On lifting of functors to the Eilenberg-Moore category of the triple generated by the functor $C_pC_p$

The second iteration of contravariant functor of the space of continuous functions with topology of pointwise convergence is functorial part of a triple on the category of Tychonoff spaces. The problem of lifting of functors to the Eilenberg-Moore category of this triple is investigated.

1. Triple generated by the functor $C_pC_p$. A triple $T = (T, \eta, \mu)$ on category $C$ consists of an endofunctor $T : C \to C$ and natural transformations $\eta : 1_C \to T$ and $\mu : TT \to T$ satisfying the following conditions: $\mu \circ \eta T = \eta T \circ \mu = \eta T \circ \mu T \circ \eta = 1_T$. Denote by $Tych$ the category of Tychonoff topological spaces and continuous maps. The contravariant functor $C_p : Tych \to Tych$ is defined as follows. The space $C_p(X)$ of all continuous real-valued functions on $X$ is equipped with topology of pointwise convergence [2]; for mapping $f : X \to Y$ (both $X$ and $Y$ are Tychonoff spaces) we define $C_pf : C_pY \to C_pX$ by the formula: $C_pf(\varphi) = \varphi \circ f$, $\varphi \in C_pY$. For every $x \in X$ denote by $ev_x : C_pX \to \mathbb{R}$ a map defined by the formula $ev_x(\varphi) = \varphi(x)$, $\varphi \in C_pX$. It is well known that the map $\eta X : X \to C_pC_pX$, $\eta X(x) = ev_x$, $x \in X$, is continuous [2]. It is
easy to see that $\eta = (\eta X)$ is natural transformation from $1_{\text{Tych}}$ into $C_{p}C_{p}$. We construct the natural transformation $\mu : C_{p}C_{p}C_{p}C_{p} \to C_{p}C_{p}$ using the following equality $\mu X (\Phi)(\psi) = \Phi (ev_{\psi})$, $\Phi \in C_{p}C_{p}C_{p}C_{p}X$, $\psi \in C_{p}X$. The map $\mu X(\Phi) : C_{p}X \to \mathbb{R}$ is continuous as the composition of $ev : C_{p}X \to C_{p}C_{p}C_{p}X$ and $\Phi : C_{p}C_{p}C_{p}C_{p}X \to \mathbb{R}$, so $\mu X$ is correctly defined. In order to see that $\mu X$ is continuous we need to verify a simple inclusion $\mu X(\Phi, ev_{\psi}, \varepsilon) \subseteq (\mu X(\Phi), \psi, \varepsilon)$ and to use the fact that the sets $(\Gamma, \psi, \varepsilon) = \{ E \in C_{p}C_{p}C_{p}X | E(\psi) - \Gamma(\psi) | < \varepsilon \}$ form a subbase in $C_{p}C_{p}C_{p}X$ (for $\Gamma \in C_{p}C_{p}C_{p}X$).

**Proposition 1.** $D_p^2 = (C_{p}C_{p}, \varepsilon, \mu)$ is a triple on the category Tych.

**Proof.** Since

$$\mu X \circ C_{p}C_{p}X(\psi)(\phi) = C_{p}C_{p}X(\psi)(\phi)(ev_{\psi}) = (\psi(\phi)) \in C_{p}C_{p}C_{p}X$$

and

$$\mu X \circ C_{p}C_{p}X(\phi)(\psi) = C_{p}C_{p}X(\phi)(\psi)(ev_{\phi}) = \psi(\phi) 
= \mu X \circ C_{p}C_{p}X(\phi) \in C_{p}C_{p}C_{p}X,$$

we obtain $\mu \circ C_{p}C_{p}C_{p}C_{p} = \varepsilon$. For any $\psi \in C_{p}C_{p}C_{p}C_{p}X$, $\phi \in C_{p}X$ we have: $C_{p}C_{p}X(\phi)(\psi) = C_{p}C_{p}X(\phi)(\psi)(ev_{\phi}) = (\mu X(\phi)) \times \times (\psi) = \psi(ev_{\phi})$, and therefore $C_{p}C_{p}X(\phi) = ev_{\phi}$. Putting this equalities together we can obtain

$$\mu X \circ C_{p}C_{p}X(\psi)(\phi) = C_{p}C_{p}X(\psi)(\phi)(ev_{\psi}) = (\psi(\phi)) \in C_{p}C_{p}C_{p}X$$

and

$$\mu X \circ C_{p}C_{p}X(\phi)(\psi) = C_{p}C_{p}X(\phi)(\psi)(ev_{\phi}) = \psi(\phi) = \mu X \circ C_{p}C_{p}X(\phi) \in C_{p}C_{p}C_{p}X.$$

The proposition is proved.

Remark that the space $C_{p}C_{p}X$ has obvious algebra structure with respect to pointwise addition and multiplication of functions and multiplication on scalars. Let $L_{p}X$ be the linear subspace of $C_{p}X$ generated by the image of $X$ under the mapping $\eta X$ and $A_{p}X$ be the least subalgebra of $C_{p}X$ including the set $\eta X$ (X). It is clear that both $L_{p}X$ and $A_{p}X$ are subobjects of $C_{p}X$.

The condition $\mu X : \eta C_{p}C_{p}X = 1_{\mathcal{X}}$ implies that $\mu X(L_{p}X) = L_{p}X$ and $\mu X(A_{p}X) = A_{p}X$. Therefore, we obtain two new triples $L_{p} = (L_{p}, \eta, \mu | L_{p}X)$, $A_{p} = (A_{p}, \eta, \mu | A_{p}X)$ on the category Tych.

2. Normal functors on the category Tych. Functor $F : \text{Tych} \to \text{Tych}$ is called normal if it is continuous, preserves weight, monomorphisms, intersections, inverse images, empty space, one point space and transforms $\varepsilon$-covering maps into surjective (see [3]). (Note that a map $f : X \to Y$ is $\varepsilon$-covering iff for arbitrary compact subspace $K \subseteq Y$ there exists a compact subspace $L \subseteq X$ such that $f(L) = K$.)

A normal functor $F : \text{Tych} \to \text{Tych}$ is said to be of degree $\leq n$ (briefly $\deg (F) \leq n$) if for every $a \in FX$ there exists $b \in Fn$ and mapping $f : n \to X$ such that $a = f(b)$. A normal functor is called finite if it preserves the class of finite spaces and is called multiplicative if it preserves products.

**Proposition 2.** Normal multiplicative functor $F$ is isomorphism to power functor $(-)^{i}$ for some $i < \infty$, if either

a) $\deg (F) = n$ (and then $i = n$); or
b) $F$ is finite

**Proof.** See [4, 5].

**Theorem 1.** Let $F$ be a normal multiplicative functor $F : \text{Tych} \to \text{Tych}$ such that $F(\text{Comp}) \subseteq \text{Comp}$. Then $F$ is a subfunctor of $(-)^{\infty}$.

**Proof.** Without loss of generality we can assume that $\deg (F) = \infty$. One can find in [6] the following result: there exists a functor isomorphism $h : F\text{Comp} \to (-)^{\infty}$ of $\text{Comp}$. First we prove that for every Tychonoff space $X$ and its compactification $i_{X} : X \to bX$ the following inclusions hold:

$$h_{X} \circ F i_{X}(FX) \subseteq X^{\infty} \subseteq (bX)^{\infty}.$$ 

Assuming the contrary, without loss of generality we can suppose $X$ to be the discrete countable space and $bX$ to be $\alpha X$ (the one point compactification of $X$). Let $a \in h_{X} \circ F i_{X}(FX) / X^{\infty}$. Then there exists a sequence $(b_{1}, b_{2}, ...)$ in $F_{\alpha}X$ converging to $a$ and such that the supports of $b_{i}$ are finite and lie in $X$.

Suppose that $a = (x_{i})_{i=1,2,...} \in (\alpha X)^{\infty}$. Without loss of generality we can assume that $x_{i} \in \alpha X \setminus X$. Suppose that $h_{X} \circ F i_{X}(b_{i}) = (y_{i})_{i=1,2,...}$. There exists a mapping $f : X \to Y$ such that the sequence $(f(y_{i}))_{i=1,2,...}$ is not
convergent in $\alpha X$. Therefore the sequence $(F_i x \otimes F (b_i))_{i=1,2,...}$ is not convergent in $F (\alpha X)$, and we get a contradiction.

Now we define the natural transformation $j : F \to (\alpha \otimes)$ by the formula: $j(x) = h_\beta x \otimes F_i x$ where $i_\beta : X \to \beta X$ is the canonical embedding $X$ into Stone-Cech compactification $\beta X$ of space $X$. Theorem is proved.

3. Lifting normal functors to the Eilenberg-Moore category. A couple $(X, \xi)$, where $\xi : TX \to X$ is $C$-morphism, is called $T$-algebra iff $\xi \circ \mu X = 1$, and $\xi \circ \mu X = \xi \circ T\xi$. A morphism $f : X \to Y$ is called morphism of $T$-algebra $(X, \xi)$ into $T$-algebra $(Y, \zeta)$ if $f \circ \xi = \xi \circ T \xi$. $T$-algebras and their morphisms form a category which is usually denoted by $C^T$ (Eilenberg-Moore category). We can define the forgetful functor $U^T : C^T \to C$ by $U^T (X, \xi) = X$, $U^T (f) = f$. (For details see [11].)

A lifting of functor $F : C \to C$ on the category $C^T$ is a functor $G : C^T \to C^T$ such that $U^T \circ G = F \circ U^T$. The following proposition gives a criterion of existence of a lifting: it is dual to a result of J. Vinárek [7] (see also [6]).

**Proposition 3** (see [9]). There exists a bijective correspondence between the liftings of functor $F$ to $C^T$ and the such natural transformations $\delta : F^T \to F^T$ that $\delta \circ \mu^T = F^T$ and $\delta \circ \mu^T = F^T \circ \delta T \delta$.

Let $T$ denote one of the triples $(\alpha \otimes, A_p, L_p)$. We use below the method used in [11] for describing functors which admit a lifting to the category of compact groups.

**Theorem 2. If a normal functor $F$ can be lifted to the category $Tych^T$, then $F$ is multiplicative.**

**Proof.** Let $T$ be one of the functors $C_p C_p, L_p, A_p$. We consider the free $T$-algebra $(TQ, \mu Q)$ denoting $TQ$ by $X$ ($Q$ is the Hilbert cube). Supposing that $F$ admit a lifting to $Tych^T$ we obtain that the mapping $f = (Fpr_1, Fpr_2) : F (X \times X) \to FX \times FX$ is bijective. Indeed, from the conditions of preserving inverse images and intersections by $F$ we obtain $\ker (f) = 0$ (here we use fact that $f$ is linear mapping of topological linear spaces). Besides, since the set $\ker (Fpr_1) = F (\ker (pr_1))$ is homeomorphically mapped onto $FX$ by the mapping $Fpr_2$ we obtain that $f$ is surjective (see [15]).

Since $Q$ can be topologically embedded into $X$, we obtain that $F$ is multiplicative (see [4]).

**Corollary.** If $F$ is a normal functor admitting a lifting to the category $Tych^T$ and $F$ is either finite or deg $(F) < \infty$, then $F$ is isomorphic to a power functor.

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