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SINGULARLY PERTURBED LINEAR BOUNDARY-VALUE PROBLEMS WITH IMPULSE EFFECTS AND REGULAR REDUCED PROBLEM

СИНГУЛЯРНО ЗБУРЕНІ ЛІНІЙНІ КРАЙОВІ ЗАДАЧІ З ІМПУЛЬСНОЮ ДІЄЮ ТА РЕГУЛЯРНОЮ ПОРОДЖУЮЧОЮ ЗАДАЧЕЮ

A singularly perturbed linear boundary-value problem with an impulse effect is considered. By using pseudo-inverse matrices, an asymptotic solution with a single boundary layer is constructed.

Розглядається сингулярно збурена лінійна крайова задача з імпульсною дією. За допомогою псевдообернених матриць побудовано асимптотичний розв'язок з одним примежовим шаром.

1. Introduction. We consider the singular differential system with impulse effects in the points

$$\varepsilon \dot{x} = Ax + \varepsilon A_1(t)x + f(t), \quad t \in [a,b], \quad t \neq \tau_i, \quad i = \overline{1,p},$$
$$\Delta x|_{t=\tau_i} = S_i x + a_i, \quad \tau_i \in (a,b), \quad i \in \mathbb{Z}, \tag{1}$$

and boundary conditions

$$lx(\cdot) = h. \tag{2}$$

We shall assume that A is a constant $(n \times n)$-matrix, have negative eigenvalues and $\det A \neq 0$; $A_1(t)$ is an $(n \times n)$-matrix with elements continuous in $[a,b]$; ε is a small positive parameter; $f(t)$ is a first-order discontinuous n-vector-function for $t = \tau_i$ and an infinitely differentiable function elsewhere, i.e.,

$$f(t) \in C^\infty([a,b]/\{\tau_i\}), \quad i = \overline{1,p}, \quad a < \tau_1 < \ldots < \tau_p < b.$$

In the impulse equations S_i are $(n \times n)$-matrices and $\det(E + S_i) \neq 0$, and $a_i \in \mathbb{R}^n$.

We consider the problem of finding a first-order discontinuous n-vector-function $x(t)$, which is a solution of (1) and satisfies the boundary conditions (2), where $l = \operatorname{col}(l_1, \ldots, l_m)$ is a linear bounded m-dimensional functional, $h = \operatorname{col}(h_1, \ldots, h_m) \in \mathbb{R}^m$.

The reduced problem is

$$Ax^0 + f(t) = 0, \quad t \neq \tau_i, \tag{3}$$

$$\Delta x^0_{|t=\tau_i} = S_i x^0 + a_i, \tag{4}$$

$$lx^0(\cdot) = h. \tag{5}$$

Since $\det A \neq 0$, we obtain by (3)

$$x^0(t) = A^{-1} f(t)$$

and we are not sure that the impulse equation (4) and condition (5) are satisfied. This is why it may happen that the boundary-value problem (1), (2) will not have a solution for an arbitrary function $f(t) \in C^\infty([a,b]/\{\tau_i\})$ and every $h \in \mathbb{R}^m$. Further, we find the conditions of existence and uniqueness of a solution of (1), (2) on the basis generalized inverse matrix [1, 2]. Exactly, by introducing boundary functions in the point $t = a$ [3, 4], we find a solution of the original problem and obtain the uniformly valid asymptotic expansions.

We shall use nonsingular boundary-value problems (see, e.g., [5–7]). The problem (1), (2) without impulse effects is discussed in [8].

2. The main result. We seek a solution of (1), (2) in the form

$$x(t, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i [x^i(t) + \Pi_i x^i(\tau)], \tag{6}$$

where $\tau = (t-a)/\varepsilon$. By $\Pi_i x^i(\tau)$ (see [3, 4]), we denote the boundary functions in a neighborhood of the point $t = a$. With τ_a, τ_b and $\overline{\tau}_i$, $i = \overline{1, p}$, we show the values of τ at $t = a$, $t = b$ and $t = \tau_i$ respectively, i.e., $\tau_a = 0$, $\tau_b = (b-a)/\varepsilon$, and $\overline{\tau}_i = (\tau_i - a)/\varepsilon$.

We denote by $X(\tau)$, $X(\tau_a) = E_n$ (E_n is an $(n \times n)$-unit matrix) a fundamental matrix of solutions of the linear system

$$\frac{d}{d\tau} x(\tau) = A x(\tau), \quad \tau \neq \overline{\tau}_i, \quad \Delta x_{|\tau = \overline{\tau}_i} = S_i x, \quad \det(E + S_i) \neq 0.$$

We introduce the $(m \times n)$-matrix $lX(\cdot) = Q$. We assume that $\operatorname{rank} Q = n_1 < \min(m, n)$. We denote by P_Q and P_{Q^*} the matrix projectors

$$P_Q : \mathbb{R}^n \to \ker Q, \quad P_Q^2 = P_Q, \quad P_Q \text{ is an } (n \times n)\text{-matrix},$$

$$P_{Q^*} : \mathbb{R}^m \to \ker Q^*, \quad P_{Q^*}^2 = P_{Q^*}, \quad Q^* = Q^T, \quad P_{Q^*} \text{ is an } (m \times m)\text{-matrix}.$$

By Q^+, we denote the $(n \times m)$ unique Mur–Penrose inverse matrix of the matrix Q [1, 2]. Let $P_{Q_d^*}$ be a $(d \times m)$-matrix with $d = m - n_1$ linear independent rows of the matrix P_{Q^*}, and let P_{Q_r} be $r = n - n_1$ linear independent columns of the matrix P_Q.

By substituting (6) in (1), (2) and equating the coefficients of terms having the same order with respect to ε, we get the equations for determining the terms of decomposition (6). For $x^i(t)$, we obtain the recursion expressions

$$x^0(t) = -A^{-1} f(t),$$

$$x^k(t) = -A^{-1}[\dot{x}^{k-1}(t) - A_1(t) x^{k-1}(t)], \; t \neq \tau_i, \; t \in [a, b], \; k = 1, 2, 3, \ldots . \tag{7}$$

Every boundary function is a general solution of a linear boundary-value problem. For $\Pi_0 x(\tau)$, it is the system

$$\frac{d}{d\tau} \Pi_0 x(\tau) = A \Pi_0 x(\tau), \quad \tau \neq \overline{\tau}_i, \quad \tau \in [\tau_a, \tau_b],$$

$$\Delta \Pi_0 x_{|\tau = \overline{\tau}_i} = S_i \Pi_0 x + b_i^0, \tag{8}$$

$$l \Pi_0 x(\cdot) = h - l x^0(\cdot),$$

where

$$b_i^0 = S_i x^0 + a_i - \Delta x^0_{|t = \tau_i}. \tag{9}$$

The other boundary functions, $\Pi_k x(\tau)$, $k = 1, 2, 3, \ldots$, describe the systems

$$\frac{d}{d\tau} \Pi_k x(\tau) = A \Pi_k x(\tau) + g_k(\tau),$$

$$\Delta \Pi_k x_{|\tau = \overline{\tau}_i} = S_i \Pi_k x + b_i^k, \tag{10}$$

$$l\Pi_k x(\cdot) = -lx^k(\cdot),$$

where

$$g_k(\tau) = \sum_{s=1}^{k} \frac{1}{(s-1)!} A_1^{(s-1)}(a)\tau^{s-1}\Pi_{k-s}x(\tau),$$

$$b_i^k = S_i x^k - \Delta x^k\big|_{t=\overline{\tau}_i}, \quad k = 1, 2, 3, \ldots . \tag{11}$$

System (8) possesses a family of solutions

$$\Pi_0 x(\tau) = X(\tau)c_0 + \sum_{i=1}^{p} X(\tau)X^{-1}(\overline{\tau}_i)(E+S_i)^{-1} b_i^0, \quad c_0 \in \mathbb{R}^n. \tag{12}$$

We substitute $\Pi_0 x(\tau)$ in boundary conditions $l\Pi_0 x(\cdot)$. The vector c_0 is obtained by means of the algebraic system

$$Q c_0 = h_0, \tag{13}$$

where

$$h_0 = h - lx^0(\cdot) - l\sum_{i=1}^{p} X(\cdot)X^{-1}(\overline{\tau}_i - 0)(E+S_i)^{-1} b_i^0. .$$

The general solution of (8) is obtained by using (13) and (12)

$$\Pi_0 x(\tau) = X_r(\tau)\, c_r^0 + X(\tau)Q^+ h_0 +$$

$$+ \sum_{i=1}^{p} X(\tau)X^{-1}(\overline{\tau}_i - 0)(E+S_i)^{-1} b_i^0, \quad c_r^0 \in \mathbb{R}^n, \tag{14}$$

if and only if

$$P_{Q_d^*} h_0 = 0. \tag{15}$$

We introduce in (14) the $(n\times r)$-matrix $X_r(\tau) = X(\tau)P_{Q_r}$.

Different vectors $c_r^k \in \mathbb{R}^r$ are to determine thé rest of the boundary functions $\Pi_k x(\tau)$, $k = 1, 2, 3, \ldots$.

From (11), it is known that the functions $g_k(\tau)$ depend on already determined boundary functions. Consequently, g_k depend also on the vectors $c_r^0, \ldots, c_r^{k-1}$,. which are to be determined.

We introduce the notation

$$h_k = -lx^k(\cdot) - l\left[\int_{\tau_a}^{\tau_b} K(\cdot, s)g_k(s)ds + \sum_{i=1}^{p} \overline{K}(\cdot, \overline{\tau}_i)b_i^k\right], \; k = 1, 2, 3, \ldots , \tag{16}$$

where $K(\tau, s)$ denotes the $(n \times n)$-matrix

$$K(\tau, s) = \begin{cases} -X(\tau)X^{-1}(s), & \tau_a \le \tau \le s \le \tau_b; \\ 0, & \tau_a \le s < \tau \le \tau_b, \end{cases}$$

$$\overline{K}(\tau, \overline{\tau}_i) = K(\tau, \overline{\tau}_i - 0)(E+S_i)^{-1}.$$

By using equalities (10), (11), (14), and the systems $Q c_k = h_k$, we get the form of the boundary functions

$$\Pi_k x(\tau) = X_r(\tau)\, c_r^k + X(\tau)\, Q^+ h_k + \int_{\tau_a}^{\tau_b} K(\tau, s)g_k(s)ds + \sum_{i=1}^{p} \overline{K}(\tau, \overline{\tau}_i)b_i^k, \tag{17}$$

where h_k and b_i^k are the vectors from (16) and (11), respectively. The boundary functions (17) are obtained from the following necessary and sufficient conditions:

$$P_{Q_d^*} h_k = 0, \quad k = 1, 2, 3, \ldots . \tag{18}$$

By induction, we obtain from equalities (11), (14), and (17) the following assertion:

Lemma. *The functions* $g_k\left(\tau, c_r^0, c_r^1, \ldots, c_r^{k-1}\right)$ *have the form*

$$g_k\left(\tau, c_r^0, c_r^1, \ldots, c_r^{k-1}\right) = \sum_{j=0}^{k} L_{k,j}(\tau) c_r^{k-1-j}, \quad c_r^{-1} = 0, \quad \tau \in \left(\overline{\tau}_i, \overline{\tau}_{i+1}\right],$$

where

$$L_{k,0}(\tau) = A_1(a) X(\tau) P_{Q_r} = A_1(a) X_r(\tau), \quad k = 1, 2, 3, \ldots ,$$

$$L_{k,q}(\tau) = \sum_{s=0}^{q-1} \frac{1}{s!} A_1^{(s)}(a) \tau^s \left\{ \int_{\tau_a}^{\tau_b} K(\tau, s) L_{k-1-s, q-1-s}(s) ds - \right.$$

$$\left. - X(\tau) Q^+ l \int_{\tau_a}^{\tau_b} K(\cdot, s) L_{k-1-s, q-1-s}(s) ds \right\} + \frac{1}{q!} A_1^q(a) \tau^q X_r(\tau),$$

$$k = 1, 2, 3, \ldots, \quad q = \overline{1, k}, \tag{19}$$

$$L_{k,k}(\tau) = \sum_{s=0}^{k-1} \frac{1}{s!} A_1^{(s)}(a) \tau^s \left\{ -X(\tau) Q^+ l x^{k-1-s}(\cdot) - \right.$$

$$\left. - X(\tau) Q^+ l \sum_{i=1}^{p} \overline{K}(\cdot, \overline{\tau}_i) b_i^{k-1-s} + \sum_{i=1}^{p} \overline{K}(\tau, \overline{\tau}_i) b_i^{k-1-s} \right\} -$$

$$- \sum_{s=0}^{k-2} \frac{1}{s!} A_1^{(s)}(a) \tau^s \left\{ X(\tau) Q^+ l \int_{\tau_a}^{\tau_b} K(\cdot, s) L_{k-1-s, k-1-s}(s) ds + \right.$$

$$\left. + \int_{\tau_a}^{\tau_b} K(\tau, s) L_{k-1-s, k-1-s}(s) ds \right\} + \frac{1}{(k-1)!} A_1^{(k-1)}(a) \tau^{k-1} X(\tau) Q^+ h, \ k = 1, 2, 3, \ldots .$$

The arbitrary $(r \times 1)$-vectors c_r^i, $i = 0, 1, 2, \ldots$, are obtained from conditions (18). We denote by D the $(d \times r)$-matrix

$$D = P_{Q_d^*} l \int_{\tau_a}^{\tau_b} K(\cdot, s) L_{k,0}(s) ds.$$

Let $P_{D^*}: \mathbb{R}^d \to \ker D^*$, $D^* = D^T$ be a matrix projector and let D^+ be the unique Mur–Person inverse matrix of the matrix D. Then the necessary and sufficient condition for determining in a unique way the vectors c_r^i is $P_{D^*} P_{Q_d^*} = 0$.

If $\operatorname{rank} D = r = n - n_1$, c_r^i have the form

$$c_r^{k-1} = D^+ P_{Q_d^*} b_{k-1}\left(c_r^0, c_r^1, \ldots, c_r^{k-2}\right), \quad k = 1, 2, 3, \ldots . \tag{20}$$

In view of (19) and (18), the expressions for b_{k-1} take the form

$$b_{k-1}\left(c_r^0, c_r^1, \ldots, c_r^{k-2}\right) =$$

$$= -l\left[\int_{\tau_a}^{\tau_b} K(\cdot, s)\sum_{j=1}^{k} L_{k,j}(s)ds\, c_r^{k-1-j} + \sum_{i=1}^{p}\overline{K}(\cdot,\overline{\tau}_i)b_i^k + x^k(\cdot)\right], \quad k = 1, 2, 3, \ldots . \quad (21)$$

By substituting the vectors c_r^k from (20) in (14), (17) and taking (16) into account, we finally obtain the form of the boundary functions $\Pi_k x(\tau)$, $\tau \in (\tau_i, \tau_{i+1}]$, $k = 0, 1, 2, \ldots$. We assume that the matrices A and $S_i = S$ are commutative. Then it is proved that $\lim\limits_{\tau\to\infty} \Pi_k x(\tau) = 0$.

Thus, the following theorem is proved.

Theorem. *Let the following conditions be satisfied:*

1) $\mathrm{Re}\,\lambda_i < 0$, *where* λ_i *are eigenvalues of* A;

2) A *and* $S_i = S$ *are commutative;*

3) $\mathrm{rank}\, Q = n_1 < \min(m, n)$ *and* $\mathrm{rank}\, D = r$, $r = n - n_1$.

Then the necessary and sufficient condition for the boundary-value problem (1), (2) *to have a unique solution presented as an asymptotic series of the form* (6) *for any* $f(t) \in C^\infty([a, b]/\{\tau_i\})$ *and any* $h \in \mathbb{R}^m$ *and satisfying condition* (15), *is*

$$P_{D*}P_{Q_d^*} = 0.$$

The coefficients $x^k(t)$ *of the decomposition have the form* (7) *and the boundary functions* $\Pi_k x(\tau)$ *have the representation*

$$\Pi_k x(\tau) = X_r(\tau)D^+P_{Q_d^*}b_k + \left(G\begin{bmatrix} g_k \\ b_i^k \end{bmatrix}\right)(\tau) - X(\tau)Q^+lx^k(\cdot), \quad k = 1, 2, 3, \ldots ,$$

$$\Pi_0 x(\tau) = X_r(\tau)D^+P_{Q_d^*}b_0 + \left(G\begin{bmatrix} g_0 \\ b_i^0 \end{bmatrix}\right)(\tau) + X(\tau)Q^+(h - lx^0(\cdot)),$$

where

$$\left(G\begin{bmatrix} g_k \\ b_i^k \end{bmatrix}\right)(\tau) \overset{def}{=} \left[\int_{\tau_a}^{\tau_b} K(\tau, s)*ds - X(\tau)Q^+l\int_{\tau_a}^{\tau_b} K(\cdot, s)*ds,\right.$$

$$\left.\sum_{i=1}^{p}\overline{K}(\tau,\overline{\tau}_i)* - X(\tau)Q^+l\sum_{i=1}^{p}\overline{K}(\cdot,\overline{\tau}_i)*\right]\begin{bmatrix} g_k(\tau) \\ b_i^k \end{bmatrix};$$

besides, $g_0(\tau) \equiv 0$, b_i^k *have the form* (9), (11), *and* b_k *has the form* (21).

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