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FIBONACCI LENGTHS OF ALL FINITE p-GROUPS OF EXPONENT p^2

ДОВЖИНИ ФІБОНАЧЧІ ДЛЯ ВСІХ СКІНЧЕННИХ p-ГРУП ЕКСПОНЕНТИ p^2

The Fibonacci lengths of finite p-groups were studied by Dikici and co-authors since 1992. All of the considered groups are of exponent p, and the lengths depend on the Wall number $k(p)$. The p-groups of nilpotency class 3 and exponent p were studied in 2004 also by Dikici. In the present paper, we study all p-groups of nilpotency class 3 and exponent p^2. We thus complete the study of Fibonacci lengths of all p-groups of order p^4, proving that the Fibonacci length is $k(p^2)$.

Довжини Фібоначчі скінченних p-груп вивчалися Дікічі та співавторами з 1992 року. Всі групи, що розглядалися, були групами експоненти p, а всі довжини залежали від числа Уолла $k(p)$. p-Групи класу нільпотентності 3 і експоненти p були також досліджені Дікічі у 2004 році. У даній статті ми вивчаємо всі p-групи класу нільпотентності 3 і експоненти p^2. Цим завершується дослідження довжини Фібоначчі всіх p-груп порядку p^4; при цьому доведено, що довжина Фібоначчі дорівнює $k(p^2)$.

1. Introduction. The study of Fibonacci sequences in groups began with the earlier work of Wall [19] in 1960, where the ordinary Fibonacci sequences in cyclic groups were investigated. In the mid-eighties, Wilcox [20] extended the problem to the abelian groups. In 1990, Campbell et al. [5] expanded the theory to some classes of finite groups. In 1992, Knox proved that the periods of k-nacci (k-step Fibonacci) sequences in the dihedral groups are equal to $2k+2$, in the article [17]. In the progress of this study, the article [2] of Aydin and Smith proves that the lengths of the ordinary 2-step Fibonacci sequences are equal to the lengths of the ordinary 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 4 and a prime exponent, in 1994.

Since 1994, the theory has been generalized and many authors had nice contributions in computations of recurrence sequences in groups and we may give here a brief of these attempts. In [7] and [8] the definition of the Fibonacci sequence has been generalized to the ordinary 3-step Fibonacci sequences in finite nilpotent groups. Then in the article [1] it is proved that the period of 2-step general Fibonacci sequence is equal to the length of the fundamental period of the 2-step general recurrence constructed by two generating elements of a group of nilpotency class 2 and exponent p. In [16] Karaduman and Yavuz showed that the periods of the 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 5 and a prime exponent, are $p.k(p)$, for $2 < p \leq 2927$, where p is a prime and $k(p)$ is the period of ordinary 2-step Fibonacci sequence. The main role of the articles [14] and [15] in generalizing the theory was to study the 2-step general Fibonacci sequences in finite nilpotent groups of nilpotency class 4 and exponent p and to the 2-step Fibonacci sequences in finite nilpotent groups of nilpotency class n and exponent p, respectively.

Going on a detailed literature in this area of research, we have to mention certain essential computation on the Fibonacci lengths of new structures like the semidirect products, the direct products and the automorphism groups of finite groups which have been studied in the articles [3, 4, 9 – 12].

Let $s = (s_i)$ be the 2-step Fibonacci sequence of numbers defined by $s_0 = 0,\ s_1 = 1,\ s_i = s_{i-2} + s_{i-1}$, for $i \geq 2$. We may extend the sequence backwards to obtain a bi-infinite sequence. The fundamental period or Wall number (see [19]) of this sequence is denoted by $k(s, p^n)$, where the sequence reduced modulo p^n, for a positive integer n and a prime p. Since now on, we denote $k(s, p^n)$ by $k(p^n)$.

A 2-step general Fibonacci sequence in a finite non-abelian 2-generated group $G = \langle a, b\rangle$ is defined by $x_0 = a,\ x_1 = b,\ x_i = x_{i-2}^m x_{i-1}^l$, for $i \geq 2$ and the integers m and l. If $m = l = 1$, the least period of this sequence is called the *Fibonacci length* of G and denoted by $k(G)$.

Among all of the p-groups of order p^4 and nilpotency class 3 (see [18]), the group

$$H = \langle a, b, c, d \mid a^p = b^p = c^p = d^p = 1,\ [a, b] = [a, c] = [a, d] = 1,$$

$$[b, d] = a,\ [c, d] = b\rangle, \qquad p \neq 3,$$

is of exponent p and studied by Dikici [6]. The remained four classes indeed, the groups

$$K = \langle a, b, c \mid a^9 = b^3 = c^3 = 1,\ [a, b] = 1,\ [a, c] = b,\ [c, b^{-1}] = a^{-3}\rangle,$$

and

$$L_\alpha = \langle a, b, c \mid a^{p^2} = b^p = 1,\ c^p = a^{\alpha p},\ [a, b] = a^p,\ [a, c] = b,\ [b, c] = 1\rangle,$$

where $\alpha = 0, 1$, or a non-residue modulo p, are of exponent p^2. The aim of this paper is to study the Fibonacci lengths of these groups. First of all we attempt to give a power-commutator presentation for the groups (see [13]) and by investigating their nilpotency class we will go to the computation of Fibonacci lengths.

Our main result is:

Main theorem. *For a group G of order p^4 and of exponent p^2 which is of nilpotency class* 3, $k(G) = k(p^2)$ *where, p is an odd prime.*

The proof of this theorem and the computation of $k(G)$ for the group $G = K$, may be checked by using a procedure in a group theoretic software like GAP (GAP-groups, Algorithms and Programming, Ver. gap4r4p12; http://www.gap-system.org). Of course, we will give the details of our calculation on $k(G)$ of the group $G = L_\alpha$ in the next section. Also, we will state a conjecture for the groups of orders of p^5, p^6 and p^7.

2. The groups L_α. *Case* $\alpha = 0$. Let $G = L_\alpha$, where $\alpha = 0$. Then $G = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, [a, b] = a^p,\ [a, c] = b, [b, c] = 1\rangle$. By the relations of group, $a^p \in [G, G']$. Therefore, G has nilpotency class 3 and $[G, G'] \leq Z(G)$. Hence a^p is a central element of G. A power-commutator presentation of G may be given as follows:

$$G = \big\langle x, y, z, w \mid x^p = y^p = z^p = 1,\ w^p = x,\ [x, y] = [x, z] = [x, w] = 1,$$

$$[z, y] = 1,\ [w, y] = x,\ [w, z] = y\big\rangle.$$

Case $\alpha = 1$. Let $G = L_\alpha$, where $\alpha = 1$. Then $G = \langle a, b, c \mid a^{p^2} = b^p = 1,\ c^p = a^p,\ [a, b] = a^p,\ [a, c] = b,\ [b, c] = 1\rangle$. We may show that G has the following power-commutator presentation:

$$G = \big\langle x, y, z, w \mid x^p = y^p = 1,\ z^p = w^p = x,\ [x,y] = [x,z] = [x,w] = 1,$$

$$[z,y] = 1,\ [w,y] = x,\ [w,z] = y\big\rangle.$$

Case where α is a non-residue modulo p. Let $G = L_\alpha$, where α is a non-residue modulo p. Then $G = \langle a, b, c \mid a^{p^2} = b^p = 1,\ c^p = a^{\alpha p},\ [a,b] = a^p,\ [a,c] = b,\ [b,c] = 1\rangle$. We may show that G has the following power-commutator presentation:

$$G = \big\langle x, y, z, w \mid x^p = y^p = 1,\ z^p = x^\alpha,\ w^p = x,\ [x,y] = [x,z] = [x,w] = 1,$$

$$[z,y] = 1,\ [w,y] = x,\ [w,z] = y\big\rangle.$$

Note that in the new presentations, the group G is generated by w and z. Moreover, x is a central element. Also, each element of G can be uniquely represented as $x^a y^b z^c w^d$, where in the first case a, b, c reduced modulo p and d reduced modulo p^2 and in the second and third cases a and b reduced modulo p and c and d reduced modulo p^2. From now on we suppose that $G = L_\alpha$, where $\alpha = 0, 1$, or a non-residue modulo p. First we prove some elementary results.

Lemma 2.1. *For every positive integers m and n,*

(i) $w^m y^n = x^{mn} y^n w^m$,

(ii) $w^m z^n = x^{\binom{m+1}{2} n} y^{mn} z^n w^m$.

Proof. Since x is a central element of G, then (i) may be proved by the induction method. To prove (ii) we may use (i) and the relation $[z,y] = 1$.

Lemma 2.2. *Let $x^a y^b z^c w^d$ and $x^{a'} y^{b'} z^{c'} w^{d'}$ be elements of G. Then*

$$(x^a y^b z^c w^d)(x^{a'} y^{b'} z^{c'} w^{d'}) = x^{a+a'+db'+\binom{d+1}{2}c'} y^{b+b'+dc'} z^{c+c'} w^{d+d'}.$$

Proof. By using Lemma 2.1, we have

$$(x^a y^b z^c w^d)(x^{a'} y^{b'} z^{c'} w^{d'}) = x^{a+a'} y^b z^c w^d y^{b'} z^{c'} w^{d'} =$$

$$= x^{a+a'} y^b z^c x^{db'} y^{b'} w^d z^{c'} w^{d'} = x^{a+a'+db'} y^{b+b'} z^c w^d z^{c'} w^{d'} =$$

$$= x^{a+a'+db'} y^{b+b'} z^c x^{\binom{d+1}{2}c'} y^{dc'} z^{c'} w^d w^{d'} = x^{a+a'+db'+\binom{d+1}{2}c'} y^{b+b'+dc'} z^{c+c'} w^{d+d'}.$$

Lemma 2.3. *Let $x^a y^b z^c w^d$ and $x^{a'} y^{b'} z^{c'} w^{d'}$ be elements of G and m and l be positive integers. Then*

(i) $(x^a y^b z^c w^d)^m = x^{ma+\binom{m}{2}bd+\binom{m}{2}c\binom{d+1}{2}+\binom{m}{3}cd^2} y^{mb+\binom{m}{2}cd} z^{mc} w^{md}$,

(ii) $(x^a y^b z^c w^d)^m (x^{a'} y^{b'} z^{c'} w^{d'})^l = x^{a''} y^{b''} z^{c''} w^{d''}$,

where

$$a'' = ma + \binom{m}{2}bd + \binom{m}{2}c\binom{d+1}{2} + \binom{m}{3}cd^2 +$$

$$+la' + \binom{l}{2}b'd' + \binom{l}{2}c'\binom{d'+1}{2} + \binom{l}{3}c'd'^2 +$$

$$+mldb' + m\binom{l}{2}dc'd' + \binom{md+1}{2}lc',$$

$$b'' = mb + \binom{m}{2}cd + lb' + \binom{l}{2}c'd' + mldc',$$

$$c'' = mc + lc',$$

$$d'' = md + ld'.$$

Proof. (i) By induction on m. (ii) By using (i) and Lemma 2.2.

Lemma 2.4. *Every element of the Fibonacci sequence in the group G may be presented by $t_n = x^{a_n}y^{b_n}z^{s_n}w^{s_{n-1}}$, where the sequences $\{a_n\}_0^\infty$ and $\{b_n\}_0^\infty$ are defined as follows*:

$$b_0 = 0, \qquad b_n = \sum_{i=0}^{n-1} s_{n-1-i}s_{i-1}s_{i+1}, \quad n \geq 1,$$

$$a_0 = 0, \qquad a_n = \sum_{i=0}^{n-1} s_{n-1-i}\left(s_{i-1}b_{i+1} + \binom{s_{i-1}+1}{2}s_{i+1}\right), \quad n \geq 1.$$

Proof. We use an induction method on n. It is obvious that $t_0 = w = x^{a_0}y^{b_0}z^{s_0}w^{s_{-1}}$ and $t_1 = z = x^{a_1}y^{b_1}z^{s_1}w^{s_0}$, for, $a_1 = b_1 = 0$. Now assume that the result holds for n and $n+1$, where $n \geq 0$. Then

$$t_{n+2} = t_nt_{n+1} = (x^{a_n}y^{b_n}z^{s_n}w^{s_{n-1}})(x^{a_{n+1}}y^{b_{n+1}}z^{s_{n+1}}w^{s_n}) =$$

$$= x^{a_n+a_{n+1}+s_{n-1}b_{n+1}+\binom{s_{n-1}+1}{2}s_{n+1}}y^{b_n+b_{n+1}+s_{n-1}s_{n+1}}z^{s_n+s_{n+1}}w^{s_{n-1}+s_n} =$$

$$= x^{a'}y^{b'}z^{s_{n+2}}w^{s_{n+1}},$$

where

$$a' = a_n + a_{n+1} + s_{n-1}b_{n+1} + \binom{s_{n-1}+1}{2}s_{n+1} =$$

$$= \sum_{i=0}^{n-1} s_{n-1-i}\left(s_{i-1}b_{i+1} + \binom{s_{i-1}+1}{2}s_{i+1}\right) +$$

$$+ \sum_{i=0}^{n} s_{n-i}\left(s_{i-1}b_{i+1} + \binom{s_{i-1}+1}{2}s_{i+1}\right) + s_{n-1}b_{n+1} + \binom{s_{n-1}+1}{2}s_{n+1} =$$

$$= \sum_{i=0}^{n} s_{n-1-i}\left(s_{i-1}b_{i+1} + \binom{s_{i-1}+1}{2}s_{i+1}\right) - s_{-1}\left(s_{n-1}b_{n+1} + \binom{s_{n-1}+1}{2}s_{n+1}\right) +$$

$$+ \sum_{i=0}^{n} s_{n-i}\left(s_{i-1}b_{i+1} + \binom{s_{i-1}+1}{2}s_{i+1}\right) + s_{n-1}b_{n+1} + \binom{s_{n-1}+1}{2}s_{n+1} =$$

$$= \sum_{i=0}^{n} s_{n+1-i}\left(s_{i-1}b_{i+1} + \binom{s_{i-1}+1}{2}s_{i+1}\right) =$$

$$= \sum_{i=0}^{n+1} s_{n+1-i}\left(s_{i-1}b_{i+1} + \binom{s_{i-1}+1}{2} s_{i+1}\right) = a_{n+2}$$

and

$$b' = b_n + b_{n+1} + s_{n-1}s_{n+1} =$$

$$= \sum_{i=0}^{n-1} s_{n-1-i}s_{i-1}s_{i+1} + \sum_{i=0}^{n} s_{n-i}s_{i-1}s_{i+1} + s_{n-1}s_{n+1} =$$

$$= \sum_{i=0}^{n} s_{n-1-i}s_{i-1}s_{i+1} - s_{-1}s_{n-1}s_{n+1} + \sum_{i=0}^{n} s_{n-i}s_{i-1}s_{i+1} + s_{n-1}s_{n+1} =$$

$$= \sum_{i=0}^{n} s_{n+1-i}s_{i-1}s_{i+1} = b_{n+2}.$$

Lemma 2.4 is proved.

From now on we shall be working modulo p^2. Let $k = k(p^2)$. The following equations hold and are easy to see:

$$s_{k-i} = s_{-i} = (-1)^{i+1}s_i, \qquad \sum_{i=0}^{k-1} s_i = \sum_{i=0}^{k-1} s_{k-i}, \qquad \sum_{i=0}^{k-1} s_{i+a} = \sum_{i=0}^{k-1} s_i, \quad a \in \mathbb{Z}.$$

The proofs of the Lemmas 2.5, 2.6 and 2.7 may be found in [2] and [6].

Lemma 2.5. *The following equations hold*:

(i) $\sum_{i=0}^{k-1} s_i = 0$,

(ii) $\sum_{i=0}^{k-1} s_i^2 = 0$,

(iii) $\sum_{i=0}^{k-1} s_i^3 = 0$.

Lemma 2.6. *If* $p > 3$, *then*

(i) $\sum_{i=0}^{k-1} s_i s_{i-1} = 0$,

(ii) $\sum_{i=0}^{k-1} s_{i-1}^2 s_i = \sum_{i=0}^{k-1} s_{i-1}s_i^2 = 0$.

Lemma 2.7. *For every integers* a, b, c, d, *and* e *the following equations hold*:

(i) $\sum_{i=0}^{k-1} s_{i+a}s_{i+b}s_{-i+c}s_i = 0$,

(ii) $\displaystyle\sum_{i=0}^{k-1}\sum_{j=0}^{i-1} s_{-i+a}s_{i+b}s_{i-j-d}s_{j+e}s_{i+c} = 0$.

Lemma 2.8. *The following equations hold*:

(i) $\sum_{i=0}^{k-1}(-1)^i s_i^3 = 0$,

(ii) $\sum_{i=0}^{k-1}(-1)^i s_{i-1}^2 s_i = \sum_{i=0}^{k-1}(-1)^i s_{i-1}s_i^2 = 0$, $p > 3$.

Proof. (i)
$$\sum_{i=0}^{k-1}(-1)^i s_{i-1}^3 = \sum_{i=0}^{k-1} s_{-(i-1)}^3 = \sum_{i=0}^{k-1} s_{k-(i-1)}^3 = \sum_{i=0}^{k-1} s_i^3 = 0.$$

(ii) We may write

$$0 = \sum_{i=0}^{k-1} s_i^3 = \sum_{i=0}^{k-1}(-1)^i s_{i+1}^3 = \sum_{i=0}^{k-1}(-1)^i (s_i + s_{i-1})^3 =$$

$$= 3\sum_{i=0}^{k-1}(-1)^i s_{i-1}s_i^2 + 3\sum_{i=0}^{k-1}(-1)^i s_{i-1}^2 s_i. \tag{1}$$

On the other hand,

$$0 = \sum_{i=0}^{k-1} s_i^3 = \sum_{i=0}^{k-1}(-1)^{i-1} s_{i-2}^3 = \sum_{i=0}^{k-1}(-1)^i (s_i - s_{i-1})^3 =$$

$$= 3\sum_{i=0}^{k-1}(-1)^i s_{i-1}s_i^2 - 3\sum_{i=0}^{k-1}(-1)^i s_{i-1}^2 s_i. \tag{2}$$

Adding (1) and (2) we obtain
$$6\sum_{i=0}^{k-1} s_{i-1}s_i^2 = 0,$$

and subtracting (2) from (1) we have

$$6\sum_{i=0}^{k-1} s_{i-1}^2 s_i = 0.$$

Since $p > 3$, (ii) follows.

Lemma 2.8 is proved.

Now we are ready to prove the main result.

Proof of main theorem. By using Lemma 2.4, it is sufficient to show that $a_k = a_{k+1} = b_k = b_{k+1} = 0$. We have

$$b_k = \sum_{i=0}^{k-1} s_{k-1-i}s_{i-1}s_{i+1} = \sum_{i=0}^{k-1} s_{-(i+1)}s_{i-1}s_{i+1} = \sum_{i=0}^{k-1}(-1)^i s_{i-1}s_{i+1}^2 =$$

$$= \sum_{i=0}^{k-1}(-1)^i s_{i-1}(s_{i-1}+s_i)^2 =$$

$$= \sum_{i=0}^{k-1}(-1)^i s_{i-1}^3 + \sum_{i=0}^{k-1}(-1)^i s_{i-1}s_i^2 + 2\sum_{i=0}^{k-1}(-1)^i s_{i-1}^2 s_i,$$

and the last three expressions vanish by Lemma 2.8. So $b_k = 0$. Similarly,

$$b_{k+1} = \sum_{i=0}^{k} s_{k-i}s_{i-1}s_{i+1} = \sum_{i=0}^{k} s_{-i}s_{i-1}s_{i+1} = \sum_{i=0}^{k}(-1)^{i+1}s_{i-1}s_is_{i+1} =$$

$$= \sum_{i=0}^{k-1}(-1)^{i+1}s_{i-1}s_is_{i+1} = \sum_{i=0}^{k-1}(-1)^{i+1}s_{i-1}s_i(s_i + s_{i-1}) =$$

$$= -\left(\sum_{i=0}^{k-1}(-1)^i s_{i-1}s_i^2 + \sum_{i=0}^{k-1}(-1)^i s_{i-1}^2 s_i\right),$$

and the last two sums vanish by Lemma 2.8. On the other hand,

$$a_k = \sum_{i=0}^{k-1} s_{k-1-i}\left(s_{i-1}b_{i+1} + \binom{s_{i-1}+1}{2}s_{i+1}\right) =$$

$$= \sum_{i=0}^{k-1} s_{k-(i+1)}\left(s_{i-1}\sum_{j=0}^{i} s_{i-j}s_{j-1}s_{j+1} + \binom{s_{i-1}+1}{2}s_{i+1}\right) =$$

$$= \sum_{i=0}^{k-1}\sum_{j=0}^{i} s_{-(i+1)}s_{i-1}s_{i-j}s_{j-1}s_{j+1} + \sum_{i=0}^{k-1}\binom{s_{i-1}+1}{2}s_{-(i+1)}s_{i+1} =$$

$$= \sum_{i=0}^{k-1}\sum_{j=0}^{i-1} s_{-i-1}s_{i-1}s_{i-j}s_{j-1}s_{j+1} + \frac{1}{2}\sum_{i=0}^{k-1}(s_{i-1}+1)s_{i-1}s_{-(i+1)}s_{i+1},$$

and the first sum vanishes by Lemma 2.7(ii). For the second sum in the above expression, we have

$$\sum_{i=0}^{k-1}(s_{i-1}+1)s_{i-1}s_{-(i+1)}s_{i+1} = \sum_{i=0}^{k-1} s_{i-1}s_{i-1}s_{-(i+1)}s_{i+1} + \sum_{i=0}^{k-1} s_{i-1}s_{-(i+1)}s_{i+1} =$$

$$= \sum_{i=0}^{k-1} s_{i-2}s_{i-2}s_{-i}s_i + \sum_{i=0}^{k-1}(-1)^i s_{i-1}s_{i+1}^2,$$

and the first sum vanishes by Lemma 2.7(i) and the second one is equal to b_k which is zero. A similar method may be used to prove $a_{k+1} = 0$. This completes the proof showing that $k(G) = k(p^2)$ for all of groups $G = L_\alpha$, where $\alpha = 0, 1$, or non-residue modulo p.

Main theorem is proved.

Conjecture. For every p-group G of order p^i, $i = 5, 6, 7$, $k(G) = k(p^2)$, where G is of nilpotency class 3 and of exponent p^2, for every odd prime p.

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