) $d(x_n,x_{n+1})(I-A)^{-1} \le A^{n+1}r$.
We show, (x_n) is a Cauchy sequence. For every $n,m \in \mathcal{N}$, $n \le m$,

$$d(x_n,x_m)(I-A)^{-1} \le d(x_n,x_{n+1})(I-A)^{-1} + d(x_{n+1},x_{n+2})(I-A)^{-1} + \ldots$$
$$\ldots + d(x_{m-1},x_m)(I-A)^{-1} \le$$
$$\le A^{n+1}r + A^{n+2}r + \ldots + A^m r \le$$
$$\le A^{n+1}(I + A + \ldots + A^{m-1} + \ldots)r =$$
$$= A^{n+1}(I-A)^{-1}r \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Therefore (x_n) is a Cauchy sequence. Since the space $\left(\widetilde{B}(x_1,r), d\right)$ is a complete, there exists an element $x^* \in \widetilde{B}(x_1,r)$ such that $x_n \longrightarrow x^*$. Now, x^* is a fixed point of f, because

$$d(x^*, f(x^*)) \le d(x^*, x_n) + d(x_n, f(x^*)) =$$
$$= d(x^*, x_n) + d(f(x_{n-1}), f(x^*)) \le$$
$$\le d(x^*, x_n) + Ad(x_{n-1}, x^*)d(x_{n-1}, x^*) +$$
$$+ Bd(x^*, f(x_{n-1}))[d(x_{n-1}, f(x_{n-1})) + d(x^*, f(x^*))] +$$
$$+ Cd(x^*, f(x_{n-1})d(x_{n-1}, x^*) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Then $\operatorname{Fix}(f) \ne \varnothing$.

An application for above stated theorem is the following operator system in Banach space X with norm $\|\cdot\|$:

$$f_1(u_1,u_2) = u_1,$$
$$f_2(u_1,u_2) = u_2, \tag{2.3}$$

where $f_i \colon X^2 \to X$, $i = 1,2$, are given nonlinear operator.

It is obvious that the system (2.3) can be viewed as a fixed point problem as follows:

$$f(u) = u$$

in the space X^2, where $u = (u_1, u_2)$ and $f = (f_1, f_2)$. Now, we have the following theorem for an application to Theorem 2.2.

Theorem 2.3. *Assume that for $i \in 1, 2$, there exist nonnegative number a_i, a'_i, b_i, b'_i, c_i, c'_i such that*

$$\|f_i(u_1, u_2) - f_i(v_1, v_2)\| \le a_i\|u_1 - v_1\|^2 + a'_i\|u_2 - v_2\|^2 +$$

$$+ b_i\|u_1 - v_1\|\big[(\|v_1 - f_i(u_1, u_2)\| + \|v_2 - f_i(u_1, u_2)\|)\times$$

$$\times(\|u_1 + v_1 - f_i(u_1, u_2) - f_i(v_1, v_2)\| + \|u_2 + v_2 - f_i(u_1, u_2) - f_i(v_1, v_2)\|)\big] +$$

$$+ b'_i\|u_2 - v_2\|\big[(\|v_1 - f_i(u_1, u_2)\| + \|v_2 - f_i(u_1, u_2)\|)\times$$

$$\times(\|u_1 + v_1 - f_i(u_1, u_2) - f_i(v_1, v_2)\| + \|u_2 + v_2 - f_i(u_1, u_2) - f_i(v_1, v_2)\|)\big] +$$

$$+ c_i\|u_1 - v_1\|\big[(\|u_1 - f_i(v_1, v_2)\| + \|u_2 - f_i(v_1, v_2)\|)(\|v_1 - f_i(u_1, u_2)\| + \|v_2 - f_i(u_1, u_2)\|)\big] +$$

$$+ c'_i\|u_2 - v_2\|\big[(\|u_1 - f_i(v_1, v_2)\| + \|u_2 - f_i(v_1, v_2)\|)\times$$

$$\times(\|v_1 - f_i(u_1, u_2)\| + \|v_2 - f_i(u_1, u_2)\|)\big] \tag{2.4}$$

for all $u_1, u_2, v_1, v_2 \in X$. In addition assume that $A = \begin{pmatrix} a_1 & a'_1 \\ a_2 & a'_2 \end{pmatrix}$ is a convergence to zero matrix, and $B = \begin{pmatrix} b_1 & b'_1 \\ b_2 & b'_2 \end{pmatrix}$ and $\begin{pmatrix} c_1 & c'_1 \\ c_2 & c'_2 \end{pmatrix}$ belong to $M_{2\times 2}(\mathcal{R}^+)$. Then (2.3) has a unique solution $u = (u_1, u_2)$ in X^2.

Proof. Condition (2.4) can be rewritten as

$$\|f(u) - f(v)\| \le A\|u - v\|^2 + B[\|u - v\|\|v - f(v)\|(\|u - f(u)\| + \|v - f(v)\|)] +$$

$$+ C[\|u - v\|(\|u - f(v)\|\|v - f(u)\|)].$$

Thus Theorem 2.2 implies our desire. Here $X^2 = R^2$ and $d(u, v) = \|u - v\|$.

3. Common fixed-point theorem. Let f and g be two self mappings on complete generalized metric space (X, d). In this section, we study the existence of a common fixed point for these mappings. Due to this, we need the following lemma.

Lemma 3.1. *Let (X, d) be a complete generalized metric space and $(y_n)_{n\in\mathcal{N}\cup\{0\}}$ be a sequence in X. If $A \in M_{m,m}(\mathcal{R}^+)$ is a matrix converging to zero, and for every $n \in \mathcal{N}$*

$$d(y_n, y_{n+1}) \le Ad(y_n, y_{n-1}). \tag{3.1}$$

Then $(y_n)_{n\in\mathcal{N}}$ is converging in X.

Proof. By (3.1) we have

$$d(y_n, y_{n-1}) \le Ad(y_{n-1}, y_{n-2}), d(y_{n-1}, y_{n-2}) \le Ad(y_{n-2}, y_{n-3}), \ldots, d(y_2, y_1) \le Ad(y_1, y_0).$$

Thus

$$d(y_{n+1}, y_n) \leq Ad(y_n, y_{n-1}) \leq A^2 d(y_{n-1}, y_{n-2}) \leq \ldots \leq A^n d(y_1, y_0) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Therefore (y_n) is a Cauchy sequence. Since (X, d) is a complete, there exists an element $x \in X$ such that $y_n \longrightarrow x$.

Theorem 3.1. *Let (X, d) be a complete generalized metric space, $f \colon X \longrightarrow X$ be a continuous function and $A \in M_{m,m}(\mathcal{R}^+)$ be a nonzero matrix converging to zero. Suppose g is a selfmap function such that $g \circ f = f \circ g$, $g(X) \subset f(X)$, and*

$$d(g(x), g(y)) \leq Ad(f(x), f(y)), \quad x, y \in X. \tag{3.2}$$

Then f and g have a unique common fixed point.

Proof. Choose the elements $x_0, x_1 \in X$ such that $f(x_1) = g(x_0)$. Construct the sequence $(x_n)_{n \in \mathcal{N} \cup \{0\}}$, as follows

$$f(x_n) = g(x_{n-1}).$$

By the assumption $g(X) \subset f(X)$, and (3.2), we have

$$d(g(x_n), g(x_{n-1})) \leq Ad(f(x_n), f(x_{n-1})),$$

$$d(f(x_n), f(x_{n-1})) = d(g(x_{n-1}), g(x_{n-2})) \leq Ad(f(x_{n-1}), f(x_{n-2})),$$

. . . .

$$d(f(x_2), f(x_1)) = Ad(g(x_1), g(x_0)) \leq Ad(f(x_1), f(x_0)).$$

Thus

$$d(f(x_{n+1}), f(x_n)) = d(g(x_n), g(x_{n-1})) \leq$$

$$\leq Ad(f(x_n), f(x_{n-1})) = Ad(g(x_{n-1}), g(x_{n-2})) \leq$$

$$\leq A^2 d(f(x_{n-1}), f(x_{n-2})) \leq \ldots$$

$$\ldots \leq A^n d(f(x_1), f(x_0)) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Therefore $(f(x_n))$ is a Cauchy sequence. Thus by Lemma 3.1, there exists a $t \in X$ such that $f(x_n) \longrightarrow t$. Also, by the definition of g, $g(x_n) \longrightarrow t$. Continuity of f implies that g is a continuous map. Thus $g(f(x_n)) \longrightarrow g(t)$ and since $g \circ f = f \circ g$, so $f(g(x_n)) \longrightarrow f(t)$. By these results, $f(t) = g(t)$ and $f(f(t)) = f(g(t)) = g(g(t))$. But,

$$d(g(t), g(g(t))) \leq Ad(f(t), f(g(t))) = Ad(g(t), g(g(t))),$$

and so

$$d(g(t), g(g(t)))(I - A) \leq 0.$$

But $I \neq A$, hence $d(g(t), g(g(t))) = 0$. This means

$$g(g(t)) = g(t) = f(g(t)) = g(f(t)).$$

Therefore $g(t) \in \mathrm{Fix}(f) \cap \mathrm{Fix}(g)$. Finally, suppose there are $x, y \in X$ such that $x, y \in \mathrm{Fix}(f) \cap \cap \mathrm{Fix}(g)$. Then by (2.2),

$$d(x,y) = d(g(x), g(y)) \leq Ad(f(x), f(y)) = Ad(x,y),$$

and so

$$d(x,y)(I - A) \leq 0.$$

Therefore $x = y$ and the common fixed point of f and g is unique.

***Corollary* 3.1.** *Let f and g be two commuting and self mappings on a complete generalized metric space (X, d). Suppose f is continuous and $g(X) \subset f(X)$. If, there exists a matrix $A \in \in M_{m,m}(\mathcal{R}^+)$ such that A converges to zero, and for each $x, y \in X$, following condition holds*

$$d(g^k(x), g^k(y)) \leq Ad(f(x), f(y)), \quad k \in \mathcal{N}.$$

Then f and g have a common fixed point.

Proof. For every $k > 1$,

$$g^k \circ f = g^{k-1} \circ g \circ f = g^{k-1} \circ f \circ g = \ldots = f \circ g^k,$$

also $g^k(X) \subseteq g(X) \subset f(X)$. Therefore, by Theorem 3.1, g^k and f have a unique common fixed point. Let $a \in X$ be the unique common fixed point of g^k and f. Thus $a = f(a) = g^k(a)$. Since f and g are commuting mappings, then

$$g(a) = g(f(a)) = g(g^k(a)) = g^k(g(a)) = f(g(a)).$$

Thus $g(a)$ is a common fixed point of g^k and f. Since, the common fixed point of g^k and f was unique. Hence, we should have $a = g(a) = f(a)$.

***Corollary* 3.2.** *Let $n \in \mathcal{N}$ and A invertible matrix which $A > I$. Suppose g is a continuous self mapping on a complete generalized metric space (X, d) satisfying:*

$$d(g^n(x), g^n(y)) \geq Bd(x,y), \quad x, y \in X,$$

where B is the matrix A^{-1}. Then g has a unique fixed point.

Proof. Take $f = g^{n+1}$, then proof is clear.

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