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S. S. Dragomir (College Eng. and Sci., Victoria Univ., Melbourne City, Australia;
School Comput. and Appl. Math., Univ. Witwatersrand, Johannesburg, South Africa)

NEW INEQUALITIES FOR THE p-ANGULAR DISTANCE IN NORMED SPACES WITH APPLICATIONS

НОВІ НЕРІВНОСТІ ДЛЯ p-КУТОВОЇ ВІДСТАНІ В НОРМОВАНИХ ПРОСТОРАХ ТА ЇХ ЗАСТОСУВАННЯ

For nonzero vectors x and y in the normed linear space $(X, \|\cdot\|)$, we can define the p-angular distance by

$$\alpha_p[x,y] := \left\| \|x\|^{p-1}x - \|y\|^{p-1}y \right\|.$$

We show (among other results) that, for $p \geq 2$,

$$\alpha_p[x,y] \leq p\|y-x\| \int\limits_0^1 \|(1-t)x + ty\|^{p-1}\, dt \leq$$

$$\leq p\|y-x\| \left[\frac{\|x\|^{p-1} + \|y\|^{p-1}}{2} + \left\| \frac{x+y}{2} \right\|^{p-1} \right] \leq$$

$$\leq p\|y-x\| \frac{\|x\|^{p-1} + \|y\|^{p-1}}{2} \leq p\|y-x\| \left[\max\{\|x\|, \|y\|\}\right]^{p-1},$$

for any $x, y \in X$. This improves a result of Maligranda from [Simple norm inequalities // Amer. Math. Month. – 2006. – **113**. – P. 256–260] who proved the inequality between the first and last terms in the estimation presented above. The applications to functions f defined by power series in estimating a more general "distance" $\|f(\|x\|)x - f(\|y\|)y\|$ for some $x, y \in X$ are also presented.

Для ненульових векторів x та y в лінійному нормованому просторі $(X, \|\cdot\|)$ можна визначити p-кутову відстань таким чином:

$$\alpha_p[x,y] := \left\| \|x\|^{p-1}x - \|y\|^{p-1}y \right\|.$$

У роботі, зокрема, показано, що

$$\alpha_p[x,y] \leq p\|y-x\| \int\limits_0^1 \|(1-t)x + ty\|^{p-1}\, dt \leq$$

$$\leq p\|y-x\| \left[\frac{\|x\|^{p-1} + \|y\|^{p-1}}{2} + \left\| \frac{x+y}{2} \right\|^{p-1} \right] \leq$$

$$\leq p\|y-x\| \frac{\|x\|^{p-1} + \|y\|^{p-1}}{2} \leq p\|y-x\| \left[\max\{\|x\|, \|y\|\}\right]^{p-1}$$

для $p \geq 2$ і будь-яких $x, y \in X$. Це покращує результат Малігранди [Simple norm inequalities // Amer. Math. Month. – 2006. – **113**. – P. 256–260], який встановив нерівність між першим та останнім членами вказаної оцінки. Також наведено застосування для функцій f, визначених степеневими рядами при оцінюванні більш загальної „відстані" $\|f(\|x\|)x - f(\|y\|)y\|$ для деяких $x, y \in X$.

1. Introduction. Following [3, p. 403] or [12], for nonzero vectors x and y in the normed linear space $(X, \|\cdot\|)$ we define the *angular distance* $\alpha[x,y]$ between x and y by

$$\alpha[x,y] := \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|.$$

In 1958, Massera and Schäffer [12] (Lemma 5.1) showed that

$$\alpha[x,y] \le \frac{2\|x-y\|}{\max\{\|x\|,\|y\|\}}, \tag{1.1}$$

which is better than the *Dunkl – Williams inequality* [7]

$$\alpha[x,y] \le \frac{4\|x-y\|}{\|x\|+\|y\|}. \tag{1.2}$$

We notice that the *Massera – Schäffer inequality* was rediscovered by Gurariĭ in [8] (see also [13, p. 516]).

In [11], Maligranda obtained the double inequality

$$\frac{\|x-y\| - |\|x\|-\|y\||}{\min\{\|x\|,\|y\|\}} \le \alpha[x,y] \le \frac{\|x-y\| + |\|x\|-\|y\||}{\max\{\|x\|,\|y\|\}}. \tag{1.3}$$

The second inequality in (1.3) is better than Massera – Schäffer's inequality (1.1).

In the recent paper [11], L. Maligranda has also considered the *p-angular distance*

$$\alpha_p[x,y] := \left\| \|x\|^{p-1}x - \|y\|^{p-1}y \right\|$$

between the vectors x and y in the normed linear space $(X, \|\cdot\|)$ over the real or complex number field $\mathbb{K}$ and showed that

$$\alpha_p[x,y] \le \|x-y\| \begin{cases} (2-p)\dfrac{\max\{\|x\|^p,\|y\|^p\}}{\max\{\|x\|,\|y\|\}} & \text{if} \quad p\in(-\infty,0) \quad \text{and} \quad x,y\ne 0, \\ (2-p)\dfrac{1}{[\max\{\|x\|,\|y\|\}]^{1-p}} & \text{if} \quad p\in[0,1] \quad \text{and} \quad x,y\ne 0, \\ p\,[\max\{\|x\|,\|y\|\}]^{p-1} & \text{if} \quad p\in(1,\infty). \end{cases} \tag{1.4}$$

The constants $2-p$ and p in (1.1) are best possible in the sense that they cannot be replaced by smaller quantities.

As pointed out in [11], the inequality (1.1) for $p\in[1,\infty)$ is better than the Bourbaki inequality obtained in 1965 [2, p. 257] (see also [13, p. 516]):

$$\alpha_p[x,y] \le 3p\|x-y\|\,[\|x\|+\|y\|]^{p-1}, \quad x,y\in X. \tag{1.5}$$

The following results concerning upper bounds for the p-angular distance have been obtained by the author in [5]:

$$\alpha_p[x,y] \le$$

$$\le \begin{cases} \|x-y\|\left[\max\{\|x\|,\|y\|\}\right]^{p-1} + \left|\|x\|^{p-1}-\|y\|^{p-1}\right|\min\{\|x\|,\|y\|\} & \text{if} \quad p\in(1,\infty), \\ \dfrac{\|x-y\|}{[\min\{\|x\|,\|y\|\}]^{1-p}} + \left|\|x\|^{1-p}-\|y\|^{1-p}\right|\min\left\{\dfrac{\|x\|^p}{\|y\|^{1-p}},\dfrac{\|y\|^p}{\|x\|^{1-p}}\right\} & \text{if} \quad p\in[0,1], \\ \dfrac{\|x-y\|}{[\min\{\|x\|,\|y\|\}]^{1-p}} + \dfrac{\left|\|x\|^{1-p}-\|y\|^{1-p}\right|}{\max\left\{\|x\|^{-p}\|y\|^{1-p},\|y\|^{-p}\|x\|^{1-p}\right\}} & \text{if} \quad p\in(-\infty,0), \end{cases} \tag{1.6}$$

and

$$\alpha_p[x,y]\leq$$

$$\leq\begin{cases}\|x-y\|\left[\min\{\|x\|,\|y\|\}\right]^{p-1}+\left|\|x\|^{p-1}-\|y\|^{p-1}\right|\max\{\|x\|,\|y\|\} & \text{if}\quad p\in(1,\infty),\\ \dfrac{\|x-y\|}{[\max\{\|x\|,\|y\|\}]^{1-p}}+\left|\|x\|^{1-p}-\|y\|^{1-p}\right|\max\left\{\dfrac{\|x\|^p}{\|y\|^{1-p}},\dfrac{\|y\|^p}{\|x\|^{1-p}}\right\} & \text{if}\quad p\in[0,1],\\ \dfrac{\|x-y\|}{[\max\{\|x\|,\|y\|\}]^{1-p}}+\dfrac{\left|\|x\|^{1-p}-\|y\|^{1-p}\right|}{\min\{\|x\|^{-p}\|y\|^{1-p},\|y\|^{-p}\|x\|^{1-p}\}} & \text{if}\quad p\in(-\infty,0),\end{cases}\tag{1.7}$$

for any two nonzero vectors x,y in the normed linear space $(X,\|\cdot\|)$.

The upper bounds for $\alpha_p[x,y]$ provided by (1.4), (1.6) and (1.7) have been compared in [5] to conclude that some of the later ones are better in certain cases. The details are omitted here.

The following result which provides a lower bound for the *p-angular distance* was stated without a proof by Gurariĭ in [8] (see also [13, p. 516]):

$$2^{-p}\|x-y\|^p\leq\alpha_p[x,y],\tag{1.8}$$

where $p\in[1,\infty)$ and $x,y\in X$. The proof of the inequality (1.8) is still an open question for the author.

Finally, we recall the results of G. N. Hile from [4]:

$$\alpha_p[x,y]\leq\frac{\|x\|^p-\|y\|^p}{\|x\|-\|y\|}\|x-y\|,\tag{1.9}$$

for $p\in[1,\infty)$ and $x,y\in X$ with $\|x\|\neq\|y\|$, and

$$\alpha_{-p-1}[x,y]\leq\frac{\|x\|^p-\|y\|^p}{\|x\|-\|y\|}\frac{\|x-y\|}{\|x\|^p\,\|y\|^p},\tag{1.10}$$

for $p\in[1,\infty)$ and $x,y\in X\setminus\{0\}$ with $\|x\|\neq\|y\|$.

2. Integral bounds for p-angular distance. The following result holds.

Theorem 2.1. *Let $(X;\|\cdot\|)$ be a normed linear space and $p\geq1$. Then for any $x,y\in X$ we have the inequality*

$$\alpha_p[x,y]\leq p\|y-x\|\int_0^1\|(1-t)x+ty\|^{p-1}\,dt.\tag{2.1}$$

If the vectors $x,y\in X$ are linearly independent and $p<1$, then we have the inequality

$$\alpha_p[x,y]\leq(2-p)\,\|y-x\|\int_0^1\|(1-t)x+ty\|^{p-1}\,dt.\tag{2.2}$$

Proof. Assume that $x \neq y$. For $p \geq 2$, consider the function $f_p : [0,1] \to [0,\infty)$ given by $f_p(t) = \|(1-t)x + ty\|^{p-1}$. The function f_p is convex on the interval $[0,1]$ for all $p \geq 2$. Therefore the lateral derivatives f'_{p+} and f'_{p-} exist on each point of the interval $[0,1)$ and $(0,1]$, respectively, and they are equal except a countably number of points in the interval $(0,1)$. The function f_p is absolutely continuos on $[0,1]$, the derivative f'_p exists almost everywhere on $[0,1]$ and (see, for instance, [14], Chapter IV)

$$f'_p(t) = (p-1)\,\|(1-t)x + ty\|^{p-2}\,\tau_{+(-)}\left((1-t)x + ty, y - x\right) \tag{2.3}$$

almost everywhere on $[0,1]$, where the *tangent functional* $\tau_{+(-)}$ is defined by

$$\tau_{+(-)}(u,v) := \begin{cases} \lim_{s\to 0+(-)} \dfrac{\|u + sv\| - \|u\|}{s} & \text{if} \quad u \neq 0, \\ +(-)\,\|v\| & \text{if} \quad u = 0. \end{cases} \tag{2.4}$$

Now, if we consider the vector valued function $g_p : [0,1] \to X$ given by

$$g_p(t) := f_p(t)\,[(1-t)x + ty]$$

then we observe that g_p is strongly differentiable almost everywhere on $[0,1]$ and (see, for instance, [1], Chapter 1)

$$g'_p(t) = f'_p(t)\,[(1-t)x + ty] + f_p(t)\,(y - x) =$$

$$= (p-1)\,\|(1-t)x + ty\|^{p-2}\,\tau_{+(-)}\left((1-t)x + ty, y - x\right) \times$$

$$\times\,[(1-t)x + ty] + \|(1-t)x + ty\|^{p-1}\,(y - x)$$

for almost every $t \in [0,1]$.

Since for any $u, v \in H$ with $u \neq 0$ we have

$$\left|\tau_{+(-)}(u,v)\right| \leq \|v\|,$$

then

$$\left\|g'_p(t)\right\| \leq (p-1)\,\|(1-t)x + ty\|^{p-1}\left|\tau_{+(-)}\left((1-t)x + ty, y - x\right)\right| +$$

$$+\,\|(1-t)x + ty\|^{p-1}\,\|y - x\| \leq$$

$$\leq (p-1)\,\|(1-t)x + ty\|^{p-1}\,\|y - x\| + \|(1-t)x + ty\|^{p-1}\,\|y - x\| =$$

$$= p\,\|(1-t)x + ty\|^{p-1}\,\|y - x\|$$

for almost every $t \in [0,1]$.

By the norm inequality for the vector-valued integral we have (see, for instance, [1], Chapter 1)

$$\left\|\|y\|^{p-1}y - \|x\|^{p-1}x\right\| = \|g_p(1) - g_p(0)\| =$$

$$= \left\| \int_0^1 g_p'(t)dt \right\| \le \int_0^1 \left\| g_p'(t) \right\| dt \le$$

$$\le p\|y - x\| \int_0^1 \|(1-t)x + ty\|^{p-1} dt$$

and the proof of (2.1) is complete.

Let $p \in (1, 2)$. The function $f_p \colon [0, 1] \to [0, \infty)$ given by $f_p(t) = \|(1-t)x + ty\|^{p-1}$ is absolutely continuous on $[0, 1]$ and the equality (2.3) also holds almost everywhere on $[0, 1]$. The above argument can then be extended to this case as well and the inequality (2.1) also holds.

If the vectors $x, y \in X$ are linearly independent and $p < 1$, then $\|(1-t)x + ty\| > 0$ for any $t \in [0, 1]$ and the function $h_p \colon [0, 1] \to [0, \infty)$ given by $h_p(t) = \|(1-t)x + ty\|^{p-1}$ is absolutely continuous on $[0, 1]$ and

$$h_p'(t) = (p-1) \|(1-t)x + ty\|^{p-2} \tau_{+(-)} ((1-t)x + ty, y - x) \tag{2.5}$$

almost everywhere on $[0, 1]$.

If we consider the vector valued function $m_p \colon [0, 1] \to X$ given by

$$m_p(t) := h_p(t) \big[(1-t)x + ty\big],$$

then we observe that m_p is strongly differentiable almost everywhere on $[0, 1]$ and

$$m_p'(t) = h_p'(t) \left[(1-t)x + ty\right] + h_p(t) (y - x) =$$

$$= (p-1) \|(1-t)x + ty\|^{p-2} \tau_{+(-)} ((1-t)x + ty, y - x) \times$$

$$\times \left[(1-t)x + ty\right] + \|(1-t)x + ty\|^{p-1} (y - x)$$

for almost every $t \in [0, 1]$.

As above we have

$$\left\|m_p'(t)\right\| \le (1-p) \|(1-t)x + ty\|^{p-1} \|y - x\| + \|(1-t)x + ty\|^{p-1} \|y - x\| =$$

$$= (2-p) \|(1-t)x + ty\|^{p-1} \|y - x\|$$

for almost every $t \in [0, 1]$, which implies the desired inequality (2.2).

Theorem 2.1 is proved.

***Remark* 2.1.** If the vectors x and y are linearly dependent and $y = \lambda x$ with $\lambda \in \mathbb{K}$, then the p-angular distance between x and y reduces to

$$\alpha_p[x, y] = \|x\|^p \left|1 - |\lambda|^{p-1} \lambda\right| = \|x\|^p \beta_p \left[1, \lambda\right].$$

The study of $\beta_p \left[1, \lambda\right] = \left|1 - |\lambda|^{p-1} \lambda\right|$ with $\lambda \in \mathbb{K}$ may be done in a similar way, however the details are omitted.

***Remark* 2.2.** If $p \geq 2$, then the function $f_p \colon [0,1] \to [0,\infty)$ given by $f_p(t) = \|(1-t)x + ty\|^{p-1}$ is convex and by the Hermite – Hadamard type inequality for the convex function $g \colon [a,b] \to \mathbb{R}$

$$\frac{1}{b-a}\int_a^b g(s)\,ds \leq \frac{1}{2}\left[\frac{g(a)+g(b)}{2} + g\left(\frac{a+b}{2}\right)\right] \leq$$

$$\leq \frac{g(a)+g(b)}{2} \leq \max\{g(a), g(b)\} \tag{2.6}$$

we have the following chain of inequalities:

$$\alpha_p[x,y] \leq p\|y-x\| \int_0^1 \|(1-t)x+ty\|^{p-1}\,dt \leq$$

$$\leq p\|y-x\| \left[\frac{\|x\|^{p-1}+\|y\|^{p-1}}{2} + \left\|\frac{x+y}{2}\right\|^{p-1}\right] \leq$$

$$\leq p\|y-x\| \frac{\|x\|^{p-1}+\|y\|^{p-1}}{2} \leq p\|y-x\| \left[\max\{\|x\|, \|y\|\}\right]^{p-1}, \tag{2.7}$$

which provides a refinement of Maligranda's inequality (1.4).

If $p \geq 1$ and since, by the triangle inequality we have

$$\|(1-t)x+ty\| \leq (1-t)\|x\| + t\|y\|,$$

then

$$\|(1-t)x+ty\|^{p-1} \leq [(1-t)\|x\| + t\|y\|]^{p-1}$$

for any $t \in [0,1]$. Integrating on $[0,1]$ we get

$$\int_0^1 \|(1-t)x+ty\|^{p-1}\,dt \leq \int_0^1 [(1-t)\|x\| + t\,\|y\|]^{p-1}\,dt = \frac{1}{p}\frac{\|y\|^p - \|x\|^p}{\|y\|-\|x\|}$$

if $\|y\| \neq \|x\|$, and by (2.1) we obtain the chain of inequalities

$$\alpha_p[x,y] \leq p\|y-x\| \int_0^1 \|(1-t)x+ty\|^{p-1}\,dt \leq \frac{\|y\|^p - \|x\|^p}{\|y\|-\|x\|}\|y-x\|, \tag{2.8}$$

which provides a refinement of Hile's inequality (1.9).

For $p \geq 2$, by the Hermite – Hadamard's type inequalities (2.6) we also have

$$\frac{1}{p}\frac{\|y\|^p - \|x\|^p}{\|y\|-\|x\|} = \int_0^1 [(1-t)\|x\| + t\|y\|]^{p-1}\,dt \leq$$

$$\leq \frac{1}{2}\left[\left(\frac{\|x\|+\|y\|}{2}\right)^{p-1}+\frac{\|x\|^{p-1}+\|y\|^{p-1}}{2}\right]\leq$$

$$\leq \frac{\|x\|^{p-1}+\|y\|^{p-1}}{2}\leq [\max\{\|x\|,\|y\|\}]^{p-1}$$

which implies the following sequence of inequalities:

$$\alpha_p[x,y]\leq p\|y-x\|\int_0^1\|(1-t)x+ty\|^{p-1}\,dt\leq$$

$$\leq \frac{\|y\|^p-\|x\|^p}{\|y\|-\|x\|}\|y-x\|\leq$$

$$\leq \frac{1}{2}p\|y-x\|\left[\left(\frac{\|x\|+\|y\|}{2}\right)^{p-1}+\frac{\|x\|^{p-1}+\|y\|^{p-1}}{2}\right]\leq$$

$$\leq p\|y-x\|\frac{\|x\|^{p-1}+\|y\|^{p-1}}{2}\leq p\|y-x\|[\max\{\|x\|,\|y\|\}]^{p-1} \tag{2.9}$$

for $\|y\|\neq\|x\|$ and $p\geq 2$.

In particular, the inequality (2.9) shows that in the case $p\geq 2$, Hile's inequality (1.9) is better than Maligranda's inequality (1.4).

***Remark* 2.3.** The case $p=0$ is of interest, since by (2.2) we have the following upper bound for the angular distance $\alpha[x,y]$:

$$\alpha[x,y]\leq 2\|y-x\|\int_0^1\|(1-t)x+ty\|^{-1}\,dt, \tag{2.10}$$

provided the vectors x and y are linearly independent.

Since for any $t\in[0,1]$

$$\|(1-t)x+ty\|=\|x-t\,(x-y)\|\geq\left|\|x\|-t\,\|x-y\|\right|\geq\|x\|-t\|x-y\|\geq\|x\|$$

and similarly

$$\|(1-t)x+ty\|\geq\|y\|,$$

then we have

$$\|(1-t)x+ty\|\geq\max\{\|x\|,\|y\|\},$$

which implies that

$$\int_0^1\|(1-t)x+ty\|^{-1}\,dt\leq\frac{1}{\max\{\|x\|,\|y\|\}}. \tag{2.11}$$

Therefore, we have the following refinement of the Massera – Schäffer's inequality (1.1):

$$\alpha[x,y]\leq 2\|y-x\|\int_0^1\|(1-t)x+ty\|^{-1}\,dt\leq\frac{2\|y-x\|}{\max\{\|x\|,\|y\|\}}.$$

***Remark* 2.4.** In [9], the authors introduced the concept of p-HH-norm on the Cartesian product of two copies of a normed space, namely

$$\|(x,y)\|_{p-HH} := \left(\int_0^1 \|(1-t)x + ty\|^p \, dt \right)^{1/p},$$

where $(x,y) \in X \times X := X^2$ and $p \geq 1$. They showed that $\|\cdot\|_{p-HH}$ is a norm on X^2 equivalent with the usual p-norms

$$\|(x,y)\|_p := (\|x\|^p + \|y\|^p)^{1/p}.$$

They also showed that completeness, reflexivity, smoothness, strict convexity etc. is inherited by X^2 with this norm.

In [10] the authors proved the following interesting lower bound for $\|(x,y)\|_{p-HH}$:

$$\left(\frac{\|x\|^p + \|y\|^p}{2(p+1)} \right)^{1/p} \leq \|(x,y)\|_{p-HH} \tag{2.12}$$

for any $(x,y) \in X^2$ and $p \geq 1$.

Now, we observe that, by (2.1) we also have

$$\alpha_{p+1}[x,y] \leq (p+1)\,\|y-x\|\,\|(x,y)\|_{p-HH}^p \tag{2.13}$$

for any $(x,y) \in X^2$ and $p \geq 1$.

For $x \neq y$ this is equivalent with

$$\left(\frac{\|\|x\|^p x - \|y\|^p y\|}{(p+1)\|y-x\|} \right)^{1/p} \leq \|(x,y)\|_{p-HH}, \tag{2.14}$$

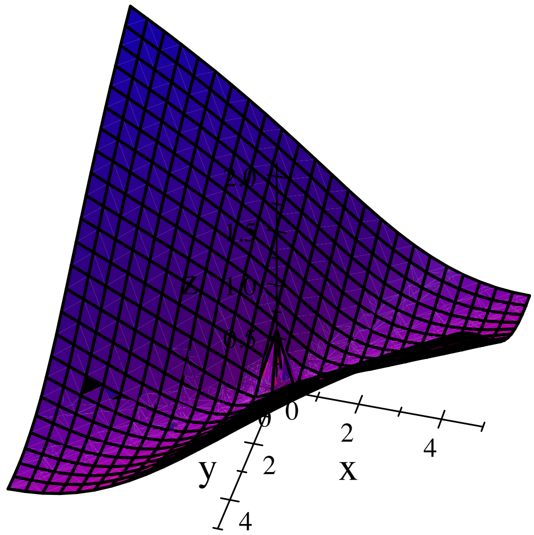
where $p \geq 1$.

It is natural to ask which lower bound from (2.12) and (2.14) for the p-HH-norm is better?

If we take $X = \mathbb{C}$, $\|\cdot\| = |\cdot|$ and $p = 2$, then by plotting the difference d given by

$$d(x,y) := \left(\frac{\left| |x|^2 x - |y|^2 y \right|}{3\,|y-x|} \right)^{1/2} - \left(\frac{|x|^2 + |y|^2}{6} \right)^{1/2}$$

for $x, y \in \mathbb{R}$ and $x \neq y$, we observe that d is nonnegative, showing that the new bound (2.14) is better than (2.12). The plot is depicted in Figure as follows:



The variation of d in the box $(x, y) \in [-4, 4] \times [-4, 4]$.

Problem 2.1. *Is the inequality*

$$\frac{\|x\|^p + \|y\|^p}{2} \leq \frac{\big\| \|x\|^p x - \|y\|^p y \big\|}{\|y - x\|} \tag{2.15}$$

true for any $(x, y) \in X^2$ *with* $x \neq y$ *and* $p \geq 1$?

3. Applications for power series. For power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with complex coefficients we can naturally construct another power series which have as coefficients the absolute values of the coefficient of the original series, namely, $f_a(z) := \sum_{n=0}^{\infty} |a_n| z^n$. It is obvious that this new power series have the same radius of convergence as the original series, and that if all coefficients $a_n \geq 0$, then $f_a = f$.

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned}
f(z) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n = \ln \frac{1}{1+z}, \quad z \in D(0,1), \\
g(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos z, \quad z \in \mathbb{C}, \\
h(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sin z, \quad z \in \mathbb{C}, \\
l(z) &= \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1+z}, \quad z \in D(0,1),
\end{aligned} \tag{3.1}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n = \ln \frac{1}{1-z}, \quad z \in D(0,1),$$

$$g_a(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z, \quad z \in \mathbb{C},$$

$$h_a(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z, \quad z \in \mathbb{C}, \tag{3.2}$$

$$l_a(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad z \in D(0,1).$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \quad z \in \mathbb{C},$$

$$\frac{1}{2} \ln \left(\frac{1+z}{1-z} \right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \quad z \in D(0,1),$$

$$\sin^{-1}(z) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!} z^{2n+1}, \quad z \in D(0,1), \tag{3.3}$$

$$\tanh^{-1}(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \quad z \in D(0,1),$$

$${}_2F_1(\alpha,\beta,\gamma,z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \qquad \alpha,\beta,\gamma > 0, \quad z \in D(0,1),$$

where Γ is Gamma function.

Theorem 3.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If $(X; \|\cdot\|)$ is a normed linear space and $x, y \in X$ with $\|x\|, \|y\| < R$, then*

$$\|f(\|x\|)\, x - f(\|y\|)\, y\| \leq$$

$$\leq \|y-x\| \int_0^1 \left[f_a(\|(1-t)x+ty\|) + \|(1-t)x+ty\|\, f_a'(\|(1-t)x+ty\|) \right] dt. \tag{3.4}$$

Proof. From the inequality (2.1) for $p = n+1$, n a natural number with $n \geq 1$, we have

$$\|\|x\|^n x - \|y\|^n y\| \leq (n+1)\|y-x\| \int_0^1 \|(1-t)x+ty\|^n \, dt. \tag{3.5}$$

We notice that the above inequality also holds for $n = 0$, reducing to an equality.

Let $m \geq 1$. Then we have, by the generalized triangle inequality and by (3.5), that

$$\left\| \left(\sum_{n=0}^{m} a_n \|x\|^n \right) x - \left(\sum_{n=0}^{m} a_n \|y\|^n \right) y \right\| \leq$$

$$\leq \sum_{n=0}^{m} |a_n| \, \| \|x\|^n x - \|y\|^n y \| \leq$$

$$\leq \|y - x\| \sum_{n=0}^{m} (n+1) \, |a_n| \int_0^1 \|(1-t)x + ty\|^n \, dt =$$

$$= \|y - x\| \int_0^1 \left(\sum_{n=0}^{m} (n+1) \, |a_n| \, \|(1-t)x + ty\|^n \right) dt. \tag{3.6}$$

Since $\|x\|, \|y\| < R$ the series

$$\sum_{n=0}^{\infty} a_n \|x\|^n, \qquad \sum_{n=0}^{\infty} a_n \|y\|^n$$

and

$$\sum_{n=0}^{\infty} (n+1) \, |a_n| \, \|(1-t)x + ty\|^n$$

are convergent.

Moreover, we obtain

$$\sum_{n=0}^{\infty} a_n \|x\|^n = f\left(\|x\|\right), \qquad \sum_{n=0}^{\infty} a_n \|y\|^n = f\left(\|y\|\right)$$

and

$$\sum_{n=0}^{\infty} (n+1) \, |a_n| \, \|(1-t)x + ty\|^n =$$

$$= \sum_{n=0}^{\infty} |a_n| \, \|(1-t)x + ty\|^n + \sum_{n=0}^{\infty} n \, |a_n| \, \|(1-t)x + ty\|^n =$$

$$= f_a\left(\|(1-t)x + ty\|\right) + \|(1-t)x + ty\| \, f_a'\left(\|(1-t)x + ty\|\right)$$

for any $\|x\|, \|y\| < R$.

Taking the limit over $m \to \infty$ in (3.6) we get the desired result (3.4).

Theorem 3.1 is proved.

***Remark* 3.1.** If we take $f(z) := \exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ then we have from (3.4) the following inequality:

$$\|\exp(\|x\|)\,x - \exp(\|y\|)y\| \leq$$

$$\leq \|y-x\| \int_0^1 \exp(\|(1-t)x+ty\|)\,(1+\|(1-t)x+ty\|)\,dt \tag{3.7}$$

for any $x, y \in X$.

If we apply the inequality (3.4) for the functions $f(z) := \dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ and $f(z) := \, := \dfrac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$, then we have

$$\left\| \frac{x}{1 \pm \|x\|} - \frac{y}{1 \pm \|y\|} \right\| \leq \|y-x\| \int_0^1 \frac{dt}{(1-\|(1-t)x+ty\|)^2} \tag{3.8}$$

for any $x, y \in X$ with $\|x\|, \|y\| < 1$.

Utilising the Hile's inequality, we can also prove the following divided difference inequality:

Proposition 3.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If $(X; \|\cdot\|)$ is a normed linear space and $x, y \in X$ with $\|x\|, \|y\| < R$ and $\|x\| \neq \|y\|$, then*

$$\frac{\|f(\|x\|)\,x - f(\|y\|)\,y\|}{\|y-x\|} \leq \frac{f_a(\|x\|)\,\|x\| - f_a(\|y\|)\,\|y\|}{\|x\| - \|y\|}. \tag{3.9}$$

Proof. The proof goes along the line of the one from Theorem 3.1 by utilizing Hile's inequality (1.9)

$$\frac{\|\|x\|^n x - \|y\|^n y\|}{\|y-x\|} \leq \frac{\|x\|^{n+1} - \|y\|^{n+1}}{\|x\| - \|y\|}$$

for any n a natural number.

***Remark* 3.2.** If we write the inequality (3.9) for the exponential function, then we get

$$\frac{\|\exp(\|x\|)\,x - \exp(\|y\|)\,y\|}{\|y-x\|} \leq \frac{\exp(\|x\|)\,\|x\| - \exp(\|y\|)\,\|y\|}{\|x\| - \|y\|}$$

for any $x, y \in X$ with $\|x\| \neq \|y\|$.

If we apply the inequality (3.9) for the functions $f(z) := \dfrac{1}{1-z}$ and $f(z) := \dfrac{1}{1+z}$, then we get

$$\left\| \frac{x}{1 \pm \|x\|} - \frac{y}{1 \pm \|y\|} \right\| \leq \frac{\|y-x\|}{(1-\|x\|)(1-\|y\|)}$$

for any $x, y \in X$ with $\|x\| \neq \|y\|$ and $\|x\|, \|y\| < 1$.

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