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CONTINUITY OF CERTAIN PSEUDODIFFERENTIAL OPERATORS IN SPACES OF GENERALIZED SMOOTHNESS

НЕПЕРЕРВНІСТЬ ДЕЯКИХ ПСЕВДОДИФЕРЕНЦІАЛЬНИХ ОПЕРАТОРІВ У ПРОСТОРАХ УЗАГАЛЬНЕНОЇ ГЛАДКОСТІ

We investigate the continuity of a pseudodifferential operator in some spaces of generalized smoothness. Some properties of the spaces of generalized smoothness and the generalized Lipschitz spaces are proved.

Досліджено неперервність псевдодиференціального оператора у деяких просторах узагальненої гладкості. Доведено деякі властивості просторів узагальненої гладкості та узагальнених просторів Ліпшиця.

1. Introduction. In this paper we investigate the continuity of a pseudodifferential operator $p(x, D)$, the symbol $p(x, \xi)$ of which for fixed x is a continuous negative definite function, in the spaces of generalized smoothness. Such a continuity result enables us to add $p(x, D)$ as a perturbation to certain generator of an L_p-sub-Markovian semigroup, and still obtain the generator of an L_p-sub-Markovian semigroup.

For the survey on pseudodifferential operators with continuous negative definite symbol we refer to [1, 2], and the literature given therein, in particular, we mention [3, 4], where the symbolic calculus in L_2 for such operators was developed. In [5][1] the symbolic calculus for pseudodifferential operators in L_p was developed, and it was proved that under certain conditions on the symbol $p(x, \xi)$ of a pseudodifferential operator $p(x, D)$, this operator is continuous between some (related) ψ-Bessel potential spaces. Such conditions are similar to those in L_2-case, but much stronger, since to guarantee the continuity of $p(x, D)$ between subspaces of $L_p(\mathbb{R}^n)$ the Lizorkin's Fourier multiplier theorem is applied.

Our approach is a bit different. We will pose the conditions on the Lévy measure of $p(x, \xi)$, and prove that under such conditions $p(x, D)$ is continuous between related generalized Lipschitz spaces. Assuming also that $p(x, D)$ is continuous between certain ψ-Bessel potential spaces, we obtain our main result by interpolation arguments. Thus, our conditions on the symbol are different from those given in [5].

In the second section we give the necessary definitions concerning the spaces of generalized smoothness. The third section contains some technical statements on such spaces as well as some properties of the generalized Lipschitz spaces. In the fourth section under some conditions on the Lévy measure of continuous negative define symbol $p(x, \xi)$ we prove the continuity of $p(x, D)$ in generalized Lipschitz spaces. In the last section we prove that $p(x, D)$ is continuous in the related spaces of generalized smoothness. This result allows us to use $p(x, D)$ as the perturbation of a generator of an L_p-sub-Markovian semigroup.

If it is not specially indicated, all the function spaces, considered in the paper, are over $\mathbb{R}^n$.

2. Preliminaries. We start with some preliminary definitions and results.

Definition 1. A. *A sequence* $(\gamma_j)_{j\in\mathbb{N}_0}$ *of positive numbers is called strongly increasing if there is a positive constant* d *and a natural number* κ *such that equations*

[1] This manuscript can also be found on the web-page of the author, http:// www.math.etzh.ch//farkas/.

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$$d\gamma_j \leq \gamma_k \quad \textit{for all} \quad j, k, \quad 0 \leq j \leq k,$$

and

$$2\gamma_j \leq \gamma_k \quad \textit{for all} \quad j, k, \quad \textit{with} \quad j + \kappa \leq k$$

are satisfied.

B. *A sequence* $(\gamma_j)_{j\in\mathbb{N}_0}$ *of positive numbers is of bounded growth if there exists a positive constant* d *and a number* $J \in \mathbb{N}_0$, *such that*

$$\gamma_{j+1} \leq d\gamma_j \quad \textit{for any} \quad j \geq J.$$

C. *A sequence* $(\sigma_j)_{j\in\mathbb{N}_0}$ *is called admissible if*

$$d_0\sigma_j \leq \sigma_{j+1} \leq d_1\sigma_j \quad \textit{for all} \quad j \in \mathbb{N}$$

holds for some positive d_0 *and* d_1.

For a fixed strongly increasing sequence $N = (N_j)_{j\in\mathbb{N}_0}$ and fixed $J \in \mathbb{N}$ define the covering

$$\Omega_j^{N,J} = \left\{\xi \in \mathbb{R}^n : |\xi| \leq N_{j+J\kappa}\right\} \quad \text{if} \quad j = 0, 1, \ldots, J\kappa,$$

and

$$\Omega_j^{N,J} = \left\{\xi \in \mathbb{R}^n : N_{j-J\kappa} \leq |\xi| \leq N_{j+J\kappa}\right\} \quad \text{if} \quad j = J\kappa, J\kappa + 1, \ldots$$

for some κ.

Definition 2. *Let* $\Phi^{N,J}$ *be a collection of functions* $\left(\varphi_j^{N,J}\right)_{j\in\mathbb{N}_0}$ *such that*

B$_1$) $\varphi_j^{N,J} \in C_0^\infty(\mathbb{R}^n)$ *and* $\varphi_j^{N,J}(\xi) \geq 0$ *if* $\xi \in \mathbb{R}^n$ *for* $J \in \mathbb{N}_0$;

B$_2$) $\operatorname{supp}\varphi_j^{N,J} \subset \Omega_j^{N,J}$;

B$_3$) *for any* $\gamma \in \mathbb{N}_0^n$ *there exists a constant* $c_\gamma > 0$ *such that for any* $J \in \mathbb{N}_0$

$$\left|D^\gamma \varphi_j^{N,J}(\xi)\right| \leq \left|\xi\right|^\gamma \quad \textit{for any} \quad \gamma \in \mathbb{R}^n;$$

B$_4$) *there exists a constant* $c_\varphi > 0$ *such that*

$$0 < \sum_{j=0}^{\infty} \varphi_j^{N,J}(\xi) = c_\varphi < \infty \quad \textit{for any} \quad \xi \in \mathbb{R}^n.$$

We will give a general definition of the spaces of generalized smoothness, $F_{pq}^{\sigma,N}$ and $B_{pq}^{\sigma,N}$, which are the generalizations of Triebel – Lizorkin and Besov spaces respectively (see [6]).

Definition 3. *Let* $(N_j)_{j\in\mathbb{N}_0}$ *be a strongly increasing sequence of bounded growth, and let* $(\sigma)_{j\in\mathbb{N}_0}$ *be an admissible sequence.*

i) *Let* $0 < p, q \leq \infty$. *The Besov space of generalized smoothness is*

$$B_{pq}^{\sigma,N} = \left\{ f \in S' : \| f \|_{B_{pq}^{\sigma,N}} = \left\| \left(\sigma_j \varphi_j^{N,J}(D) f\right)_{j\in\mathbb{N}_0} \right\|_{l_q(L_p)} < \infty \right\}.$$

ii) *The* $0 < p < \infty$, $0 < q \leq \infty$. *The Triebel – Lizorkin space of generalized smoothness is*

$$F_{pq}^{\sigma,N} = \left\{ f \in S' : \| f \|_{F_{pq}^{\sigma,N}} = \left\| \left(\sigma_j \varphi_j^{N,J}(D) f\right)_{j\in\mathbb{N}_0} \right\|_{L_p(l_q)} < \infty \right\}.$$

***Remark* 1.** For $0 < p,\ q < \infty$ the spaces $B_{pq}^{\sigma,N}$ and $F_{pq}^{\sigma,N}$ can be defined without the restriction that $(N_j)_{j\in\mathbb{N}_0}$ is of bounded growth.

Such spaces are the generalizations (due to the standardization theorem) of those introduced by Kaljabin [7, 8]; they are equivalent to the spaces given in [7, 8] if σ is strongly increasing and of bounded growth (see [6] for details).

Denote by σ^s the sequence $(\sigma_j^s)_{j\in\mathbb{N}_0} = (2^{sj})_{j\in\mathbb{N}_0}$.

***Remark* 2.** For $N_j = 2^j$ and $\sigma = \sigma^s$ the spaces $F_{pq}^{\sigma^s,N}$ and $B_{pq}^{\sigma^s,N}$ coincide with F_{pq}^s and B_{pq}^s respectively (see [6]).

We are interested in the situation when the strongly increasing sequence N is associated with a function which satisfies some additional conditions. For the following definition we refer to [6] (see also [5]).

Definition 4. *Let $\mathcal{A}$ be the class of all nonnegative functions $a: \mathbb{R}^n \to \mathbb{R}$ of class C^∞ with the following properties*:

A) $\lim\limits_{|\xi|\to\infty} a(\xi) = \infty$;

B) *$a(\xi)$ is almost increasing in $|\xi|$, i.e., there exist constants $\delta_0 \geq 1$ and $R > 0$ such that $a(\xi) \leq \delta_0 a(\eta)$ if $R \leq |\xi| \leq \eta$;*

C) *there exists $m > 0$ such that $a(\xi)|\xi|^{-m}$ is almost decreasing in $|\xi|$, i.e., there exists a constant δ_m, $0 < \delta_m \leq 1$, and $R > 0$ such that*

$$a(\xi)|\xi|^{-m} \geq \delta_m a(\eta)|\eta|^{-m} \quad \textit{if} \quad R \leq |\xi| \leq \eta;$$

D) *for every multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{N} \cup \{0\}$, $i = 1, \ldots, n$, there exists some $c_\alpha > 0$ such that*

$$\left|D^\alpha a(\xi)\right| \leq c_\alpha a(\xi)\left(1+|\xi|^2\right)^{-|\alpha|}, \quad \textit{if} \quad |\xi| \geq R.$$

The functions from $\mathcal{A}$ are called admissible functions.

It was proved that for an admissible symbol $a(\xi)$ the sequence $N^a = \left(N_j^{a,r}\right)_{j\in\mathbb{N}_0}$, where

$$N_j^{a,r} = \sup\left\{|\xi|\colon a(\xi) \leq 2^{rj}\right\}, \quad j \in \mathbb{N}_0, \tag{1}$$

is strongly increasing, see Lemma 3.1.16 from [6]. Note that the definition of the strongly increasing sequence by (1) does not require the radial symmetry of the symbol.

Recall the definition of the ψ-Bessel potential space of order s:

$$H_p^{\psi,s}(\mathbb{R}^n) = \left\{u\colon \left\| F^{-1}\left((1+\psi(\xi))^{s/2} Fu(\xi)\right)\right\|_{L_p} < \infty\right\},$$

where ψ is a continuous negative definite function, Fu is the Fourier transform of function u, and $s > 0$ (see [9] for details).

For the admissible continuous negative functions $a(\xi)$

$$F_{p,2}^{\sigma^s,N} = H_p^{a,s}, \quad s > 0$$

(see [6]).

For the following definition we refer to [2, p. 293, 294], or [10] (§ 1.9.1).

Let $G = \{ z \in \mathbb{C};\ 0 < \operatorname{Re} z < 1 \}$. For two complex Banach spaces $\left(X_0, \|\cdot\|_{X_0}\right)$ and $\left(X_1, \|\cdot\|_{X_1}\right)$ both embedded into some Hausdorff space χ, set $X := X_0 + X_1$, equipped with the norm $\|\cdot\|_X := \max\left(\|\cdot\|_{X_0}, \|\cdot\|_{X_1}\right)$ which is equivalent to the norm $\|\cdot\|_{X_0} + \|\cdot\|_{X_1}$, and which turns X into a Banach space. Denote by $W(G, X)$ the space of all continuous functions $\omega : \overline{G} \to X$ with the following properties:

1) $\omega|_G$ is analytic and $\sup\limits_{z \in \overline{G}} \|\omega(z)\|_X < \infty$;

2) $\omega(iy) \in X_0$ and $\omega(1 + iy) \in X_1$, for $y \in \mathbb{R}$ with continuous maps $y \mapsto \omega(iy)$ and $y \mapsto \omega(1 + iy)$;

3) $\|\omega\|_{W(G,X)} := \max\left(\sup\|\omega(iy)\|_{X_0}, \sup\|\omega(1+iy)\|_{X_1}\right) < \infty$.

By the maximum principle $\left(W(G, X), \|\cdot\|_{W(G, X)}\right)$ is the Banach space. We call $\{X_0, X_1\}$ an *interpolation couple,* and for any interpolation couple define its *complex interpolation space*

$$[X_0, X_1]_\theta := \left\{ u \in X;\ \text{there exists } \omega \in W(G, X) \text{ such that } \omega(\theta) = u \right\}.$$

On $[X_0, X_1]_\theta$ we introduce the norm

$$\|u\|_{[X_0, X_1]_\theta} := \inf\left(\|\omega\|_{W(G,X)},\ \omega \in W(G, X) \text{ and } \omega(\theta) = u \right). \tag{2}$$

With this norm the space $\left([X_0, X_1]_\theta, \|\cdot\|_{[X_0, X_1]_\theta}\right)$ is a Banach space.

3. Some properties of the spaces of generalized smoothness and generalized Lipschitz spaces. In this section we give some auxiliary technical results, which will be necessary to prove our main results in Section 5.

Lemma 1. *Let* $0 < p_0, q_0, p_1, q_1 \leq \infty$, *and* $0 < \theta < 1$. *Then for* $\dfrac{1}{p} = \dfrac{1-\theta}{p} + \dfrac{\theta}{p_1}$, $\dfrac{1}{q} = \dfrac{1-\theta}{q_0} + \dfrac{\theta}{q_1}$ *and* $s = (1-\theta)s_0 + \theta s_1$ *we have*

$$\left[B^{\sigma^{s^0}, N}_{p_0 q_0}, B^{\sigma^{s^1}, N}_{p_1 q_1} \right]_\theta = B^{\sigma^s, N}_{pq}.$$

Proof. Let $\left\{\varphi_j^{N,J}\right\}_{j \geq 0} \in \Phi^{N,J}$, and $Sf = \left\{ f * \varphi_j^{N,J} \right\}_{j \geq 0}$. From the definition of $B^{\sigma^s, N}_{pq}$ we have

$$f \in B^{\sigma^s, N}_{pq} \Leftrightarrow Sf \in l_q^s(L_p),$$

where $l_q^s = \left\{ (a_j)_{j \geq 0} : (2^s a_j)_{j \geq 0} \in l_q \right\}$. Therefore, by the definition of $W(G, X)$ and (2) we see that for $0 < \theta < 1$

$$\left[B^{\sigma^{s^0}, N}_{p_0 q_0}, B^{\sigma^{s^1}, N}_{p_1 q_1} \right]_\theta = \left\{ f \in B^{\sigma^{s^0}, N}_{p_0 q_0} + B^{\sigma^{s^1}, N}_{p_0 q_0} : \exists\, \omega \in W\left(G, B^{\sigma^{s^0}, N}_{p_0 q_0} + B^{\sigma^{s^1}, N}_{p_1 q_1}\right), \right.$$

$$\left. \text{such that } \omega(\theta) = f \right\} =$$

$$= \left\{ Sf \in l_{q_0}^{s^0}(L_{p_0}) + l_{q_1}^{s^1}\left(L_{p_1}\right) : \exists\, \omega \in W\left(G, l_{q_0}^{\sigma^{s^0}}\left(L_{p_0}\right) + l_{q_1}^{s^1}\left(L_{p_1}\right)\right) \right.$$

$$\text{such that } g(\theta) = Sf \Big\} =$$

$$= \left\{ Sf : Sf \in \left[l_{q_0}^{s^0}\left(L_{p_0}\right), l_{q_1}^{s^1}\left(L_{p_1}\right)\right]_\theta \right\} = \left\{ Sf : Sf \in l_q^s(L_p) \right\} = B_{pq}^{\sigma^s, N}.$$

The lemma is proved.

***Remark* 3.** Similarly, for $0 < p_0, q_0, p_1, q_1 < \infty$ and $0 < \theta < 1$, $\dfrac{1}{p} = \dfrac{1-\theta}{p} + \dfrac{\theta}{p_1}$, $\dfrac{1}{q} = \dfrac{1-\theta}{q_0} + \dfrac{\theta}{q_1}$, $s = (1-\theta)s_0 + \theta s_1$, we can prove that

$$\left[F_{p_0 q_0}^{\sigma^{s_0}, N}, F_{p_1 q_1}^{\sigma^{s_1}, N} \right]_\theta = F_{pq}^{\sigma^s, N}.$$

The embedding properties of the spaces $F_{pq}^{\sigma^s, N}$ and $B_{pq}^{\sigma^s, N}$ are similar to those of F_{pq}^s and B_{pq}^s, since the construction of the spaces of generalized smoothness is different to those of F_{pq}^s and B_{pq}^s only in the choice of the decomposition of unity and the sequence σ^s. We will give a chain of embeddings, which will be useful for us when we prove the mapping properties of certain pseudodifferential operators.

Lemma 2. *Let* $N = \left(N_j^{a,2}\right)_{j \in \mathbb{N}_0}$, $0 < q \le p < \infty$, *and* $0 < \varepsilon < s < \infty$. *Then the following embedding take place*:

$$F_{p,2}^{\sigma^s, N} \subset F_{p,q}^{\sigma^s, N} \subset B_{p,p}^{\sigma^s, N} \subset F_{p,q}^{\sigma^{s-\varepsilon}, N}.$$

Proof. Two left-hand embeddings follow from $l_q \subset l_p$ for $q \le p$ and the observation that $F_{p,p}^{\sigma^s, N} = B_{p,p}^{\sigma^s, N}$, $2 \le p < \infty$. Indeed, for $f \in F_{p,q}^{\sigma^s, N}$

$$\| f \|_{B_{p,p}^{\sigma^s, N}} \le c \| f \|_{F_{p,p}^{\sigma^s, N}} = c \left\| \left(\sum_{j=0}^{\infty} \left| 2^{js} \psi_j^{NJ}(D) f \right|^p \right)^{1/p} \right\|_{L_p} \le$$

$$\le c \left\| \left(\sum_{j=0}^{\infty} \left| 2^{js} \psi_j^{NJ}(D) f \right|^q \right)^{1/q} \right\|_{L_p} = c \| f \|_{F_{p,q}^{\sigma^s, N}} \le$$

$$\le c \left\| \left(\sum_{j=0}^{\infty} \left| 2^{js} \psi_j^{NJ}(D) f \right|^2 \right)^{1/2} \right\|_{L_p} = c \| f \|_{F_{p,2}^{\sigma^s, N}},$$

i.e., we obtain two left-hand embeddings.

To obtain the right-hand embedding, we will follow Proposition 2.3.2/2 [7]. Note, that for a sequence $\left(a_j\right)_{j \ge 0}$

$$\left(\sum_{j=0}^{\infty} \left| 2^{j(s-\varepsilon)} a_j \right|^q \right)^{1/q} \le \sup_{j \ge 0} \left| 2^{js} a_j \right| \left(\sum_{j=0}^{\infty} 2^{-j\varepsilon q} \right)^{1/q} \le$$

$$\le c \sup_{j \ge 0} \left| 2^{js} a_j \right|.$$

Then for $f \in B_{pp}^{\sigma^s, N}$

$$\|f\|_{F_{pq}^{\sigma^{s-\varepsilon},N}} = c\left\|\left(\sum_{j=0}^{\infty}\left|2^{j(s-\varepsilon)}\psi_j^{NJ}(D)f\right|^q\right)^{1/q}\right\|_{L_p} \le$$

$$\le c\left\|\sup_{j\ge 0}\left|2^{js}\psi_j^{NJ}(D)f\right|\right\|_{L_p} \le c\left\|\left(\sum_{j=0}^{\infty}\left|2^{js}\psi_j^{NJ}(D)f\right|^p\right)^{1/p}\right\|_{L_p} = c\|f\|_{B_{pp}^{\sigma^s,N}},$$

which proves the right-hand embedding.

The lemma is proved.

***Remark* 4.** Since $S(\mathbb{R}^n)$ is dense in $H_p^{a,s} = F_{p,2}^{\sigma^s,N}$, $(N_j)_{j\in\mathbb{N}_0} = \left(N_j^{a,2}\right)_{j\in\mathbb{N}_0}$, then all the embeddings in Lemma 2 are dense.

Denote by $a(D)$ the operator with symbol $a(\xi)$. For the following theorem we again refer to [6] (see also Theorem 2.3.8 from [12]). Let $a(\xi)$ and $(N_j)_{j\in\mathbb{N}_0}$ be as before. For any real s we have

$$(1+a(D))^{\mu/r} : B_{pq}^{\sigma^{t+\mu},N} \to B_{p,q}^{\sigma^t,N}, \quad 0 < p, q \le \infty,$$

isomorphically.

Similarly to the Lipschitz spaces of order α one can define the spaces $\Lambda_{\lambda^\mu(t)}$ (see [11]), where the function $|t|^\alpha$ is substituted by some function $\lambda^\mu(t)$. In what follows, we denote by Λ_1 the space of Lipschitz functions on $\mathbb{R}^n$.

Definition 5. *Let* $\lambda(t): \mathbb{R}^n \to \mathbb{R}_+$ *be a continuous nonnegative function and* $\mu >$ > 0 *be a real number such that*

1) $\lambda(t) \to 0$ *as* $|t| \to 0$, *and* $\lambda(t) \to \infty$ *as* $|t| \to \infty$;

2) $\lim\limits_{t\to 0}\dfrac{\lambda^{\mu+1}(t)}{|t|} = \infty$;

3) $\lambda(t)$ *has no other zeros except at* $t = 0$.

Define

$$\Lambda_{\lambda^\mu(t)} = \left\{f \in C_\infty : \|f(x-t) - f(x)\|_{C_\infty} \le A\lambda^\mu(t)\right\},$$

and let

$$\|f\|_{\Lambda_{\lambda^\mu(t)}} = \|f\|_\infty + \sup_{|t|>0}\frac{\|f(x-t)-f(x)\|_{C_\infty}}{\lambda^\mu(t)}$$

be the norm in $\Lambda_{\lambda^\mu(t)}$.

We need to make an additional assumption about the relation between the operator $a(D)$ and the function $\lambda^\mu(t)$.

Assumption 1. Assume, that $\lambda^\mu(t)$ is such that $(1+a(D))^{-\mu/r}$ maps C_∞ continuously into $\Lambda_{\lambda^\mu(t)}$.

Of course the choice of such a function $\lambda^\mu(t)$ may be not unique.

Lemma 3. *Let* $\sigma^s = (2^{js})_{j\ge 0}$, $N = \left(N_j^{a,r}\right)_{j\ge 0}$ *be defined as in* (1), *and the function* $(\lambda^\mu(t))$ *satisfy Assumption* 1.

Then

$$B^{\sigma^0,N}_{\infty,1} \subset C_\infty \subset B^{\sigma^0,N}_{\infty,\infty} \tag{3}$$

and

$$B^{\sigma^\mu,N}_{\infty,1} \subset \Lambda_{\lambda(t)^\mu} \subset B^{\sigma^\mu,N}_{\infty,\infty}. \tag{4}$$

Proof. The proof of (3) is similar to the proof of Proposition 2.5.7 [12]. Let $\left(\varphi_j^{N,J}\right)_{j\in\mathbb{N}_0} \in \Phi^{N,J}$, and $f \in B^{\sigma^0,N}_{\infty,1}$. Then

$$\|f\|_{L_\infty} = \left\| \sum_{j=1}^{\infty} \varphi_j^{N,J}(D) f(x) \right\|_{L_\infty} \le \sum_{j=1}^{\infty} \left\| \varphi_j^{N,J}(D) f(x) \right\|_{L_\infty} = \|f\|_{B^{\sigma^0,N}_{\infty,1}}.$$

Since $\varphi_j^{N,J}(D) f(x)$ is bounded and continuous (by Paley – Wiener theorem), and hence uniformly continuous in $\mathbb{R}^n$, then $f \in C_\infty$, and thus we have the left-hand embedding in (3).

Now let $f \in C_\infty$. Since by B_3 the function $\varphi_j^{N,J}(x)$ is a Fourier multiplier, then

$$\|f\|_{B^{\sigma^0,N}_{\infty,\infty}} = \sup_{j\in\mathbb{N}_0} \left\| \varphi_j^{N,J}(D) f \right\|_{L_\infty} \le \sup_{j\in\mathbb{N}_0} \left\| \varphi_j^{N,J}(D) \right\|_{L_\infty} \|f\|_{L_\infty} \le c\|f\|_{C_\infty},$$

which proves the right-hand embedding. (By $\left\| \varphi_j^{N,J}(D) \right\|_{L_\infty}$ we understand the norm of the operator which corresponds to the symbol $\varphi_j^{N,J}(\xi)$.)

By the lifting property of $(1 + a(D))^{-\mu/r}$ (see [6]) we get (4) from (3).

The lemma is proved.

Remark 5. *Similarly to Proposition* 2.3.2/2 [12], *one can prove that* $B^{\sigma^{\mu+\varepsilon},N}_{\infty,\infty} \subset$ $\subset B^{\sigma^\mu,N}_{\infty,1}$ *for some* $\varepsilon > 0$, *which leads to*

$$B^{\sigma^{\mu+\varepsilon}}_{\infty,\infty} \subset \Lambda_{\lambda(t)^\mu} \subset B^{\sigma^\mu,N}_{\infty,\infty}. \tag{5}$$

Proof. We only need to show that $B^{\sigma^{\mu+\varepsilon},N}_{\infty,\infty} \subset B^{\sigma^\mu,N}_{\infty,1}$ holds for $\varepsilon > 0$, then (5) will follow from (4).

Since for a sequence $(a_j)_{j\ge 0}$ with $|a_j|$-finite

$$\sum_{j=0}^{\infty} 2^{js}|a_j| \le \sup_{j\ge 0}\left|2^{j(s+\varepsilon)} a_j\right| \sum_{j=0}^{\infty} 2^{-j\varepsilon} \le c \sup_{j\ge 0}\left|2^{j(s+\varepsilon)} a_j\right|$$

holds, then for $f \in B^{\sigma^{\mu+\varepsilon},N}_{\infty,\infty}$ we obtain

$$\|f\|_{B^{\sigma^\mu,N}_{\infty,1}} = \sum_{j=1}^{\infty} 2^{js} \left\| \varphi_j^{N,J}(D) f \right\|_{L_\infty} \le \sup_{j\ge 0}\left(2^{j(s+\varepsilon)} \left\| \varphi_j^{N,J}(D) f \right\|_{L_\infty} \right) \sum_{j=0}^{\infty} 2^{-j\varepsilon} \le$$

$$\le c \sup_{j\ge 0}\left(2^{j(s+\varepsilon)} \left\| \varphi_j^{N,J}(D) f \right\|_{L_\infty} \right) = c\|f\|_{B^{\sigma^{\mu+\varepsilon},N}_{\infty,\infty}}.$$

Let $0 < \mu < 1$. Define now by $\Lambda^0_{\lambda(t)^\mu}$ the closure of C_0^∞ with respect to $\|\cdot\|_{\Lambda_{\lambda(t)^\mu}}$-norm. Similarly to Theorem III.3.3 from [13], we have the following lemma.

Lemma 4. *The space* $\Lambda^0_{\lambda(t)^\mu}$, $0 < \mu < 1$, *coincides with the space of functions from* $\Lambda_{\lambda(t)^\mu}$ *for which*

$$\lim_{|t|\to 0} \frac{\| f(\cdot - t) - f(\cdot)\|_{C_\infty}}{\lambda(t)^{\mu}} = 0. \tag{6}$$

Proof. Since $C_0^\infty \subset \Lambda_1 \subset \Lambda_{\lambda(t)^\mu}$ and C_0^∞ is dense in Λ_1, we see, that the space $\overline{\Lambda_1}^{\|\cdot\|_{\Lambda_{\lambda^\mu(t)}}}$ is equivalent to $\overline{C_0^\infty}^{\|\cdot\|_{\Lambda_{\lambda^\mu(t)}}}$.

We will follow the proof given in [13]. Clearly, for all functions from Λ_1 (6) holds. Let $f_m \in \Lambda_1$ and $f_m \to f$ in $\Lambda_{\lambda^\mu(t)}$. Fix $\varepsilon > 0$. There exists N_0, such that

$$\| f_m \|_{\Lambda_1} < c, \qquad \| f_m - f \|_{\Lambda_{\lambda^\mu(t)}} < \frac{\varepsilon}{2}$$

for all $m \geq N_0$ and let δ be such that for $|t| < \delta$ we have

$$\frac{|t|}{\lambda^\mu(t)} < \frac{\varepsilon}{2c}.$$

Then

$$\frac{|f(x-t) - f(x)|}{\lambda^\mu(t)} \leq \frac{|f(x-t) - f_m(x-t) - (f(x) - f_m(x))|}{\lambda^\mu(t)} +$$

$$+ \frac{|f_m(x-t) - f_m(x)|}{\lambda^\mu(t)} \leq \| f_m - f \|_{\Lambda_{\lambda^\mu(t)}} + \| f_m \|_{\Lambda_1} \frac{|t|}{\lambda^\mu(t)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and f satisfies (6).

Let now $f \in \Lambda_{\lambda^\mu(t)}$ and satisfy (6). We will show that f belongs to $\Lambda^0_{\lambda^\mu(t)}$, i.e., we will show that there exists a sequence $(f_m)_{m\geq 0}$ from Λ_1, that converges to f in $\|\cdot\|_{\Lambda_{\lambda^\mu(t)}}$-norm.

Consider

$$f_m(x) = m \int_x^{x+1/m} f(\tau)\,d\tau - m \int_0^{1/m} f(\tau)\,d\tau.$$

Here and further we denote by

$$\int_x^{x+1/m} \ldots d\tau = \int_{x_1}^{x_1+1/m} \ldots \int_{x_m}^{x_n+1/m} \ldots d\tau_1 \ldots d\tau_m,$$

and analogously for $\int_0^{1/m} \ldots d\tau$.

The functions f_m, $m \geq 0$, are once continuously differentiable (by each x_i), and therefore belong to Λ_1. Changing the variables, we obtain

$$f_m(x) = m \int_0^{1/m} [\, f(x+\theta) - f(\theta)\,]\,d\theta,$$

which gives

$$f_m(x) - f(x) = m \int_0^{1/m} [\, f(x+\theta) - f(\theta) - f(x)\,]\,d\theta.$$

Define

$$[\, f_m(x+t) - f(x-t)\,] - [\, f_m(x) - f(x)\,] = \Psi(f_m - f, t);$$

in this notation

$$\| f_m - f \|_{\Lambda_{\lambda^\mu(t)}} = \| f_m - f \|_{C_\infty} + \sup_{|t|>0} \frac{|\Psi(f_m - f, t)|}{\lambda^\mu(t)}.$$

Since $f \in \Lambda_{\lambda^\mu(t)}$ and (6) holds, then for chosen $\varepsilon > 0$ there exists $\delta > 0$ such that for $|t| < \delta$ f satisfies

$$\frac{|f(x+t) - f(x)|}{\lambda^\mu(t)} < \frac{\varepsilon}{2} \quad \forall x \in \mathbb{R}^n.$$

Then

$$\frac{|\Psi(f_m - f, t)|}{\lambda^\mu(t)} = \frac{m}{\lambda^\mu(t)} \left| \int_0^{1/m} [f(x+\theta+t) - f(x+t) - f(x+\theta) + f(x)] d\theta \right| \le$$

$$\le m \int_0^{1/m} \left(\frac{|f(x+\theta+t) - f(x+\theta)|}{\lambda^\mu(t)} + \frac{|f(x+t) - f(x)|}{\lambda^\mu(t)} \right) d\theta < m \frac{1}{m} \left(\frac{\varepsilon}{2} + \frac{\varepsilon}{2} \right) = \varepsilon.$$

Let $|t| \ge \delta$. Then

$$\frac{|\Psi(f_m - f, t)|}{\lambda^\mu(t)} = m \int_0^{1/m} \frac{\lambda^\mu(\theta)}{\lambda^\mu(t)} \left(\frac{|f(x+\theta+t) - f(x+t)|}{\lambda^\mu(\theta)} + \frac{|f(x+\theta) - f(x)|}{\lambda^\mu(\theta)} \right) d\theta \le$$

$$\le 2m \| f \|_{\Lambda_{\lambda^\mu(t)}} \int_0^{1/m} \frac{\lambda^\mu(\theta)}{\lambda^\mu(t)} d\theta \le 2C \| f \|_{\Lambda_{\lambda^\mu(t)}} m \int_0^{1/m} \lambda^\mu(\theta) d\theta,$$

where C is such that $\lambda^{-\mu}(t) \le C$ for $|t| \ge \delta$, and we again may chose large N_0 such that for all $m \ge N_0$

$$m \int_0^{1/m} \lambda^\mu(\theta) d\theta < \frac{\varepsilon}{2C \| f \|_{\Lambda_{\lambda^\mu(t)}}}.$$

Therefore $\| f_m - f \|_{\Lambda_{\lambda^\mu(t)}} < \varepsilon$ for $m \ge N_0$, and in such a way $f \in \Lambda^0_{\lambda^\mu(t)}$.

Lemma 4 is proved.

Let $\lambda : (0, 1) \to \mathbb{R}_+$ be a nondecreasing, continuous function, $\lim_{t\to 0} \lambda(t) = 0$ and let for $1 \le p, q \le \infty$ and $M \in \mathbb{N}$

$$B^\lambda_{pq}(\mathbb{R}^n) = \left\{ f \in L_p : \left(\int_0^1 \left(\frac{\omega_p^M(f, t)}{\lambda(t)} \right)^q \frac{d\lambda(t)}{\lambda(t)} \right)^{1/q} < \infty \right\},$$

where

$$\omega_p^M(f, t) = \sup_{|h|<t} \left\| \Delta_p^M u(\cdot) \right\|_{L_p},$$

and Δ_p^M is the finite difference of order M in h.

In addition, let $t \mapsto \dfrac{\lambda(t)}{t^M}$ be increasing, and $t \mapsto \dfrac{\lambda(t)}{t^\delta}$ be almost decreasing, then

$$B^\lambda_{pq}(\mathbb{R}^n) = B^{\sigma^1, N}_{pq},$$

where $(\sigma^1) = (2^j)_{j\geq 0}$, $N_j = \dfrac{1}{h_j}$, where h_j is such that $\lambda(h_j) = 2^{-j}\lambda(1)$ (see [14]). In the following we suppose that $\lambda(1) = 1$.

Clearly, if the function $\lambda^{\mu}(t)$ satisfies the conditions above, we obtain from the definition of Λ_{λ^μ} that

$$\Lambda_{\lambda^\mu}(\mathbb{R}^n) = B^{\lambda^\mu}_{\infty\infty}(\mathbb{R}^n). \tag{7}$$

Moreover, if an operator $(1 + a(D))$ is an isomorphism between $B^{\sigma^{1+\mu},N}_{pq}$ and $B^{\sigma^\mu,N}_{pq}$, it is an isomorphism between $B^{\sigma^{1+\mu}}_{\infty\infty}$ and $B^{\sigma^\mu}_{\infty\infty}$, where the function λ is uniquely determined by $a(\xi)$.

4. The continuity of a pseudodifferential operator in generalized Lipschitz spaces. Let the pseudodifferential operator be of the form

$$p(x, D)f(x) = \int\limits_{\mathbb{R}^n\setminus\{0\}} (f(x-y) - f(x))\nu(x, dy), \tag{8}$$

where $\nu(x, dy)$ is a Lévy measure, which depends on x.

Theorem 1. *Let $p(x, D)$ be as in* (8), $\nu(x, dy) = g(x, y)dy$, *where the function $g(x, y)$ is differentiable in x and satisfies*

$$\sup_{x\in\mathbb{R}^n}\left(\int\limits_{|y|<1}\lambda^{\mu+1}(y)|g'_h(x, y)|dy + \int\limits_{|y|\geq 1}|g'_h(x, y)|dy\right) < \infty \tag{9}$$

for any direction h, and

$$\sup_{x\in\mathbb{R}^n}\left(\lambda^{-\mu}(h)\int\limits_{|y|<|h|}\lambda^{\mu+1}(y)|g(x, y)|dy + \lambda(h)\int\limits_{|y|>|h|}|g_h(x, y)|dy\right) \to 0 \quad as \quad |h| \to 0, \tag{10}$$

where $g'_h(x, y)$ is the derivative of $g(x, y)$ with respect to x in direction h.

Then

$$p(x, D)\colon \Lambda^0_{\lambda^{\mu+1}(t)} \to \Lambda^0_{\lambda^{\mu}(t)}.$$

Proof. Let $f \in \Lambda^0_{\lambda^{\mu+1}(t)}$. For such f we have

$$\sup_{|y|>0}\frac{\|f(x-y) - f(x)\|_{C_\infty}}{\lambda^{\mu+1}(y)} < \infty.$$

Since

$$\|p(x, D)f\|_{C_\infty} \leq c\|f\|_{\Lambda^{\mu+1}_{\lambda(t)}}, \tag{11}$$

we will check weather the following inequality is satisfied:

$$\sup_{|h|>0}\frac{1}{\lambda^\mu(h)}\left\|\int\limits_{\mathbb{R}^n}(f(x-h-y) - f(x-h))\nu(x-h, dy) -\right.$$

$$\left. - \int\limits_{\mathbb{R}^n}(f(x-y) - f(x))\nu(x-h, dy)\right\|_{C_\infty} \leq c\|f\|_{\Lambda^{\mu+1}_{\lambda(t)}}.$$

Again, for $|h| \geq 1$ we obtain

$$\frac{1}{\lambda^\mu(h)}\left| \int_{\mathbb{R}^n} (f(x-h-y) - f(x-h))\nu(x-h, dy) - \int_{\mathbb{R}^n} (f(x-y) - f(x))\nu(x, dy)\right| \leq$$

$$\leq C\left| \int_{\mathbb{R}^n} (f(x-h-y) - f(x-h))\nu(x-h, dy)\right| +$$

$$+ C\left| \int_{\mathbb{R}^n} (f(x-y) - f(x))\nu(x, dy)\right| \leq 2C\| p(x, D) f\|_{C_\infty},$$

and then for $f \in \Lambda^0_{\lambda^{\mu+1}(t)}$

$$\sup_{|h|\geq 1} \frac{\| p(x-h, D) f(x-h) - p(x, D) f(x)\|_{C_\infty}}{\lambda^\mu(h)} \leq c\| f\|_{\Lambda^{\mu+1}_{\lambda(t)}}. \tag{12}$$

Next consider the case $|h| < 1$. It is convenient to decompose

$$p(x-h, D) f(x-h) - p(x, D) f(x) =$$

$$= \int_{\mathbb{R}^n} \{ (f(x-h-y) - f(x-h)) - (f(x-y) - f(x))\} g(x, y) dy +$$

$$+ \int_{\mathbb{R}^n} (f(x-h-y) - f(x-h))(g(x-h, y) - g(x, y)) dy = I_1 + I_2,$$

and consider I_1 and I_2 separately.

From the mean-value theorem we have

$$|g(x-h, y) - g(x, y)| = |h| g'_h(x_0, y)$$

for some x_0, where $g'_h(x, y)$ is the derivative of g with respect to x in direction h, and consequently in view of (9) we obtain

$$\| I_2\|_{C_\infty} \leq |h| \sup_{|y|>0} \frac{\| f(x-h-y) - f(x-h)\|_{C_\infty}}{\lambda^{\mu+1}(y)} \int_{|y|<1} \lambda^{\mu+1}(y) |g'_h(x_0, y)| dy +$$

$$+ 2|h| \| f\|_{C_\infty} \int_{|y|\geq 1} |g'_h(x_0, y)| dy \leq |h| \| f\|_{\Lambda_{\lambda^{\mu+1}(t)}}.$$

Therefore for I_2 we obtain

$$\frac{\| I_2\|_{C_\infty}}{\lambda^\mu(h)} \leq \frac{|h|}{\lambda^\mu(h)} \| f\|_{\Lambda_{\lambda^{\mu+1}(t)}} = o(|h|) \| f\|_{\Lambda_{\lambda^{\mu+1}(t)}} \quad \text{as} \quad |h| \to 0.$$

For $\dfrac{\| I_1\|_{C_\infty}}{\lambda^\mu(h)}$ we have using (9) and (10)

$$\frac{\| I_1\|_{C_\infty}}{\lambda^\mu(h)} =$$

$$= \frac{1}{\lambda^\mu(h)} \left\| \left(\int_{|y|<|h|} \{ (f(x-h-y) - f(x-h)) - (f(x-y) - f(x))\} g(x, y) \mu(dy) + \right.\right.$$

$$+ \int\limits_{|y| \geq |h|} \{ (f(x-h-y) - f(x-h)) - (f(x-y) - f(x)) \} g(x, y) \mu(dy) \Bigg) \Bigg\|_{C_\infty} \leq$$

$$\leq \sup_{|y|>0} \frac{\| (f(x-h-y) - f(x-h)) - (f(x-y) - f(x)) \|_{C_\infty}}{\lambda^{\mu+1}(y)} \times$$

$$\times \frac{1}{\lambda^{\mu}(h)} \int\limits_0^{|h|} \lambda^{\mu+1}(y) |g(x, y)| dy \; +$$

$$+ \sup_{0<|h|<1} \frac{\| (f(x-h-y) - f(x-y)) - (f(x-h) - f(x)) \|_{C_\infty}}{\lambda^{\mu+1}(h)} \lambda(h) \int\limits_{|h|}^{\infty} g(x, y) dy \leq$$

$$\leq C \| f \|_{\Lambda_{\lambda^{\mu+1}(t)}} o(|h|) \quad \text{as} \quad |h| \to 0,$$

and thus

$$\sup_{|h| \leq 1} \frac{\| p(x-h, D) f(x-h) - p(x, D) f(x) \|_{C_\infty}}{\lambda^{\mu}(h)} \leq C \| f \|_{\Lambda_{\lambda^{\mu+1}(t)}}. \tag{13}$$

Combining (13) with (12) and (11), we arrive at

$$\| p(x, D) f \|_{\Lambda_{\lambda^{\mu}(t)}} \leq C \| f \|_{\Lambda_{\lambda^{\mu+1}(t)}}.$$

Moreover, we can see from (13) that

$$\lim_{|h| \to 0} \frac{\| p(x-h, D) f(x-h) - p(x, D) f(x) \|_{C_\infty}}{\lambda^{\mu}(h)} = 0,$$

which completes the proof.

5. Continuity of a pseudodifferential operator in some spaces of generalized smoothness. In this section we give the theorem on the continuity of some pseudodifferential operator in the Besov spaces of generalized smoothness.

We start with an auxiliary theorem, see [13] or [1] for the reference.

Theorem 2. *Let* $\left(X_0, \|\cdot\|_{X_0}\right)$ *and* $\left(X_1, \|\cdot\|_{X_1}\right)$ *be two Banach spaces as above, and let* $\left(Y_0, \|\cdot\|_{Y_0}\right)$ *and* $\left(Y_1, \|\cdot\|_{Y_1}\right)$ *be two Banach spaces satisfying the same conditions as* X_0 *and* X_1. *Suppose that* $T: X_0 \to X_1$ *is a bounded linear operator such that* $Af \in Y_k$ *for* $f \in X_k$, *and*

$$\| Af \|_{Y_k} \leq M_k \| f \|_{X_k}, \quad k = 0, 1.$$

Then A *maps continuously* $X_\theta = [X_0, X_1]_\theta$ *into* $Y_\theta = [Y_0, Y_1]_\theta$, *and we have the estimate*:

$$\| Af \|_{Y_\theta} \leq M_0^{1-\theta} M_1^{\theta} \| f \|_{X_\theta}, \quad \theta \in [0, 1].$$

Next we need a theorem which gives the continuity of pseudodifferential operator between the generalized Bessel potential spaces in L_2. For our convenience we quote the necessary conditions.

Let us split the symbol $p(x, D)$ into two parts:

$$p(x, \xi) = p_1(\xi) + p_2(x, \xi), \tag{14}$$

where $p_1: \mathbb{R}^n \to \mathbb{C}$ is a continuous negative definite function, and $p_2: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is continuous.

Assumption 2. We assume that the function $p(x, \xi)$ admits the decomposition (14), where $p_1 : \mathbb{R}^n \to \mathbb{C}$ is some continuous negative definite function, and $p_2 : \mathbb{R}^n \times \times \mathbb{R}^n \to \mathbb{C}$ is continuous, and suppose that the following conditions are satisfies:

C_1. The function p_1 satisfies for some $\gamma_0 > 0$ and $\gamma_1, \gamma_2 \geq 0$:

$$\gamma_0 a(\xi) \leq \operatorname{Re} p_1(\xi) \leq \gamma_1 a(\xi) \quad \text{for all} \quad |\xi| \geq 1$$

and

$$|\operatorname{Im} p_1(\xi)| \leq \gamma_2 \operatorname{Re} p_1(\xi) \quad \text{for all} \quad \xi \in \mathbb{R}^n.$$

C_2. For $m \in \mathbb{N}_0$ the function $x \mapsto p_2(x, \xi)$ belongs to C^m, and the estimate

$$\left|\partial_x^\alpha q_2(x, \xi)\right| \leq \varphi_\alpha(x)(1 + a(\xi))$$

holds for all $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq m$, with $\varphi_\alpha \in L_1$.

(See Assumption 2.3.5 from [2] for the reference.)

For the following theorem we refer to Proposition 2.3.6 and Theorem 2.3.11 from [2].

Theorem 3. *Let the conditions* C_1 *and* C_2 *with* $m \geq n + [t] + 1$ *of Assumption* 2 *hold for the symbol* $p(x, \xi)$ *of the pseudodifferential operator* $p(x, D)$. *Then* $p(x, D)$ *is continuous from* $H^{a,t+2}$ *to* $H^{a,t}$ *for any* $t \geq 0$.

Theorem 4. *Let* $p(x, D)$ *be as in Theorem* 1, *and in addition assume that it satisfies the conditions of Theorem* 3, *and for large* $|\xi|$ *it holds*

$$a(\xi) \geq |\xi|^\alpha, \quad 0 < \alpha < 2.$$

Then for $s > p^{-1}\left(\dfrac{2+n}{\alpha} + (p-2)\mu\right)$, $p \geq 2$,

$$p(x, D) : B_{p,p}^{\sigma^{s+1}, N} \to B_{p,p}^{\sigma^s, N}$$

continuously, where $N = (N_j)_{j \geq 0}$, $N_j = \sqrt{(a)^{-1}}(2^{2j})$.

Proof. From Theorem 3 we have

$$p(x, D) : H_2^{a,t+2} \to H_2^{a,t} \tag{15}$$

continuously, in particular, (15) holds for all $t > \dfrac{2+n}{2\alpha}$. For such t the space $H^{a,2}$ is continuously embedded into C_∞.

We know that $B_{\infty\infty}^{\sigma^{1+\mu}, N} = B_{\infty\infty}^{\lambda^{1+\mu}} = \Lambda_{\lambda^{\mu+1}(t)}$, $\mu \geq 0$. By Theorem 1 the operator $p(x, D)$ is continuous from $\mathring{\Lambda}_{\lambda^{\mu+1}(t)} = \mathring{B}_{\infty\infty}^{\sigma^{1+\alpha}, N}$ to $\mathring{\Lambda}_{\lambda^\mu(t)} = \mathring{B}_{\infty\infty}^{\sigma^\alpha, N}$, i.e.,

$$\| p(x, D) f \|_{\mathring{B}_{\infty\infty}^{\sigma^\alpha, N}} \leq c \| f \|_{\mathring{B}_{\infty\infty}^{\sigma^{1+\alpha}, N}}.$$

Further, for $t > \dfrac{2+n}{2\alpha}$, we have

$$B_{22}^{\sigma^{t+1}, N} \subset \Lambda_1 \subset \mathring{B}_{\infty\infty}^{\sigma^{1+\alpha}, N},$$

$$B_{22}^{\sigma^t, N} \subset \Lambda_1 \subset \mathring{B}_{\infty\infty}^{\sigma^\alpha, N},$$

and these embeddings are dense.

Since the norms in the interpolation spaces $\left[B_{22}^{\sigma^t,N}, B_{\infty\infty}^{\sigma^s,N}\right]_\theta$ and $\left[B_{22}^{\sigma^t,N}, \mathring{B}_{\infty\infty}^{\sigma^s,N}\right]_\theta$ coincide for $t, s > 0$, $0 < \theta < 1$, we obtain by Theorem 2 that $p(x, D)$ is continuous from $B_{p,p}^{\sigma^{s+1},N}$ to $B_{p,p}^{\sigma^s,N}$ for $p = \dfrac{2}{1-\theta}$ and $s = (1-\theta)t + \theta\mu = \dfrac{2t + (p-2)\mu}{p}$. Since $t > \dfrac{2+n}{2\alpha}$ then $s > p^{-1}\left(\dfrac{2+n}{\alpha} + (p-2)\mu\right) = \tilde{s}$, and since s depends on t linearly then for any $s_0 > \tilde{s}$ there exists $t_0 > \dfrac{2+n}{2\alpha}$ such that $s_0 = \dfrac{2t_0 + (p-2)\mu}{p}$. Therefore $p(x, D)$ is continuous between $B_{p,p}^{\sigma^{s+1},N}$ and $B_{p,p}^{\sigma^s,N}$ for all $s > p^{-1}\left(\dfrac{2+n}{\alpha} + (p-2)\mu\right)$ and $p \ge 2$.

Theorem 4 is proved.

Theorem 4 together with Lemma 2 give us that under the conditions of Theorem 4

$$p(x, D):\ F_{p,2}^{\sigma^{s+1},N} \to B_{p,p}^{\sigma^s,N} \quad \text{for} \quad s > p^{-1}\left(\frac{2+n}{\alpha} + (p-2)\mu\right) \quad \text{and} \quad p \ge 2$$

continuously, or to make our notation easier we will write

$$p(x, D):\ H_p^{a,s+1} \to B_{p,p}^{\sigma^s,N},$$

where $H_p^{a,s} = F_{p,2}^{\sigma^s,N}$ is an a-Bessel potential space.

Theorem 4 allows us to use such operators $p(x, D)$ as perturbations of some generators of L_p-sub-Markovian semigroups. To do this, we will quote Theorem 2.8.1 from [2], from where our result easily follows.

Theorem 5. *Let* $(-A, D(A))$ *be a pseudodifferential operator which generates a sub-Markovian semigroup in* L_p, $1 < p < \infty$. *If an operator* $p(x, D)$ *is* L_p-*dissipative, A-bounded, i.e.,* $D(p(x, D)) \subset D(A)$, *and for some* $\varepsilon \in [0, 1)$ *and* $\delta > 0$

$$\| p(x, D)u \|_{L_p} \le \varepsilon \| Au \|_{L_p} + \delta \| u \|_{L_p}, \quad u \in D(A),$$

and in addition $(-A - p(x, D), D(A))$ *is an* L_p-*Dirichlet operator, then* $(-A - p(x, D), D(A))$ *is a generator of an* L_p-*sub-Markovian semigroup.*

We arrive at the following theorem:

Theorem 6. *Let* $\left(-\psi(D), H_p^{\psi,2}\right)$ *be the generator of an* L_p-*sub-Markovian semigroup, and let* $p(x, D)$ *satisfy conditions of Theorem* 4. *Assume that for* ψ, *such that*

$$(\tilde{\psi})^{-1}(x)(\lambda)^{-1}\left(\frac{1}{x}\right) = 1, \tag{16}$$

the operator $\left(\tilde{\psi}(D), H^{\tilde{\psi},2}\right)$ *is* $\psi(D)$-*bounded. Then the operator* $(-\psi(D) - p(x, D), H_p^{\psi,2})$ *is the generator of an* L_p-*sub-Markovian semigroup.*

Proof. From (7) we see, that if $\tilde{\psi}$ satisfies (16) and $p(x, D)$ satisfies the conditions of Theorem 4, then $p(x, D)$ is continuous from $H_p^{\tilde{\psi},s+1}$ to $B_{p,p}^{\sigma^s,\tilde{N}}$, where $\tilde{N} = \left(\tilde{N}_j\right)_{j\ge 0}$, $\tilde{N}_j = \sup\left\{|\xi| : \tilde{\psi}(\xi) \le 2^{2j}\right\}$.

Since our operator is a Dirichlet operator (as an operator with continuous negative define symbol), and therefore it is dissipative, the statement of the theorem follows from Theorem 5.

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1. *Jacob N.* Pseudodifferential operators and Markov processes. Vol. 1. Fourier analysis and semigroups. – London: Imper. Coll. Press, 2001. – 493 p.
2. *Jacob N.* Pseudodifferential operators and Markov processes. Vol. 2. Generators and their potential theory.– London: Imper. Coll. Press, 2002. – 453 p.
3. *Hoh W.* A symbolic calculus for pseudodifferential operators generating Feller semigroups // Osaka J. Math. – 1998. – **35**. – P. 798–820.
4. *Hoh W.* Pseudodifferential operators generating Markov processes. – Bielefeld: Habilitationsschrift, 1998.
5. *Farkas W.* Function spaces of generalized smoothness and pseudodifferential operators associated to a continuous negative definite function. – Munich: Habilitationsschrift, 2003. – 150 p.
6. *Farkas W., Leopold H.-G.* Characterization of function spaces of generalized smoothness // Ann. mat. pures et appl. – 2004.
7. *Kaljabin G. A.* Description of the traces for anisotropic Triebel – Lizorkin type spaces // Tr. Mat. Inst. Akad. Nauk SSSR. – 1979. – **150**. – P. 160–173.
8. *Kaljabin G. A.* Theorems on extension, multiplicators and diffeomorphisms for generalized Sobolev – Liouville classes on domains with Lipschitz boundary // Tr. Mat. Inst. Akad. Nauk SSSR. – 1985. – **172**. – P. 173–186.
9. *Farkas W., Jacob N., Schilling R.* Function spaces related to continuous negative definite functions: ψ-Bessel potential spaces // Diss. Math. – 2001. – **393**. – P. 1–62.
10. *Triebel H.* Interpolation theory, functional spaces, differential operators. – Amsterdam: North Holland Publ. Co., 1978. – 207 p.
11. *Kufner A., John O., Fucik S.* Function spaces. – Leyden: Noordhoff Int. Publ., 1977. – 454 p.
12. *Triebel H.* Theory of function spaces // Monogr. Math. – 1983. – **78**. – 284 p.
13. *Krein S., Petunin Yu., Semenov E.* Interpolation of linear operators. – Moscow: Nauka, 1978. – 400 p.
14. *Kaljabin G. A., Lizorkin P. I.* Spaces of functions of generalized smoothness // Math. Nachr. – 1987. – **133**. – P. 7–32.

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