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ON GIBBS QUANTUM AND CLASSICAL PARTICLE SYSTEMS WITH THREE-BODY FORCES

ПРО ГІББСІВСЬКІ КВАНТОВІ ТА КЛАСИЧНІ СИСТЕМИ ЧАСТИНОК З ТРИЧАСТИНКОВИМИ СИЛАМИ

For equilibrium quantum and classical systems of particles, interacting via ternary and pair (nonpositive) infinite-range potentials, a low activity convergent cluster expansion for their grand canonical reduced density matrices and correlation functions are constructed in the thermodynamic limit.

Для рівноважних квантових та класичних систем частинок, що взаємодіють завдяки тернарному і парному (непозитивним) далекосяжним потенціалам, побудовано кластерний розклад для їх редукованих матриць щільності та кореляційних функцій великого канонічного ансамблю, збіжний при низьких активностях у термодинамічній границі.

1. Introduction and main result. We consider classical and quantum systems of particles with the Maxwell – Boltzmann statistics that are characterized by the n-particle potential energy (see Remark in the end of the paper)

$$U(x_{(n)}) = \sum_{1\le j_1<j_2\le n} \phi^0(x_{j_2}-x_{j_1}) + \sum_{1\le j_1\ne j_2\ne j_3\le n} \phi^1(x_{j_2}-x_{j_1}, x_{j_3}-x_{j_1}), \quad (1.1)$$

where $x_{(n)} = (x_1,\ldots,x_n) \in \mathbb{R}^{dn}$, $x_j = (x_j^1,\ldots,x_j^d)$ he three-body translation invariant potential ϕ^1 is given by $\phi^1(x,y) = 2\sum_{l=1}^{d'} \phi_l(x)\phi_l(y)$. The pair (two-body) and three-body potentials are Euclidean invariant functions. U can be rewritten as

$$U(x_{(n)}) = \sum_{1\le j_1<j_2\le n} \phi_0(x_{j_2}-x_{j_1}) + U'(x_{(n)}),$$

where $\phi_0 = -2\sum_{l=1}^{d'} \phi_l^2 + \phi^0$, and

$$U'(x_{(n)}) = \sum_{l=1}^{d'}\sum_{j_1=1}^{n}\left[\sum_{j_2\ne j_1, j_2=1}^{n} \phi_l(x_{j_2}-x_{j_1})\right]^2.$$

The conditions of stability (see [1 – 7]) for it is reduced to the stability condition for the pair potential ϕ_0:

$$U_0(x_{(n)}) = \sum_{1\le j_1<j_2\le n} \phi_0(x_{j_2}-x_{j_1}) \ge -Bn, \quad (1.2)$$

where B is a constant. U is superstable [2 – 4] if ϕ_0 is superstable.

The equilibrium systems of d-dimensional classical and quantum particles, enclosed in a compact domain $\Lambda \in \mathbb{R}^d$, are described by the sequence of the grand canonical correlation functions $\rho^\Lambda = \{\rho^\Lambda(x_{(m)}),\, m \in \mathbb{Z}^+\}$ and reduced density matrices (RDMs) $\rho^\Lambda = \{\rho^\Lambda(x_{(m)}; y_{(m)}),\, m\in\mathbb{Z}^+\}$. The former are given by

$$\rho^\Lambda(x_{(m)}) = \Xi_\Lambda^{-1}\sum_{n\ge 0}\frac{z^{n+m}}{n!}\int e^{-\beta U(x_{(m)},x'_{(n)})}\chi_\Lambda(x_{(m)},x'_{(n)})dx'_{(n)},$$

$$x_{(m)} = (x_1,\ldots,x_m),$$

where the integration is performed over $\mathbb{R}^{nd}$, β is the inverse temperature, z is the particle activity, the grand partition function Ξ_Λ is given by the denominator in which $m = 0$ and

$\chi_\Lambda(x)$ is the characteristic(indicator) function of the compact domain $\Lambda \in \mathbb{R}^d$. The RDMs are expressed (see [1, 6, 7])

$$\rho^\Lambda(x_{(m)}|y_{(m)}) = \int P^\beta_{x_{(m)},y_{(m)}}(dw_{(m)})\rho^\Lambda(w_{(m)}), \tag{1.3}$$

where the path (quantum) correlation functions $\rho^\Lambda(w_{(m)})$ for the Dirichlet boundary conditions are determined by

$$\rho^\Lambda(w_{(m)}) = \Xi_\Lambda^{-1} \sum_{n\geq 0} \frac{z^{n+m}}{n!} \int \exp\{-\beta U(w_{(m)}, w'_{(n)})\}\times$$

$$\times\chi_\Lambda(w_{(m)}, w'_{(n)})dx'_{(n)} P^\beta_{x'_{(n)},x'_{(n)}}(dw'_{(n)}), \tag{1.4}$$

$$U(w_{(m)}) = \beta^{-1} \int_0^\beta U(w_{(m)}(\tau))d\tau,$$

Ξ_Λ is the grand partition function, coinciding with the numerator in the r.h.s. of the expression when $m = 0$, $P^t_{x,y}(dw)$ is the conditional Wiener measure, concentrated on paths, starting from x and arriving into y at the time t, $\chi_\Lambda(w_{(m)})$ is the product of the characteristic functions of paths w_j localized in Λ on all the interval $[0, \beta]$,

$$P^\beta_{x_{(m)},y_{(m)}}(dw_{(m)}) = P_{x_{(m)}}(dw_{(m)}|w_{(m)}(\beta) = y_{(m)}) =$$

$$= P_{x_{(m)}}(dw_{(m)}) \prod_{j=1}^m \delta(y_j - w_j(\beta)),$$

$\delta(x)$ is the point measure concentrated in x, $P_x(dw)$ is the Wiener measure concentrated on paths starting from the the point x. These measures are defined on the probability space Ω_0^d which may be considered as the Banach space of continuous functions [8] with the σ-algebra of the Borel sets (see also [9]). One easily checks with the help of the Feynman – Kac formula (see [6, 7, 10]) that every term in the sum in (1.3) after substitution of (1.4) in it is equal to the integral over $\mathbb{R}^{dn}$ of the kernel of the $(n + m)$-particle quantum Hamiltonain with the Dirichlet boundary condition, the usual kinetic term and the potential energy U.

There is a standard derivation of the KS equation for the sequence of the correlation functions and path correlation functions corresponding to a pair interaction potential (see [1 – 7])

$$\rho^\Lambda = zK_\Lambda\rho^\Lambda + z\alpha_\Lambda, \tag{1.5}$$

where $K_\Lambda = \hat\chi_\Lambda K \hat\chi_\Lambda$, $\alpha_\Lambda = \hat\chi_\Lambda\alpha$, α is the sequence whose first component is the unity and other components are zero, $\hat\chi_\Lambda$ is the operator of multiplication by the characteristic function of Λ, $(\hat\chi_\Lambda F)(\omega_{(n)}) = \chi_\Lambda(\omega_{(n)})F(\omega_{(n)})$ and ω is either a vector from $\mathbb{R}^d$ or a Wiener path. The KS operator K will be introduced by us in the end of this section. The denominator and numerator in the expression for correlation function diverge in the thermodynamic limit $\Lambda \to \mathbb{R}^d$ but the equation (1.5) is well defined in the limit

$$\rho = zK\rho + z\alpha. \tag{1.6}$$

Its solution determines the equilibrium state of the infinite-particle system.

It is well known that the norm of the classical KS operator K [11, 12] is bounded in the Banach space of sequences of bounded functions $\mathbb{E}_\xi$ for a positive short-range interaction (many-body potentials have finite supports). This Banach space is determined by the following norm $\|F\|_\xi = \max_{n\geq 1} \xi^{-n} \operatorname*{ess\,sup}_{\omega_{(n)}} \left|F(\omega_{(n)})\right|$, where $\omega = x \in \mathbb{R}^d$ for classical systems. In this case the solutions of (1.5), (1.6) are expressed in terms of the perturbation expansion of the resolvent of K acting on $z\alpha$. The norm of the classical KS operator is also bounded for a potential energy generated by an infinite range positive integrable pair potential [1 – 3]. If the interaction is mediated only by an integrable and stable pair potential ϕ then Ruelle proposed to symmetrize the classical KS operator to in order to make it bounded in $\mathbb{E}_\xi$. There were no examples of potential energies with nonpositive many-body (nonpair) potentials yielding the bounded KS operator in the Banach space $\mathbb{E}_\xi$. For quantum systems there were no results at all concerning solutions of the KS equation in $\mathbb{E}_\xi$ for the case of many-body potentials (in the definition of its norm one has to put $\omega = w$, where w is a Wiener path). In this paper we consider such the systems, i.e., classical and quantum systems for which the potential ϕ^0, ϕ^1 are not necessarily positive and have finite supports, and construct their (path) correlation function in the thermodynamic limit.

Let's deal with quantum systems at first.If one drops the ternary term U' in the expression of U then by the standard technique of [1, 5 – 7] the series in powers of z for the correlation functions is found converging if $|z| < c^{-1}(\beta)$, where $c(\beta) = \operatorname{ess\,sup}_w c_\beta(w) = |c_\beta|_0,\ c_\beta(w) = \int dx P^\beta_{x,x}(dw')\left|e^{-\beta\phi_0(w-w')} - 1\right|$. It'll be shown that the presence of U' leads to the necessity of dealing with the additional function

$$c_{*\beta}(w) = \sum_{l=1}^{d'} l^\theta \int dx P^\beta_{x,x}(dw') \left[\int_0^\beta \phi_l^2(w(\tau) - w'(\tau))d\tau\right]^{\frac{1}{2}}, \quad \theta \geq 0. \tag{1.7}$$

The functions

$$C_\beta(w) = c_\beta(w) + 16\sqrt{2}\left(\sum_{l=1}^{d'} l^{-\theta}\right) c_{*\beta}(w), \quad C(\beta) = |C_\beta|_0,$$

play a prominent role in the convergence of a cluster expansion for the correlation functions when U' is non-zero. The function $C^\Lambda_\beta(w) = c^\Lambda_\beta(w) + 16\sqrt{2}\left(\sum_{l=1}^{d'} l^{-\theta}\right) c^\Lambda_{*\beta}(w)$ will determine the character of a convergence to the thermodynamic limit, where the expressions for the functions $c^\Lambda_\beta, c^\Lambda_{*\beta}$ are obtained from the expressions of the corresponding functions without dependence on Λ by inserting $(1-\chi_\Lambda(w'))$ under the sign of the Wiener integral. The following theorem will enable us to prove our main result formulated in the subsequent two theorems (expression for the constant c in Theorem 1.1 is given in Section 5).

Theorem 1.1. *Let ϕ_0 be an integrable stable potential and the condition* (A) $l^{2\theta}|\phi_l(x)| \leq h(|x|)$, $l \geq 1$, *hold, where h is a monotone integrable function and $\theta \geq 0$. Then*

$$c(\beta) \le \beta e^{2\beta B}\|\phi_0\|_1, \qquad c_*(\beta) \le \sqrt{\beta}(4\pi\beta)^{-\frac{d}{2}}\left(\sum_{l=1}^{d'} l^{-\theta}\right)c,$$

where $\|\cdot\|_1$ *is the norm of the space of integrable functions* $L^1(\mathbb{R}^d)$ *and* c *is a positive constant. Moreover, if the condition* (B) $h(|x|) \le \bar{h}|x|^{-d-2\varepsilon}$, $|x| \ge R$, *holds, where* $R, \bar{h}, \varepsilon$ *are positive constants then there exists a positive function* $C'(\beta)$ *independent of* w *such that*

$$C^{\Lambda}_{\beta}(w) \le \delta^{-\varepsilon}C'(\beta), \qquad w \in \Lambda(\delta), \quad \delta \ge 2R, \tag{1.8}$$

where $\Lambda(\delta) = \{x \in \Lambda\colon \mathrm{dist}(x, \partial\Lambda) \ge \delta\}$, $\partial\Lambda$ *is a boundary of* Λ.

The second theorem is the first step towards proving an existence of the thermodynamic limit of the path correlation functions.

Theorem 1.2. *Let the condition* (A) *of Theorem* 1.1 *be satisfied. Then for an arbitrary bounded or unbounded* Λ *there exist functions* $\rho^{\Lambda}_n(w_{(m)})$ *such that*

$$\rho^{\Lambda}(w_{(m)}) = \sum_{n\ge 0} z^{n+m}\rho^{\Lambda}_n(w_{(m)}), \qquad \left|\rho^{\Lambda}_n(w_{(m)})\right| \le \xi^{m-1}(e_0 e^{2\beta B}C(\beta))^{n+m-1}$$

for $\xi^{-1} = 4C(\beta)$ *and the series converges in the norm of the space of bounded functions if*

$$|z| \le (e_0 e^{2\beta B}C(\beta))^{-1}, \qquad e_0 = 4\sum_{n\ge 1}(n!)^{-\frac{1}{2}}.$$

It is natural to expect the correlation functions $\rho(w_{(m)}) = \rho^{\Lambda}(w_{(m)})$, $\Lambda = \mathbb{R}^d$, determined in Theorem 1.2, to be the thermodynamic limit of the finite volume correlation functions. The third theorem confirms this establishing the character of a convergence of the finite volume correlation functions to the functions ((1.8) plays a crucial role).

Theorem 1.3. *Let the conditions* (A), (B) *of Theorem* 1.1 *be true then there exist positive functions* $\epsilon(\lambda)$, $\epsilon'(\lambda)$, *decreasing at infinity, such that if* $\rho_n(w_{(m)}) = \rho^{\Lambda}_n(w_{(m)})$ *for* $\Lambda = \mathbb{R}^d$ *and*

$$\rho(w_{(m)}) = \sum_{n\ge 0} z^{n+m}\rho_n(w_{(m)}) \tag{1.9}$$

then for $\xi^{-1} = 4C(\beta)$ *and* $\lambda = \max_j \mathrm{dist}(x_j \in A, \partial\Lambda)$, $A \subset \Lambda$, *the following inequalities are true:*

$$\left|\rho(w_{(m)}) - \rho^{\Lambda}(w_{(m)})\right| \le \xi^m\epsilon(\lambda), \quad w_{(m)} \in A^m, \tag{1.10}$$

$$\left|\rho(x_{(m)} \mid y_{(m)}) - \rho^{\Lambda}(x_{(m)} \mid y_{(m)})\right| \le (4\pi\beta)^{-\frac{dm}{2}}m\xi^m\epsilon'(\lambda), \quad x_j,\ y_j \in A, \tag{1.11}$$

where $\rho(x_m \mid y_m)$ *are given by* (1.3) *with* $\Lambda = \mathbb{R}^d$.

***Corollary* 1.1.** *Let* $\rho_n(x_{(m)}|y_{(n)}) = \int \rho_n(w_{(m)})P^{\beta}_{x_{(m)},y_{(m)}}(dw_{(m)})$. *Then the thermodynamic limits of the RDMs of the quantum system with the potential energy* (1.1), *satisfying conditions of Theorem* 1.1, *are given by*

$$\rho(x_{(m)}|y_{(m)}) = \sum_{n\ge 0} z^{n+m}\rho_n(x_{(m)}|y_{(m)}),$$

$$\left|\rho_n(x_{(m)}|y_{(m)})\right| \le \xi^{m-1}\xi'^{n+m-1}\prod_{j=1}^{m} P^\beta(x_j - y_j),$$

for $|z| \le \xi'^{-1}$, $\xi' = e_0 e^{2\beta B}C(\beta)$, $P^\beta(x) = (4\pi\beta)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4\beta}}$.

In our approach the Gibbs factor containing the quadratic term U' is transformed into the Gibbs factor with an imaginary potential energy, generated by a pair potential, depending on additional d' Wiener paths, labeled by a star, with the help of the Fourier-type transformation (see (2.1)). In this way we simplify our system and have to deal with a new Gibbs path system with a potential energy expressed through a complex pair potential. It easy to write down the KS equation and R-symmetrized KS equation for the correlation functions but it is not obvious that the operator in its right-hand side (KS operator)is bounded in the Banach space of sequences of bounded functions.

It turns out that the bounds of the norm of the Ruelle symmetrized (R-symmetrized) KS operator in the Banach space of sequences of bounded functions is difficult to obtain because of the presence of a stochastic integral in the expression of the imaginary pair potential. We circumvent this difficulty by introducing a new L^p-type norm $\|\Psi\|_{\xi,q}$ on sequences of functions with an increasing number of variables (integrating out dependence in the star paths) and establish the expected boundedness of the R-symmetrized KS operator in the corresponding Banach space $\mathbb{E}_{\xi,q}$. This implies, as usual, existence of the thermodynamic limit of the correlation functions. The corner stone for the results of this paper is boundedness of the function $\int dx\left[\int_0^\beta h^2\big(|w(\tau)-x|\big)\,d\tau\right]^{\frac{1}{2}}$, where w is a Wiener path and h is a monotonic integrable function. Our main tool is the generalized Holder inequality (3.7) and an inequality for the weighted convolution of two monotone functions (5.3). Without difficulty our result can be generalized to the case of infinite d' since for $\theta > 1$ our bounds are uniform in d'. The proposed approach is inspired by the results of the papers [13 – 16] in which diffusion Gibbs path particle systems with three-body forces were introduced. The thermodynamic limit of their path correlation functions allow to calculate the thermodynamic limit of correlation functions of the nonequilibrium systems of interacting Brownian particles. In this papers only KS recursion relation was used for this purpose. The results of this paper can be applied without difficulty for the systems.

The similar simplified technique we develop for classical systems. The new variables which are finite-dimensional vectors are introduced with the help of (6.1). With their help we pass to a new Gibbs system with a pair but complex potential described by the KS equation having the unique solution in the Banach space $\tilde{\mathbb{E}}_\xi$ with the norm given by (6.3). For classical systems we have the following analog of the previous theorems.

Theorem 1.4. *Let* $|\phi'(x)|_2 = \left(\sum_{l=1}^{d'} l^2\phi_l^2(x)\right)^{\frac{1}{2}}$, $\|\phi'\| = \int|\phi'(x)|_2dx$. *If* $\|\phi'\| < \infty$ *and* ϕ_0 *is a stable potential then there exist functions* $\rho_n(x_{(m)})$*, a positive function* $\epsilon(\lambda)$ *decreasing at infinity, positive numbers* η_0, η_1 *such that the series*

$$\rho(x_{(m)}) = \sum_{n\ge 0} z^{n+m}\rho_n(x_{(m)}),$$

converges in the disc $|z| < R$

$$R = \frac{\eta_1}{\eta_0 C_\eta}e^{-\beta(2B+\frac{\eta_1}{\eta_0})}, \qquad C_\eta = 2\sqrt{2\beta}\eta_1\|\phi'\| + \eta_0\|e^{-\beta\phi_0} - 1\|_1,$$

and

$$|\rho(x_{(m)}) - \rho^{\Lambda}(x_{(m)})| \le \left(\frac{\eta_1}{\eta_0 C_\eta}\right)^m \epsilon(\lambda),$$

where $\lambda = \max_j \operatorname{dist}(x_j, \partial\Lambda)$ *and* $\partial\Lambda$ *is a boundary of* Λ. *Moreover,*

$$\rho(x_{(m)}) \le \left(|z|\eta_0 e^{\beta(2B+\frac{\eta_1}{\eta_0})}\right)^m c, \qquad c = e^{-\beta(2B+\frac{\eta_1}{\eta_0})}\left(1-|z|R^{-1}\right)^{-1}.$$

This theorem follows directly from Theorem 6.1 and Corollaries 6.1, 6.2, Lemma 6.1 and standard arguments from [1] (see Theorem 4.2.3, formula (2.48) and its proof with the help of the inequality(2.39)) if one puts

$$\xi = \frac{\eta_1}{\eta_0 C_\eta}.$$

This choice of ξ satisfies the condition of Theorem 6.1 and Lemma 6.1, i.e., $1-2\sqrt{2\beta}\,\eta_0\times$ $\times\|\phi'\|\xi > 0$. From Lemma 6.1 it is easy to see that $\epsilon(\lambda)$ is proportional to the function $C_{\eta,\lambda}\left(\frac{\eta_1}{\eta_0 C_\eta}\right)$ determined in this lemma. It is evident that the growth of R for small β is proportional to $\sqrt{\beta^{-1}}$. Our estimates allow to consider infinite d'.

We'll employ the following abstract KS operator on the measure space (Ω, μ) describing classical and quantum systems with complex pair potential ϕ given by

$$(KF)(\omega_{(m)}) = \exp\left\{-\beta W(\omega_1|\omega_{(m\backslash 1)})\right\}\times$$

$$\times\left[F(\omega_{(m\backslash 1)})(1-\delta_{m,1}) + \sum_{n\ge 1}\frac{1}{n!}\int K(\omega_1|\omega'_{(n)})F(\omega_{(m\backslash 1)},\omega'_{(n)})\mu(d\omega'_{(n)})\right], \quad (1.12)$$

where the integration is performed over Ω^n, $(n\backslash j)$ is a sequence $(1,\ldots,n)$ without the integer j and $\delta_{m,k}$ is the Kronecker symbol, the function W determines the interaction of one particle with others,

$$W(\omega_1|\omega_{(m\backslash 1)}) = U(\omega_{(m)}) - U(\omega_{(m\backslash 1)}),$$

$$K(\omega|\omega_{(n)}) = \prod_{j=1}^{n}\left(\exp\{-\beta\phi(\omega|\omega_j)\} - 1\right).$$

Now, its important to introduce the Ruelle's symmetrization for the KS equation. The following inequality $\operatorname{Re} W(\omega_j|\omega_{(m\backslash j)}) \ge -2B$ holds in some nonempty set of Ω^m. Let's denote by $\chi_{(j,m)}$ the characteristic function (it does not depend on the paths w_j^*) of the set and put

$$\sum_{j=1}^{m}\chi^*_{(j,m)} = 1, \;\; \chi^*_{(j,m)} = \left(\sum_{j=1}^{m}\chi_{(j,m)}\right)^{-1}\chi_{(j,m)}. \quad (1.13)$$

The first equality follows from the stability condition $\operatorname{Re} U(\omega_{(m)}) \ge -Bm$. After multiplying both sides of the KS equation by $\chi^*_{(j,m)}$ and summing over j from 1 to m, we obtain the R-symmetrized KS equation

$$(\tilde{K}F)(\omega_{(m)}) = \sum_{j=1}^{m} \chi^{*}_{(j,m)}(\omega_{(m)}) \exp\left\{-\beta W(\omega_j|\omega_{(m\backslash j)})\right\}\times$$

$$\times\left[F(\omega_{(m\backslash j)})(1-\delta_{m,1}) + \sum_{n\geq 1}\frac{1}{n!}\int K(\omega_j|\omega'_{(n)})F(\omega_{(m\backslash j)},\omega'_{(n)})\mu(d\omega'_{(n)})\right], \quad m\geq 1. \tag{1.14}$$

Our paper is organized as follows. In the next section we reduce our Gibbs systems to new Gibbs path systems with a complex pair potential and prove important bounds (2.7), (2.8) in Lemma 2.1 connected with the functions $C(\beta), C^{\Lambda}_{\beta}(w)$. The first of them permits to express the norm of the symmetrized KS equation in terms of $C(\beta)$. In the third, fourth and fifth sections we prove Theorems 1.2, 1.3 and 1.1, respectively. In the last sectio we prove Theorem 1.4.

2. New Gibbs path systems. Our systems can be reduced to the Gibbs path system with a complex pair potential with the help of the transformation

$$\exp\left\{-\int_0^{\beta}\left[\sum_{k\neq j,k=1}^{n}\phi_l(w_j(\tau)-w_k(\tau))\right]^2 d\tau\right\} =$$

$$= \int P_0(d\omega^l_{*j})\exp\left\{-i\sum_{k\neq j,k=1}^{n}\int_0^{\beta}\phi_l(w_j(\tau)-w_k(\tau))dw^l_{*j}\right\}, \tag{2.1}$$

where $\int dw^l_{*j}$ is the stochastic integral over the one-dimensional Wiener process and $P_0(dw^l_{*j})$ is the Wiener measure concentrated on paths starting from the origin. As a result

$$\exp\{-\beta U'(w_{(n)})\} = \int \exp\left\{-\beta\sum_{l=1}^{d'}U_l(\omega_{(n)})\right\}P_0(dw_{*(n)}),$$

where $\omega = (w, w^1_*, \ldots, w^l_*) = (w, w_*)$, $P_0(dw_{*(n)}) = \prod_{j=1}^{n} P_0(dw_{*j})$, $P_0(dw_*) = = \prod_{l=1}^{d'} P_0(dw^l_*)$,

$$U_l(\omega_{(n)}) = \sum_{1\leq k<j\leq n}\phi_l(\omega_j|\omega_k), \quad l = 1,\ldots,d', \tag{2.2}$$

$$\phi_l(\omega|\omega') = i\beta\big(\varphi_l(w-w'|w_*) + \varphi_l(w'-w|w'_*)\big), \qquad \varphi_l(w|w_*) = \int_0^{\beta} dw^l_*\phi_l(w(\tau)). \tag{2.3}$$

The stochastic integral $\varphi_l(w|w_*)$ is the measurable function in $L^2(\Omega_0^{d+1}, P_{0,x}(dw))$, $x \in \in \mathbb{R}^d$. Indeed, this function almost everywhere in $w \in \Omega_0^d$ (w is a continuous function) is defined as a limit in the topology of $L^2(\Omega_0, P_0)$ of integral Riemannian sums, so it is a measurable function in w_*. This function is, also, measurable in w, since it is defined as a limit of almost everywhere convergent subsequence of measurable functions (a sequence of functions, converging in the topology of $L^2(\Omega_0, P_0)$ has a subsequence, converging almost everywhere). These sums are cylinder functions in w, w_*. Hence the limit is a measurable function.

So, we reduced the initial Gibbs path system to the Gibbs system with the complex pair potential, described by the correlation functions and one-particle extended by the new Wiener paths w_*^l, $l = 1, \ldots, d'$, phase space $\Omega = \Omega_0^{d'+d}$

$$\rho^\Lambda(\omega_{(m)}) =$$

$$= \Xi_\Lambda^{-1} \sum_{n \geq 0} \frac{z^{n+m}}{n!} \int \exp\left\{-\beta U(\omega_{(m)}, \omega'_{(n)})\right\} \chi_\Lambda(\omega_{(m)}, \omega'_{(n)}) dx'_{(n)} P_{*x_{(n)}}(d\omega'_{(n)}), \tag{2.4}$$

where $\chi_\Lambda(\omega_n) = \chi_\Lambda(w_n)$, the integration is performed over $\Lambda^n \times \Omega^n$, i.e., n-fold Cartesian product of $\Lambda \times \Omega$, and

$$U(\omega_{(n)}) = \sum_{l=0}^{d'} U_l(\omega_{(n)}), \qquad U_l(\omega_{(n)}) = \sum_{1 \leq k \neq j \leq n} \phi_l(\omega_k | \omega_j),$$
$$P_{*x}(dw dw_*) = P_{x,x}^\beta(dw) P_0(dw_*), \qquad \phi_0(\omega|\omega') = \phi_0(w - w') \tag{2.5}$$

and Ξ_Λ coincides with the numerator for $m = 0$ in (2.4). The path correlation function are expressed in terms of the new ones as follows:

$$\rho^\Lambda(w_{(m)}) = \int \rho^\Lambda(\omega_{(m)}) P_0(dw_{*(m)}). \tag{2.6}$$

So, we'll deal with the KS operator in (1.12) and (1.14) for $\mu(d\omega) = dx P_{*x}(dw dw_*)$. Our main estimates will concern the functions

$$K_{n,q}^0 = |K_{n,q}|_0,$$

$$K_{n,q}(w) = \int dx_{(n)} P_{x_{(m)}, x_{(n)}}^\beta(dw_{(n)}) K_q(w|w_{(n)}), \quad K_{0,q}^0 = 1,$$

$$K_q(w|w_{(n)}) = \left(\int P_0(dw_*) |K_q(\omega|w_{(n)})|^q\right)^{\frac{1}{q}} =$$

$$= \left(\int P_0(dw_*) P_0(dw_{*(n)} |K(\omega|\omega_{(n)})|^q\right))^{\frac{1}{q}}.$$

The first two important bounds are given in the following lemma.

Lemma 2.1. *Let the condition* (A) *of Theorem* 1.1 *be satisfied. Then*

$$K_{q,n}^0 \leq (n!)^{\frac{1}{2}} (4C_q^0(\beta))^n, \quad q \in 2\mathbb{Z}^+, \tag{2.7}$$

where $C_q^0(\beta) = c(\beta) + 16\sqrt{q}\left(\sum_{l=1}^{d'} l^{-\theta}\right) c_*(\beta)$, *and for* $\Lambda' \subseteq \Lambda''$

$$\int dx P_{x,x}^\beta(dw_{(n)}) K_q(w|w_{(n)}) [\chi_{\Lambda''}(w_{(n)}) - \chi_{\Lambda'}(w_{(n)})] \leq$$

$$\leq 4n C_{\beta,q}^\Lambda(w) ((n-1)!)^{\frac{1}{2}} (4C_q^0(\beta))^{n-1}, \tag{2.8}$$

where $C_{\beta,q}^\Lambda(w) = c_\beta^\Lambda(w) + 32\sqrt{q}\left(\sum_{l=1}^{d'} l^{-\theta}\right) c_{*\beta}^\Lambda(w)$.

Proof. The generalized Helder inequality (3.7) yields

$$K_q(w|w_{(n)}) \le \prod_{j=1}^{n} \left[\int\int P_0(dw_*)P_0(dw_{*j})|K(\omega|\omega_j)|^{qn} \right]^{\frac{1}{nq}}. \tag{2.9}$$

Integrating by $dx_{(n)}P^{\beta}_{x_{(n)},x_{(n)}}(dw_{(n)})$ the left-hand side of the inequality we derive

$$K_{n,q}(w) \le \left[\int dxP^{\beta}_{x,x}(dw') \left(\int\int P_0(dw^*)P_0(dw'_*)|K(\omega|\omega')|^{qn} \right)^{\frac{1}{qn}} \right]^n.$$

From the inequality $|e^{b+ia}-1| \le |e^b-1|+2|a|$ it follows that (by ϕ_0, ϕ_* we denote the real and imaginary parts of the pair potential, respectively)

$$K_{n,q}(w) \le 2^n \Bigg\{ \int dxP^{\beta}_{x,x}(dw') \bigg[|e^{-\beta\phi_0(w-w')}-1|+$$

$$+2\bigg(\int\int P_0(dw_*)P_0(dw'_*)|\beta\phi_*(\omega|\omega')|^{qn} \bigg)^{\frac{1}{qn}} \bigg] \Bigg\}^n.$$

Here we used the inequalities $(a+b)^n \le 2^n(a^n+b^n)$, $(a+b)^{\frac{1}{m}} \le a^{\frac{1}{m}}+b^{\frac{1}{m}}$. The Helder inequality

$$\left(\sum_s a_s \right)^m \le \left(\sum_s s^{-\theta\frac{m}{m-1}} \right)^{m-1} \sum_s s^{m\theta}a_s^m, \quad a_s \ge 0,$$

for $a_s = \beta|\phi_s|$, and $m = qn$ implies

$$K_{n,q}(w) \le 2^n \Bigg[c_\beta(w) + 2\bigg(\sum_{s=1}^{d'} s^{-\theta} \bigg) \int dxP^{\beta}_{x,x}(dw') \times$$

$$\times \bigg(\int\int P_0(dw_*)P_0(dw'_*) \sum_{s=1}^{d'} s^{nq\theta}|\beta\phi_s(\omega|\omega')|^{qn} \bigg)^{\frac{1}{qn}} \Bigg]^n.$$

Now, the inequality $\left[\sum_{s=1}^{d'} a_s \right]^{\frac{1}{nq}} \le \sum_{s=1}^{d'} a_s^{\frac{1}{nq}}$, yields

$$K_{n,q}(w) \le 2^n \Bigg[c_\beta(w) + 4\bigg(\sum_{l=1}^{d'} l^{-\theta} \bigg) \sum_{s=1}^{d'} s^{\theta} \int dxP^{\beta}_{x,x}(dw') \times$$

$$\times \bigg(\int P_0(dw_*)|\varphi_s(w-w'|w_*)|^{qn} \bigg)^{\frac{1}{qn}} \Bigg]^n.$$

Let $q \in 2\mathbb{Z}^+$, then the function in the round brackets is equal to

$$\int P_0(dw_*)|\varphi_s(w|w_*)|^{qn} = \frac{(qn)!}{\frac{qn}{2}!} \left[\int_0^{\beta} \phi_s^2(w(\tau)d\tau \right]^{\frac{qn}{2}}.$$

The inequality $4^{-n}n^n \le n! \le n^n$ leads to

$$\left(\left(\frac{qn}{2}\right)!\right)^{-1} \leq \left(\frac{8}{q}\right)^{\frac{nq}{2}} n^{-\frac{nq}{2}}, \qquad \left(\frac{(qn)!}{(\frac{qn}{2})!}\right)^{\frac{1}{q}} \leq \left(\sqrt{8qn}\right)^n \leq \left(4\sqrt{2q}\right)^n \sqrt{n!}.$$

The inequality (2.7) is proved. The inequality (2.8) follows from (2.9) after repeating the above arguments (instead of the Wiener measure its product with the characteristic function is substituted in the bounds) and the inequality $0 \leq \chi_{\Lambda''}(w'_{(n)}) - \chi_{\Lambda'}(w'_{(n)}) \leq$ $\leq \sum_{j=1}^{n}(1-\chi_{\Lambda'}(w'_j))$.

3. Proof of Theorem 1.2. Let's introduce the Banach space $\mathbb{E}_{\xi,q}$ of sequences of measurable functions $F = \{F(\omega_{(n)}), n \geq 1\}$ with the norm $\|F\|_{\xi,q}$

$$\|F\|_{\xi,q} = \max_n \xi^{-n} \operatorname{ess}\sup_{w_{(n)}} |F|_q(w_{(n)}),$$

$$|F|_q(w_{(n)}) = \left[\int |F(\omega_{(n)})|^q P_0(dw_{*(n)})\right]^{\frac{1}{q}}.$$

In this section we'll prove that the norm $\|\tilde{K}\|_{\xi,q}$ of the KS operator is bounded in the Banach space for positive even integers $q > 1$ and a positive number ξ. This fact will allow us to prove the convergence of the low activity cluster expansion for the correlation functions $\rho^{\Lambda}(\omega_{(m)})$, $m \geq 1$, which results from the expansion of the KS resolvent in the series in powers of the KS operator, and represent them in the form

$$\rho^{\Lambda}(w_{(m)}) = \sum_{n\geq 0} z^{n+m} \int P_0(dw_{*(m)})(\tilde{K}_{\Lambda}^{n+m-1}\alpha_{\Lambda})(w_{(m)}, w_{*(m)}). \tag{3.1}$$

Here we took into account that $(\tilde{K}_{\Lambda}^{n}\alpha_{\Lambda})(\omega_{(m)}) = 0$, $n < m-1$. One proves Theorem 1.2 with the aid of Theorem 3.1 and Corollary 3.1 putting $q = 2$ and taking into account that $C(\beta) = C_2^0(\beta)$. This gives

$$\rho_n^{\Lambda}(w_{(m)}) = \int P_0(dw_{*(m)})(\tilde{K}_{\Lambda}^{n+m-1}\alpha)(w_{(m)}, w_{*(m)}) \tag{3.2}$$

and after applying the Schwartz inequality one derives ($\|\alpha_{\Lambda}\|_{\xi,q} = \xi^{-1}$)

$$|\rho_n^{\Lambda}(w_{(m)})| \leq \xi^{m-1}\|\tilde{K}\|_{\xi,q}^{n+m-1}. \tag{3.3}$$

This inequality and (3.4) imply the bound for ρ_n in Theorem 1.2.

Theorem 3.1. *Let the condition* (A) *of Theorem* 1.1 *be satisfied. Then*

$$\|\tilde{K}\|_{\xi,q} \leq \xi^{-1}e^{2\beta B}\xi_q^0, \qquad \xi_q^0 = \sum_{n\geq 0}\frac{(4\xi C_q^0(\beta))^n}{\sqrt{n!}}, \tag{3.4}$$

and for $\Lambda \subseteq \Lambda' \subseteq \Lambda''$

$$\|\hat{\chi}_{\Lambda}\tilde{K}(\hat{\chi}_{\Lambda''} - \hat{\chi}_{\Lambda'})\|_{\xi,q} \leq e^{2\beta B}|\chi_{\Lambda}C_{\beta,q}^{\Lambda'}|_0\xi_q^0. \tag{3.5}$$

If one puts $\xi^{-1} = 4C_q^0(\beta)$ then $\|\tilde{K}\|_{\xi,q} \leq e^{2\beta B}e_0C_q^0(\beta)$.

***Corollary* 3.1.** *If the condition* $|z|e_0C_q^0(\beta)e^{2\beta B} < 1$ *is satisfied for even positive integer* q *then there exists the unique solution of the symmetrized KS equations* (1.5), (1.6), *in* $\mathbb{E}_{\xi,q}, \xi^{-1} = 4C_q^0(\beta)$ *given by perturbation expansion convergent in the uniform operator norm*

$$\rho^\Lambda = \sum_{n\geq 0} z^{n+1}\tilde{K}^n_\Lambda \alpha_\Lambda, \qquad \rho = \sum_{n\geq 0} z^{n+1}\tilde{K}^n \alpha \tag{3.6}$$

and $\|\rho^\Lambda\|_{\xi,q} \leq |z|\xi^{-1}(1-|z|\|\tilde{K}\|_{\xi,q})^{-1}$. *If the potential* ϕ_0 *is positive then the same is true for the KS equations* (1.5), (1.6) *and the expansion for the sequence of their solutions in terms of powers of the operators* K_Λ *and* K *instead of* $\tilde{K}_\Lambda$ *and* $\tilde{K}$.

***Proof of Theorem* 3.1.** Applying the Helder inequality for a probability measure $P_0(dw^*)$ for the case when one of the function is the unity, we obtain for $m > 1$

$$\big|(\tilde{K}F)(\omega_{(m)})\big| \leq \sum_{j=1}^m \chi^*_{j,m}(w_{(m)})e^{2\beta B}\times$$

$$\times \sum_{n\geq 0}\frac{1}{n!}\int dx'_{(n)}P^\beta_{x'_{(n)},x'_{(n)}}(dw'_{(n)})K_{\frac{q}{q-1}}(\omega_j|w'_{(n)})|F|_q(\omega_{(m\backslash j)};w'_{(n)}),$$

$$|F|_q(\omega_{(m\backslash j)};w'_{(n)}) = \left[\int |F(\omega_{(m\backslash j)},\omega'_{(n)})|^q P_0(dw'_{*(n)})\right]^{\frac{1}{q}}.$$

For $m = 1$ the summation in n is restricted to $n \geq 1$. Let $\chi^*_{(j_{(q)},m)} = \prod_{l=1}^q \chi^*_{(j_l,m)}$. Taking the both sides to the q-th power, we have

$$\int |(\tilde{K}F)|^q(\omega_{(m)})P_0(dw_{*(m)}) \leq$$

$$\leq e^{2\beta Bq}\sum_{j_1,\ldots,j_q}\chi^*_{(j_{(q)},m)}(w_{(m)})\sum_{n_1,\ldots,n_q}(n_1!\ldots n_q!)^{-1}\times$$

$$\times\int\prod_{s=1}^q dx'_{(n_s)}P^\beta_{x'_{(n_s)},x'_{(n_s)}}\left(dw^{(s)\prime}_{(n_s)}\right)\times$$

$$\times\int P_0\left(dw_{*(m)}\right)\prod_{r=1}^q K_{\frac{q}{q-1}}\left(\omega_{j_r}|w^{(r)\prime}_{(n_r)}\right)|F|_q\left(\omega_{(m\backslash j_r)};w^{(r)\prime}_{(n_r)}\right).$$

The integral over $w_{*(m)}$ can be taken independently over $w_{*(m\backslash j_s)}$ and w_{*j_s}. Applying the generalized Holder inequality

$$\int\prod_{j=1}^n |F_j(x)|\mu(dx) \leq \prod_{j=1}^n\left(\int |F_j(x)|^n\mu(dx)\right)^{\frac{1}{n}} \tag{3.7}$$

for $n = q$ and the two integrals, we see that the first integral coincides with the function $|F|_q(w_{(m\backslash j_s)},w'_{(n_s)})$ and the second with $\int P_0(dw_{*j_s})\big|K_{\frac{q}{q-1}}(\omega_{j_s}|w'_{(n)})\big|^q \leq$ $\leq K_q(w_{j_s}|w'_{(n)})$. Here we applied the ordinary Helder inequality for the integral in a probability measure for the power $q-1$. Taking ess sup and using (1.13) the following bound is derived:

$$|\tilde{K}F|_q(w_{(m)}) \leq \|F\|_{\xi,q}\xi^{m-1}e^{2\beta B}\sum_{n\geq 0}\frac{\xi^n}{n!}K^0_{n,q}. \tag{3.8}$$

The same is true for $m = 1$. The inequality (3.4) follows from Lemma 2.1. The similar arguments yield

$$\hat{\chi}_\Lambda(w_{(m)})|\tilde{K}(\hat{\chi}_{\Lambda''} - \hat{\chi}_{\Lambda'})F|_q(w_{(m)}) \le$$

$$\le \|F\|_{\xi,q}\xi^{m-1}e^{2\beta B}\sum_{n\ge 0}\frac{\xi^n}{n!}\text{ess}\sup_w \chi_\Lambda(w)\int dx_{(n)}P^\beta_{x_{(n)},x_{(n)}}(dw_{(n)})\times$$

$$\times K_q(w|w_{(n)})[\chi_{\Lambda''}(w_{(n)}) - \chi_{\Lambda'}(w_{(n)})] \le$$

$$\le \|F\|_{\xi,q}\xi^{m-1}e^{2\beta B}|\chi_\Lambda C^{\Lambda'}_{\beta,q}|_0\sum_{n\ge 0}\frac{\xi^n}{n!}K^0_{n,q}.$$

Here we applied (2.8). Now (3.5) is proven with the help of (2.7). Theorem 3.1 is proved.

4. Proof of Theorem 1.3. Application of inequalities (1.8), (3.5) yields for $\Lambda = \Lambda'(\delta)$ (remark that $\xi^{-1} = 4C(\beta)$)

$$\|\hat{\chi}_\Lambda\tilde{K}(\hat{\chi}_{\Lambda''} - \hat{\chi}_{\Lambda'})\|_{\xi,2} \le \eta(\delta), \qquad \eta(\delta) = \delta^{-\varepsilon}e^{2\beta B}C'(\beta)e_0.$$

Then the following estimate is proved in a standard way (see (2.39), (2.48) in [1] and Section 4 in [6])

$$\left[\int |\rho(\omega_{(m)}) - \rho^\Lambda(\omega_{(m)})|^2 P_0(dw_{*(m)})\right]^{\frac{1}{2}} \le$$

$$\le \xi^m\|\hat{\chi}_\Lambda(\rho - \rho^\Lambda)\|_{\xi,2} \le \xi^m\epsilon(\lambda), \quad w_{(m)} \in A^m.$$

The inequality (1.10) follows after an application of the Schwartz inequality. To prove (1.11) one has to deal with the set $\Gamma_\beta(2^{-1}\delta)$ of paths that on the time interval $[0,\beta]$ depart at the distance $2^{-1}\delta$, that is for all its paths $|w(t') - w(t)| \ge \delta$, $t', t \in [0,\beta]$. It is well known (see Appendix 1 in [6]) that for the characteristic function $\chi(w|\Gamma_\beta(R))$ of this set the inequality holds

$$\int P^\beta_{x,y}(dw)\chi(w|\Gamma_\beta(R)) \le \gamma(R,\beta) = \frac{\gamma_0}{(4\pi\beta)^{\frac{d}{2}}}\int\limits_{u\ge(4\sqrt{t})^{-1}R} e^{-u^2}u^{d-1}du, \tag{4.1}$$

where γ_0 is a constant. From the definition we derive for $\xi^{-1} = 4C(\beta)$

$$\chi_A(x_{(m)}, y_{(m)})|\rho(x_{(m)}|y_{(m)}) - \rho^\Lambda(x_{(m)}|y_{(m)})| \le$$

$$\le \chi_A(x_{(m)}, y_{(m)})\int P^\beta_{x_{(m)},y_{(m)}}(dw_{(m)})P_0(w_{*(m)})\times$$

$$\times\left[(1 - \chi_{\Lambda(2^{-1}\lambda)})(w_{(m)}) + \chi_{\Lambda(2^{-1}\lambda)})(w_{(m)})\right]\ |\rho(\omega_{(m)}) - \rho^\Lambda(\omega_{(m)})| \le$$

$$\le \chi_A(x_{(m)}, y_{(m)})\int P^\beta_{x_{(m)},y_{(m)}}(dw_{(m)})\times$$

$$\times(1 - \chi_{\Lambda(2^{-1}\lambda)}(w_{(m)}))(\|\rho\|_{\xi,2} + \|\rho^\Lambda\|_{\xi,2})\xi^m +$$

$$+\xi^m(4\pi\beta)^{-\frac{md}{2}}\|\hat{\chi}_{\Lambda(2^{-1}\lambda)}(\rho - \rho^\Lambda)\|_{\xi,2}. \tag{4.2}$$

Here we applied the Schwartz inequality for the integral by the measure $P_0(w^*_{(m)})$ and

$$\int P^{\beta}_{x_{(m)},y_{(m)}}(dw_{(m)}) = \prod_{j=1}^{m} P^{\beta}(x_j - y_j) =$$

$$= \prod_{j=1}^{m} (\sqrt{4\pi\beta})^{-d} e^{-\frac{|x_j-y_j|^2}{4\beta}} \le (\sqrt{4\pi\beta})^{-md}. \tag{4.2a}$$

For the second term in inequality (4.2) above the inequality (4.1) has to be applied. To bound the first term one has to use Corollary 3.1 and the inequality $1-\chi_{\Lambda(2^{-1}\lambda)}(w_{(m)}) \le \le \sum_{j=1}^{m}(1-\chi_{\Lambda(2^{-1}\lambda)}(w_j))$. As a result

$$\chi_A(x_{(m)}, y_{(m)}) \int P^{\beta}_{x_{(m)},y_{(m)}}(dw_{(m)})(1-\chi_{\Lambda(2^{-1}\lambda)}(w_{(m)})) \le$$

$$\le m(4\pi\beta)^{\frac{-m+1}{2}d} \operatorname{ess\,sup}_{x,y} \chi_A(x,y) \int P^{\beta}_{x,y}(dw)(1-\chi_{\Lambda(2^{-1}\lambda)}(w)) \le$$

$$\le m(4\pi\beta)^{\frac{-m+1}{2}d} \operatorname{ess\,sup}_{x,y} \int P^{\beta}_{x,y}(dw)\chi(w|\Gamma_\beta(2^{-1}\lambda)) \le$$

$$\le m(4\pi\beta)^{\frac{-m+1}{2}d}\gamma(2^{-1}\lambda,\beta).$$

Here we employed (4.2a), took into account that $1-\chi_{\Lambda(2^{-1}\lambda)}(w)$ is concentrated on paths that enter $\Lambda^c(2^{-1}\lambda)$ and $\{\operatorname{dist}(A, \Lambda^c(2^{-1}\lambda))\} \ge 2^{-1}\lambda$. So, inequality (1.11) holds with

$$\epsilon'(\lambda) = \epsilon(2^{-1}\lambda) + 2(4\pi\beta)^{\frac{d}{2}}\gamma(2^{-1}\lambda,\beta)|z|\xi^{-1}(1-|z|\|\tilde{K}\|_{\xi,2})^{-1}. \tag{4.3}$$

5. Proof of Theorem 1.1. From the bound $|e^{-\beta\phi_0} - 1| \le e^{2\beta B}\beta|\phi_0|$ (see formula (3.16) in [6]), it follows that $c_\beta(w) \le e^{2\beta B}\int dx P_{x,x}(dw')\int_0^\beta |\phi_0(w(\tau) - - w'(\tau))|d\tau\}$. After changing the order of integrations (the Fubini theorem is used) one easily derives the bound taking into account translation invariance of $P_{x,x}(dw)$ in x it follows that $c(\beta) \le e^{2\beta B}\beta\|\phi_0\|_1$. The first step in the proof of the second bound for $c_{*\beta}$ is the following bound:

$$c_{*\beta}(w) \le (4\pi\beta)^{-\frac{d}{2}} \left(\sum_{l=1}^{d'} l^{-\theta}\right) h_*(w), \tag{5.1}$$

where

$$h_*(w) = \int dx' \left[\int_0^\beta d\tau h_0(|w(\tau) - x'|)\right]^{\frac{1}{2}},$$

$$h_0(|x|) = h^2\left(\frac{|x|}{2}\right) + \exp\left\{-\frac{|x|^2}{16\beta}\right\} 2^{\frac{d}{2}}|h^2|_0.$$

The second step is the proof of the bound

$$\int (h_*(w))^n P_0(dw) \le \left(\beta 2^{d+1}[1+\sqrt{2^d}]\,\|\sqrt{h_0}\|_1^2\right)^{\frac{n}{2}}. \tag{5.2}$$

The following proposition shows that h_* is bounded by the square root of the term in the round brackets in the right-hand side of this inequality and (5.1), (5.2) lead to the needed inequality for $c_{*\beta}$ with $c = \left(2^{2d+3}(1+\sqrt{2^d})\right)^{\frac{1}{2}} \|\sqrt{h_0}\|_1$.

Proposition 5.1. *Let $f(x)$, $x \in X$, be a positive function on the probability space (X,μ). Then $|f|_0 \leq a$ iff $\int_X f^n(x)\mu(dx) \leq a^n$, $n \in \mathbb{Z}^+$.*

Proof. The first condition implies the second one. Now, suppose that the last condition holds. Then for arbitrary $\epsilon > 0$ we have

$$\int\limits_X ((a+\epsilon)^{-1}f)^n(x)\mu(dx) \leq ((a+\epsilon)^{-1}a)^n, \quad n \in \mathbb{Z}^+.$$

The right-hand side is less than arbitrary small number δ for sufficiently large n. This means that $f(x) \geq a+\epsilon$, $x \in A$, holds for the set A such that $\mu(A) \leq \delta$. Tending δ to zero, i.e., n to infinity, we prove the proposition.

***Proof of* (5.1).** Applying the Schwartz inequality for the integral in the measure $P_{x,x}(dw')$ we obtain, taking into account the condition for ϕ_l from Theorem 1.1,

$$c_{*\beta}(w) \leq \left(\sum_{l=1}^{d'} l^{-\theta}\right) \int dx \left[\int P^{\beta}_{x,x}(dw')\right]^{\frac{1}{2}} \left[\int\limits_0^{\beta} d\tau G(w(\tau);x)\right]^{\frac{1}{2}}, \tag{5.2a}$$

$$G(w(\tau);x) = \int P^{\beta}_{x,x}(dw')h^2(|w(\tau)-w'(\tau)|).$$

We'll prove now that

$$G(w(\tau);x') \leq (4\pi\beta)^{-\frac{d}{2}} h_0(|w(\tau)-x'|). \tag{5.2b}$$

It is not difficult to derive the following equalities $G(w(\tau);x) = G(w(\tau)-x)$, $G(x) = \int dy P^{\tau}(y-x)P^{\beta-\tau}(y-x)h^2(|y|)$. Here we used the definition of the Wiener measure $(t_{j-1} \leq t_j \leq t)$ for $n=1$ and $f(y_1) = h^2(y_1 - x)$

$$\int P^t_{x,y}(dw)f(w(t_1),\ldots,w(t_n)) =$$

$$= \int dy_{(n)}P^{t_1}(x-y_1)\prod_{j=2}^{n} P^{t_j-t_{j-1}}(y_j-y_{j-1})P^{t-t_n}(y_n-y)f(y_{(n)}).$$

In all our following estimates we 'll use that for monotone integrable bounded positive functions f,g and positive ψ the following bound is valid (monotonicity inequality)

$$(f\psi * g)(x) = \int f(|x-y|\psi(y-x))g(|y|)dy \leq$$

$$\leq g\left(\frac{|x|}{2}\right)\|\psi f\|_1 + f\left(\frac{|x|}{2}\right)(\psi * g)(x) \leq g\left(\frac{|x|}{2}\right)\|\psi f\|_1 + f\left(\frac{|x|}{2}\right)|g|_0\|\psi\|_1. \tag{5.3}$$

The bounds are obtained, representing the integral into the sum of two integrals over two domains: $|y| \geq \frac{|x|}{2}$, $|y| \leq \frac{|x|}{2}$, using the monotonicity of the functions, the fact that in these domains either $|y| \geq \frac{|x|}{2}$ or $|x - y| \geq \frac{|x|}{2}$, and after that enlarging the domains to the whole space. Now, let's put $P^{[2]t}(x) = P^t\left(\frac{x}{2}\right)$, $\psi(x) = P^{[2]\tau}(x)P^{[2](\beta-\tau)}(x)$, $g = h^2$, $f = \exp\left\{-\frac{|x|^2}{8\tau} - \frac{|x|^2}{8(\beta-\tau)}\right\}$ in (5.3). Then

$$\int dy P^\tau(y-x)P^{\beta-\tau}(y-x)h^2(|y|) \leq h^2\left(\frac{|x|}{2}\right)\|P^\tau P^{\beta-\tau}\|_1 +$$

$$\exp\left\{-\frac{|x|^2}{16\tau} - \frac{|x|^2}{16(\beta-\tau)}\right\}\left\|P^{[2]\tau}P^{[2](\beta-\tau)}\right\|_1 |h^2|_0.$$

The equalities $\|P^\tau P^{\beta-\tau}\|_1 = P^\beta(0) = (4\pi\beta)^{-\frac{d}{2}}$, $\|P^{[2]\tau}P^{[2](\beta-\tau)}\|_1 = 2^{\frac{d}{2}}\|P^\tau P^{\beta-\tau}\|_1 = 2^{\frac{d}{2}}(4\pi\beta)^{-\frac{d}{2}}$ and inequalities $\tau \leq \beta, \beta - \tau \leq \beta$ yield (5.2b). It and (5.2a) lead to (5.1).

***Proof of* (5.2).** We have

$$|h_*|_0 = \operatorname{ess} \sup_{w \in \Omega_0^d, w(0)=0} h_*(w). \tag{5.4}$$

This equality follows from translation invariance of h_*: $h_*(w + x) = h_*(w)$. With the help of the Schwartz inequality for the probability measure and the definition of the Wiener integral

$$\int P_x(dw) f(w(t_1), \ldots, w(t_n)) =$$

$$= \int dy_{(n)} P^{t_1}(x - y_1) \prod_{j=2}^n P^{t_j - t_{j-1}}(y_j - y_{j-1}) f(y_{(n)})$$

we derive

$$\int P_0(dw)(h_*(w))^n = \int dx_{(n)} \int P_0(dw) \prod_{j=1}^n \left[\int_0^\beta d\tau h_0(|x_j - w(\tau)|)\right]^{\frac{1}{2}} \leq$$

$$\leq \int dx_{(n)} \left[\int P_0(dw) \prod_{j=1}^n \int_0^\beta d\tau h_0(|x_j - w(\tau)|)\right]^{\frac{1}{2}} =$$

$$= \int dx_{(n)} \left[\int_0^\beta d\tau_{(n)} \int P_0(dw) \prod_{j=1}^n h_0(|x_j - w(\tau_j)|)\right]^{\frac{1}{2}} = \int dx_{(n)} I(x_{(n)}) = I_n,$$

where

$$I^2(x_{(n)}) = n! \int\limits_{[\beta]_{(n)}} d\tau_{(n)} \int P^{\tau_1}(|y_1|) h_0(|y_1 - x_1|) \times$$

$$\times \prod_{j=2}^{n} P^{\tau_j - \tau_{j-1}}(|y_j - y_{j-1}|) h_0(|y_j - x_j|) dy_{(n)}$$

and $[t]_{(n)} = 0 \leq \tau_1 \leq \tau_2 \ldots \leq \tau_n \leq t \in \mathbb{R}^n$. From inequality (5.3) with $\psi = p^{[2]\tau}$, $g(x) = h_0\left(\frac{|x|}{2^l}\right)$, $f(x) = e^{-\frac{|x|^2}{8\tau}}$ and the elementary inequality $\exp\left\{-\frac{|x|^2}{a}\right\} \leq \exp\left\{-\frac{|x|^2}{b}\right\}$, $b > a$, it follows that

$$\int P^{\tau}(|x-y|) h_0\left(\frac{|y|}{2^l}\right) dy \leq 2\big(1 + \sqrt{2^d}\big) h_0\left(\frac{|x|}{2^{l+1}}\right), \quad l \in \mathbb{Z}^+. \tag{5.5}$$

Here we took into account that $|h_0|_0 \leq 2\big(1 + \sqrt{2^d}\big)|h^2|_0$. Applying this inequality for the integral in y_n, translating y_n by x_n at first, one gets (x stands for $x_n - y_{n-1}$)

$$I^2(x_{(n)}) \leq n! 2\big(1 + \sqrt{2^d}\big) \int\limits_{[\beta]_{(n)}} d\tau_{(n)} \int P^{\tau_1}\big(|y_1|\big) h_0\big(|y_1 - x_1|\big) \times$$

$$\times \prod_{j=2}^{n-1} P^{\tau_j - \tau_{j-1}}\big(|y_j - y_{j-1}|\big) h_0\big(|y_j - x_j|\big) h_0\big(|y_{n-1} - x_n|\big) dy_{(n-1)}.$$

From the first inequality in (5.3) with $\psi = P^{\tau}$ and (5.5) we obtain for $\tau \leq \beta$ translating the variable y_{n-1} by x_n and then, on the second step, by $-x_n$

$$\int P^{\tau}\big(|y_{n-1} - y_{n-2}|\big) h_0\big(|y_{n-1} - x_{n-1}|\big) h_0\left(\frac{|y_{n-1} - x_n|}{2^l}\right) dy_{n-1} \leq$$

$$\leq h_0\left(\frac{|x_n - x_{n-1}|}{2^{l+1}}\right) \int P^{\tau}\big(|y_{n-1} - y_{n-2}|\big) h_0\big(|y_{n-1} - x_{n-1}|\big) dy_{n-1} +$$

$$+ h_0\left(\frac{|x_n - x_{n-1}|}{2}\right) \int P^{\tau}\big(|y_{n-1} - y_{n-2}|\big) h_0\left(\frac{|y_{n-1} - x_n|}{2^l}\right) dy_{n-1} \leq$$

$$\leq 2\left(1 + \sqrt{2^d}\right)\left[h_0\left(\frac{|x_n - x_{n-1}|}{2^{l+1}}\right) h_0\left(\frac{|y_{n-2} - x_{n-1}|}{2}\right) + \right.$$

$$\left. + h_0\left(\frac{|x_n - x_{n-1}|}{2}\right) h_0\left(\frac{|y_{n-2} - x_n|}{2^{l+1}}\right)\right].$$

Let's substitute this inequality into the expression for I_n for $l = 1$ (here $\tau_j - \tau_{j-1} \leq \beta$) then

$$I_n \leq \big(2\big(1 + \sqrt{2^d}\big)\big)^{\frac{1}{2}} \left[I_{n-1}^{(1)}(n) \int dx \left(h_0\left(\frac{|x|}{4}\right)\right)^{\frac{1}{2}} + I_{n-1}^{(2)}(n) \int dx \left(h_0\left(\frac{|x|}{2}\right)\right)^{\frac{1}{2}}\right],$$

$$I_{n-1}^{(l)}(n) = \sqrt{n!} \int dx_{(n-1)} \Bigg[\int\limits_{[\beta]_{(n)}} d\tau_{(n)} \int dy_{(n-1)} P^{\tau_1}\big(|y_1|\big) h_0\big(|y_1 - x_1|\big) \times$$

$$\times \prod_{j=1}^{n-2} P^{\tau_j - \tau_{j-1}}\big(|y_j - y_{j-1}|\big) h_0\big(|y_j - x_j|\big) h_0\left(\frac{|y_{n-2} - x_{n-1}|}{2^l}\right)\Bigg]^{\frac{1}{2}}, \quad l = 1, 2.$$

Here we used the fact that the square root of a finite sum is less than the sum of square roots of its elements. Iterating this bound $n-1$ times we'll obtain

$$I_n \le \sqrt{n!}\big(2\big(1+\sqrt{2^d}\big)\big)^{\frac{n}{2}} \left(\int\limits_{[\beta]_n} d\tau_{(n)}\right)^{\frac{1}{2}} 2^{(n-1)(d+1)} \big\|\sqrt{h_0}\big\|_1.$$

The integral in $\tau_{(n)}$ is equal to $(n!)^{-1}\beta^n$. That is, $I_n \le \big(\sqrt{\beta a}\,\|\sqrt{h_0}\|_1\big)^n$, $a = 2^{2d+3}(1+\sqrt{2^d}\,)$ and (5.2) is proved.

***Proof of* (1.8).** Let's insert the equality under the sign of the integral in the expression for $C_\beta^\Lambda(w)$ the equality

$$1 = \big(1 - \chi_{\Lambda^c(2^{-1}\delta)}(w')\big) + \chi_{\Lambda^c(2^{-1}\delta)}(w'). \tag{5.6}$$

By $c_\beta^{\Lambda-(+)}(w)$, $c_{*\beta}^{\Lambda-(+)}(w)$ we'll denote the functions which are obtained from the expressions for the corresponding functions without the marks $-(+)$ by inserting the first and and the second terms in this equality. The function $1 - \chi_{\Lambda^c}(w')$ is concentrated on paths that intersect Λ at least once. We have $\big(1 - \chi_{\Lambda^c(2^{-1}\delta)}(w')\big)\big(1 - \chi_\Lambda(w')\big) \le \chi\big(w'|\Gamma_\beta(2^{-1}\delta)\big)$, since the left-hand side is non-zero on the set $\Gamma_\beta(2^{-1}\delta)$. Now, applying the Schwartz inequality for the integral in the measure $P_{x,x}^\beta(dw')$ we obtain,taking into account the condition for A in Theorem 1.1 and (4.1),

$$c_{*\beta;\sigma}^{\Lambda-}(w) \le \gamma^{\frac{1}{2}}\left(2^{-1}\delta, \beta\right) \left(\sum_{l=1}^{d'} l^{-\theta}\right) \int dx \left[\int\limits_0^\beta G\left(w(\tau); x\right) d\tau\right]^{\frac{1}{2}}.$$

The inequality (5.2b) gives

$$c_{*\beta;\sigma}^{\Lambda-}(w) \le \gamma^{\frac{1}{2}}(2^{-1}\delta, \beta) c_*^-, \qquad c_*^- = (4\pi\beta)^{-\frac{d}{4}} \left(\sum_{l=1}^{d'} l^{-\theta}\right) |h_*|_0. \tag{5.7}$$

c_*^- is finite since h_* is a bounded function as it is proved at the beginning of this section. Applying (4.1) and the Schwartz inequality for the measure $P_{x,x}^\beta(dw')$ we derive $c_\beta^{\Lambda-}(w) \le \gamma^{\frac{1}{2}}(2^{-1}\delta, \beta) \int_0^\beta d\tau \int dx [G(w(\tau); x)]^{\frac{1}{2}}$. The inequality (5.2b) leads to

$$c_\beta^{\Lambda-}(w) \le \gamma^{\frac{1}{2}}(2^{-1}\delta, \beta) c^-, \qquad c^- = \beta(4\pi\beta)^{-\frac{d}{4}} \|\sqrt{h_0}\|_1. \tag{5.8}$$

From the the inequality $|w(\tau) - w'(\tau)| \ge 2^{-1}\delta$ we derive for $w \in \Lambda(\delta)$ employing once more the Schwartz inequality for the measure $P_{x,x}^\beta(dw')$

$$c_\beta^{\Lambda+}(w) \le \delta^{-\varepsilon} \int\limits_0^\beta d\tau \int dx \left[\int P_{x,x}^\beta(dw')\right]^{\frac{1}{2}} [G'(w(\tau); x)]^{\frac{1}{2}},$$

where G' is derived from h' in the same way as G from h, $h'(x) = h(x)$, $|x| \le \dfrac{\delta}{2}$; $h'(x) = \bar{h}|x|^{-d-\varepsilon}$, $|x| \ge \dfrac{\delta}{2}$, $\delta \ge 2R$. As a result with the help of (5.2b) we derive

$$c_{\beta}^{\Lambda+}(w) \leq \delta^{-\varepsilon} c^{+}, \qquad c^{+} = \beta(4\pi\beta)^{-\frac{d}{4}} \| \sqrt{h_0'} \|_1, \quad w \in \Lambda(\delta), \tag{5.9}$$

where h_0' is determined in the same way as h_0. Repeating the above arguments we obtain for $w \in \Lambda(\delta)$ the bound $c_{*\beta;\sigma}^{\Lambda+}(w) \leq \delta^{-\varepsilon}\left(\sum_{l=1}^{d'} l^{-\theta}\right) \int dx \left[\int P_{x,x}^{\beta}(dw')\right]^{\frac{1}{2}} \times \times \left[\int_0^{\beta} G'(w(\tau);x)d\tau\right]^{\frac{1}{2}}$. As a result

$$c_{*\beta}^{\Lambda+}(w) \leq \delta^{-\varepsilon} c_*^{+}, \qquad c_*^{+} = (4\pi\beta)^{-\frac{d}{4}} \left(\sum_{l=1}^{d'} l^{-\theta}\right) |h_*'|_0, \quad w \in \Lambda(\delta), \tag{5.10}$$

where h_*' is determined in the same way as h_*. Hence c_*^{+} is finite. Let $r(\varepsilon) = \max\left\{\delta^{-\varepsilon} \times \times \gamma^{\frac{1}{2}}(2^{-1}\delta, \beta)\right\}$. Then combining (5.7) – (5.10) we conclude that (1.8) holds with $C'(\beta) = = c^{+} + r(\varepsilon)c^{-} + 16\sqrt{2}\left(\sum_{l=1}^{d'} l^{-\theta}\right)(c_*^{+} + r(\varepsilon)c_*^{-})$. This ends the proof of Theorem 1.1.

6. Classical correlation functions. For classical systems we have the following simple analog of (2.1):

$$e^{-\beta U'(x_{(n)})} = \int e^{-i\sqrt{\beta}U'(x_{(n)};s_{(n)})} \mu_0(ds_{(n)}), \tag{6.1}$$

where the integration is performed over $\mathbb{R}^{d'n}$,

$$\mu_0(ds) = (\sqrt{2\pi})^{-d'} e^{-2^{-1}\|s\|^2} ds, \qquad \|s\|^2 = (s,s) = \sum_{l=1}^{d'} s_l^2,$$

$$U'(x_{(n)}; s_{(n)}) = \sqrt{2} \sum_{j_1=1}^{n} \left(s_{j_1}, \sum_{0 \leq j_2 \leq n, j_2 \neq j_1} \phi'(x_{j_2} - x_{j_1})\right) =$$

$$= \sqrt{2} \sum_{1 \leq k < j \leq n} (\phi'(x_j - x_k), s_j + s_k). \tag{6.2}$$

In our formulas d' may be infinite and we'll omit it in our formulas. Details about existence of the Gaussian measure on $\mathbb{R}^{\infty}$ and finite character of η_j a reader may find in [8]. The equality (6.1) reduces the classical Gibbs system with the potential energy U to the Gibbs particle system with with an additional $\mathbb{R}^{\nu}$-valued degree of freedom, smeared by the measure μ_0, and a pair complex interaction potential $\phi(x; s, s') = = \phi_0(x) + i\sqrt{2\beta^{-1}}(\phi'(x), s + s')$. The function $(s, \phi(x))$ is a limit of finite sums in the topology of $L^2(\mathbb{R}^{\infty}, \mu_0)$ if d' is infinite.

Now we have to deal with the KS equations (1.5), (1.6) determined by (1.12), (1.14) with $\Omega = \mathbb{R}^{d'+d}$, $\omega = (x, s)$, $\mu(d\omega) = \mu_0(ds)dx$. The most natural space for a description of the KS operator is the Banach space $\tilde{\mathbb{E}}_{\xi}$ of sequences $F = \left\{F(x_{(n)}; s_{(n)})\right\}_{n \geq 1}$ of measurable functions with the following norm:

$$\|F\|_\xi = \max_{n\ge1} \xi^{-n} \operatorname{ess} \sup_{x_{(n)},s_{(n)}} \exp\left\{-\sum_{j=1}^{n}\|s\|_1\right\}\left|F(x_{(n)},s_{(n)})\right|,$$
$$\|s\|_1^2 = \sum_{l=1}^{d'} l^{-2}|s_l|^2. \tag{6.3}$$

Our main result for the classical systems is formulated as follows.

Theorem 6.1. *If the conditions of Theorem* 1.4 *are satisfied,* $\eta_j = \int \|s\|_1^j e^{\|s\|_1} \times$ $\times \mu_0(ds)$, *then the KS operator* $\tilde K$ *is bounded in* $\tilde{\mathbb{E}}_\xi$ *if* $1 - 2\sqrt{2\beta}\eta_0\|\phi'\|\xi \ge 0$ *and its norm* $\|\tilde K\|_\xi$ *is given by* $\|\tilde K\|_\xi \le \xi^{-1}e^{2\beta B + \xi C_\eta}$.

Proof. From the definition of the R-symmetrized KS operator and (1.12)–(1.14) we obtain

$$\|\tilde K F\|_\xi \le \operatorname{ess}\sup_{x,s} \xi^{-1} e^{-\|s\|_1 + 2\beta B}\sum_{n\ge0}\frac{\xi^n}{n!}\int \left|K(x;s|(x'_{(n)},s'_{(n)})\right|dx'_{(n)}\mu_1(ds'_{(n)})\,\|F\|_\xi,$$

where

$$\mu_1(ds) = e^{\|s\|_1}\mu_0(ds),$$

$$K(x;s|x'_{(n)},s'_{(n)}) = \prod_{j=1}^{n}\left(e^{-\beta\phi_0(x-x'_j) - i\sqrt{2\beta}(\phi(x-x_j),s+s'_j)} - 1\right).$$

We'll need further to employ the following relation $|e^{b+ia} - 1| \le |e^b - 1| + 2|a|$. Putting $b = -\beta\phi_0(x - x'_j)$, $a = -\sqrt{2\beta}\,(\phi'(x - x'_j), s + s'_j)$ one derives with the help of the Schwartz inequality

$$\int \left|K(x;s|x'_j,s'_j)\right|\mu_1(ds'_j)dx'_j \le \eta_0\|e^{-\beta\phi_0} - 1\|_1 +$$
$$+ 2\sqrt{2\beta}\left[\int \left|(\phi'(x'_j),s)\right|dx'_j\mu_1(ds'_j) + \int \left|(\phi'(x'_j),s'_j)\right|dx'_j\mu_1(ds'_j)\right] \le$$
$$\le \eta_0\|e^{-\beta\phi_0} - 1\|_1 + 2\sqrt{2\beta}\,\|\phi'\|(\eta_0\|s\|_1 + \eta_1).$$

This results in

$$\int \left|K(x;s|x'_{(n)},s'_{(n)})\right|dx'_{(n)}\mu_1(ds'_{(n)}) \le \left(C_\eta + 2\sqrt{2\beta}\eta_0\,\|\phi'\|\,\|s\|_1\right)^n,$$

where $\|\tilde K\|_\xi = \xi^{-1}\max_s e^{2\beta B + \xi C_\eta - \|s\|_1(1 - 2\sqrt{2\beta}\eta_0\|\phi'\|\xi)}$. The theorem is proved.

Hence we came to the following conclusion.

***Corollary* 6.1.** *If the conditions of Theorem* 1.4 *are satisfied and* $1 - 2\sqrt{2\beta}\eta_0 \times$ $\times\|\phi'\|\xi \ge 0$ *then the sequence* $\rho = \sum_{n\ge0} z^{n+1}\tilde K^n\alpha$ *belongs to the Banach space* $\tilde{\mathbb{E}}_\xi$ *if* $|z| \le \xi e^{-\beta(2B + C_\eta\xi)}$ *and is the unique solution of the* R*-symmetrized KS equation in the space. If* ϕ_0 *is nonnegative then the sequence*

$$\rho = \sum_{n\ge0} z^{n+1}K^n\alpha$$

is the unique solution of the KS equation (1.6) *in the same Banach space and the same values of* z *with* $B = 0$.

Taking into account that $(\tilde{K}^n\alpha)(x_{(m)}) = 0$, $n < m-1$ it is seen that

$$\rho(x_{(m)}; s_{(m)}) = \sum_{n \geq m-1} z^{n+1}(\tilde{K}^n\alpha)(x_{(m)}; s_{(m)}).$$

Thus the following conclusion holds if one takes into account that $\|\alpha\|_\xi = \xi^{-1}$.

***Corollary* 6.2.** *Functions ρ_n from Theorem* 1.1 *are given by*

$$\rho_n(x_{(m)}) = \int (\tilde{K}^{m+n-1}\alpha)(x_{(m)}; s_{(m)})\mu_0(ds_{(m)})$$

and satisfy the inequality

$$\left|\rho_n(x_{(m)})\right| \leq \xi^{-1}\|\tilde{K}\|_\xi^{m+n-1}(\eta_0\xi)^m \leq \eta_0^m(e^{\beta(2B+C_\eta\xi)})^{n+m-1}\xi^{-n}.$$

If ϕ_0 is nonnegative then the equality and inequality hold after substituting K instead of $\tilde{K}$ and putting $B = 0$, respectively, in them.

Lemma 6.1. *Let the conditions of Theorem* 1.4 *hold,* $1 - 2\sqrt{2\beta}\eta_0\|\phi'\|\xi > 0$ *and* $\Lambda \subset \Lambda' \subset \Lambda''$ *and δ be the distance of Λ to the boundary of Λ' then*

$$\left\|\chi_\Lambda\tilde{K}(\chi_{\Lambda''} - \chi_{\Lambda'})\right\|_\xi \leq C_{\eta,\delta}(\xi)$$

where $0 < C_{\eta,\delta}(\xi)$ tends to zero if δ tends to infinity.

Proof. Applying the following relations:

$$\chi_\Lambda(x_{(m)})(\chi_{\Lambda''}(x_{(m)}, x'_{(n)}) - \chi_{\Lambda'}(x_{(m)}, x'_{(n)})) = \chi_\Lambda(x_{(m)})(\chi_{\Lambda''}(x'_{(n)}) - \chi_{\Lambda'}(x'_{(n)})),$$

$$0 \leq \chi_{\Lambda''}(x'_{(n)}) - \chi_{\Lambda'}(x'_{(n)}) \leq \sum_{j=1}^{n}(1 - \chi_{\Lambda'}(x'_j))$$

one obtains, taking into account that the considered functions are symmetric, the inequalities

$$\chi_\Lambda(x)\chi_\Lambda(x_{(m-1)}) \int \left|K(x; s|x'_{(n)}, s'_{(n)})\right|\big(\chi_{\Lambda''}(x_{(n)}) - \chi_{\Lambda'}(x_{(n)})\big)\times$$

$$\times\left|F(x_{(m-1)}, x'_{(n)}; s_{(m-1)}, s'_{(n)})\right|dx'_{(n)}\mu_0(ds'_{(n)}) \leq$$

$$\leq n\xi^{m+n-1}\|F\|_\xi e^{\sum_{j=1}^{m-1}\|s_j\|_1}\left(\int dx'\mu_1(ds')\left|K(x, s|x's')\right|\right)^{n-1}\times$$

$$\times \operatorname*{ess\,sup}_x \chi_\Lambda(x)\int dx'\big(1 - \chi_{\Lambda'}(x')\big)\mu_1(ds')\left|K(x, s|x's')\right| \leq$$

$$\leq n\xi^{m+n-1}\|F\|_\xi e^{\sum_{j=1}^{m-1}\|s_j\|_1}\big(C_\eta + 2\sqrt{2\beta}\eta_0\|\phi'\|\,\|s\|_1\big)^{n-1}\times$$

$$\times\left[\eta_0\int\limits_{|x|\geq\delta}\left|e^{-\beta\phi_0(x)} - 1\right|dx + 2\sqrt{2\beta}(\eta_0\|s\|_1 + \eta_1)\int\limits_{|x|\geq\delta}\left|\phi'(x)\right|_2 dx\right].$$

As a result, we have

$$\left\|\chi_\Lambda \tilde{K}(\chi_{\Lambda''} - \chi_{\Lambda'})\right\|_\xi \leq e^{\beta B + \xi C_\eta} e^{-\|s\|_1(1-2\sqrt{2\beta}\eta_0\|\phi'\|\xi)} \times$$

$$\times \left[\eta_0 \int\limits_{|x|\geq\delta} |e^{-\beta\phi_0(x)} - 1|dx + 2\sqrt{2\beta}\big(\eta_0\|s\|_1 + \eta_1\big) \int\limits_{|x|\geq\delta} \big|\phi'(x)\big|_2 dx\right].$$

That is

$$C_{\eta,\delta}(\xi) =$$

$$= e^{2\beta B + \xi C_\eta} \left[\eta_0 \int\limits_{|x|\geq\delta} |e^{-\beta\phi_0(x)} - 1|dx + 2\sqrt{2\beta}(\eta_0\theta_\xi + \eta_1) \int\limits_{|x|\geq\delta} |\phi'(x)|_2 dx\right],$$

where $\theta_\xi = \left(\max\limits_{q\geq 0} q e^{-q}\right)\left(1 - 2\sqrt{2\beta}\eta_0\|\phi'\|\xi\right)^{-1}$. The lemma is proved.

Remark **6.1.** A general potential energy is represented as

$$U(x_{(n)}) = \sum_k \sum_{1\leq j_1\neq j_2\ldots\neq j_k\leq n} \phi_k(x_{j_2} - x_{j_1}, x_{j_3} - x_{j_1}, \ldots x_{j_k} - x_{j_1}),$$

where ϕ_k are k-particle potentials.

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