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DESCRIPTIVE CLASSES OF SETS AND TOPOLOGICAL FUNCTORS

ДЕСКРИПТИВНІ КЛАСИ МНОЖИН І ТОПОЛОГІЧНІ ФУНКТОРИ

It is proved that the image of a normal functor from the Stone – Cech compactification of the projective class of sets also belongs to this class.

Доведено, що образ нормального функтора від стоун-чехівської компактифікації з класу проективних множин також належить цьому класу.

In this article, we discuss the relationship between descriptive set theory and the theory of topological functors.

Let $\sigma - \mathcal{M}_1 = \{X = \bigcup_{n=1}^{\infty} X_n \mid X_n,\ n \in \mathbb{N},$ are complete-metrizable closed sets in $X\}$. By $\mathcal{A}_\alpha$ and $\mathcal{M}_\alpha$, $1 \le \alpha < \omega_1$, we denote the additive and multiplicative classes of absolute Borelian sets corresponding to a countable ordinal α, respectively; and by $\mathcal{P}_n$, $n > 0$, we denote the classes of projective sets [1, 2]. Recall that $\mathcal{P}_0$ is the class of analytic sets (i.e., the class of continuous images of absolute Borelian sets) and $\mathcal{P}_2$ is the class of coanalytic sets (i.e., the class consisting of complements to analytic sets).

Information concerning the theory of functors in the category of compacta Comp can be found in [3–6].

Let F: Comp $\to$ Comp be a monomorphic functor. Recall that for a compactum X and $A \in FX$, $\operatorname{supp}(a) = \bigcap \{A \subset X \mid a \in FA\}$. We denote by F_β: Tych $\to$ Tych the extension of the functor F onto the category Tych of Tychonoff spaces offered by A. Chigogidze [7]: for a space $X \in$ Tych, let $F_\beta X = \{a \in F(\beta x) \mid \operatorname{supp}(a) \subset X\}$, where βX is the Stone – Cech compactification of X. It is known [7] that the topology of the space $F_\beta X$ does not depend on a particular compactification of X, i.e., the space

$$F_\gamma X = \{a \in F(\gamma X) \mid \operatorname{supp}(a) \subset X\} \subset F(\gamma X)$$

is homeomorphic to the space $F_\beta X$ for any compactification γX of X. So, sometimes, we shall denote the space $F_\beta(X)$ by FX.

Assumption. *From now on, all spaces are metrizable and separable and* F: Comp $\to$ Comp *is a monomorphic continuous functor preserving weight, intersections, point, and the empty set.*

Recall that exp: Comp $\to$ Comp is the hyperspace functor. Then, for a metric space X, $\exp_\beta X = \exp X$ is the space of compact subsets of X endowed with the Vietoris topology. By $O(x, \varepsilon)$, we denote an open ε-ball with the center $x \in X$.

The article was initiated by the following problem due to V. V. Fedorchuk.

***Problem* 3.23** [8]. *Is the subset* $P(\mathbb{Q}) \subset P[0, 1]$ *Borelian? Analytic? (Here,* $\mathbb{Q}$ *is the set of all rationales in* $[0, 1]$ *and* P *is the probability measure functor).*

We answer this problem negatively. Namely, we show that the space $P(\mathbb{Q})$ is coanalytic and nonanalytic. In fact, we prove more general results.

Theorem 1. *For* $n \ge 1$, $X \in \mathcal{P}_{2n}$ *if and only if* $FX \in \mathcal{P}_{2n}$.

Proof. If $FX \in \mathcal{P}_{2n}$, then, since X is a closed subset in FX ([3], Proposition 2.8), $X \in \mathcal{P}_{2n}$ as well.

Assume now $X \in \mathcal{P}_{2n}$. Let us consider the complement $F(\overline{X})\setminus FX$, where $\overline{X}$ is any metric compactification of X. (Recall that $FX = \{a \in F(\overline{X}) \mid \operatorname{supp}(a) \subset X\} \subset \subset F(\overline{X})$). Then

$$F(\overline{X})\setminus FX = \{a \in F(\overline{X}) \mid \operatorname{supp}(a) \cap (\overline{X}\setminus X) \neq \varnothing\} =$$
$$= \{a \in F(\overline{X}) \mid \exists\, x \in \overline{X}\setminus X \ \ \forall\, n \in \mathbb{N} \ \ O(x, 1/n) \cap \operatorname{supp}(a) \neq \varnothing\} = \operatorname{pr}_1(E),$$

where $\operatorname{pr}_1 : F(x) \times X \to F(x)$ is the projection and $E = \{(a, x) \in F(\overline{X}) \times \overline{X} \mid x \notin X$ and $\forall\, n \in \mathbb{N} \quad O(x, 1/n) \cap \operatorname{supp}(a) \neq \varnothing\}$. Note that $E = F(\overline{X}) \times (\overline{X}\setminus X) \cap \cap (\bigcap_{n=1}^{\infty} E_n)$, where

$$E_n = \{(a, x) \in F(\overline{X}) \times \overline{X} \mid O(x, 1/n) \cap \operatorname{supp}(a) \neq \varnothing\}.$$

Since the map $\operatorname{supp} : F(\overline{X}) \to \exp(\overline{X})$ is lower semi-continuous [4], the set $E_n \subset \subset F(\overline{X}) \times \overline{X}$ is open for every $n \in \mathbb{N}$. Since $X \in \mathcal{P}_{2n}$, by definition, $\overline{X}\setminus X \in \in \mathcal{P}_{2n-1}$ and $F(\overline{X})\setminus FX = \operatorname{pr}_1(E) \in \mathcal{P}_{2n-1}$. Consequently, $FX \in \mathcal{P}_{2n}$ and the theorem is proved.

Let us note that Theorem 1, in the case where $F = \exp$ and $n = 1$, has been known (see Theorem 2 [2], § 42, III).

Theorem 2. *Let F be a functor with a finite support and let $\mathcal{D}$ be any of the classes $\sigma - \mathcal{M}_1$, $\mathcal{A}_\alpha$, $\mathcal{M}_\alpha$, $1 \le \alpha < \omega_1$, $\mathcal{P}_n$, $n \ge 0$. Then $X \in \mathcal{D}$ if and only if $FX \in \mathcal{D}$.*

Proof. Sufficiency follows from the fact that X is a closed subset in FX [3]. It is easily seen that a functor with finite support preserves countable unions and intersections of monotone sequences of sets. Therefore, to prove that F preserves the Borelian classes $\sigma - \mathcal{M}_1$, $\mathcal{A}_\alpha$, and $\mathcal{M}_\alpha$, $1 \le \alpha < \omega_1$, it suffices to prove that $FX \in \mathcal{M}_1$ provided that $X \in \mathcal{M}_1$. This was proved by M. Zarichnyi in [9].

By Theorem 1, $X \in \mathcal{P}_{2n}$ if and only if $FX \in \mathcal{P}_{2n}$, $n \ge 1$. Besides, since F preserves the Borelian sets, F preserves the class $\mathcal{P}_0$. To prove that F preserves the classes $\mathcal{P}_n$ for even n, note that the functor F, being a functor with finite support, preserves subjective maps. Consequently, F preserves the classes $\mathcal{P}_n$ for all $n \ge 0$. Theorem 2 is proved.

On the other hand, if F is not a functor with finite support, the situation is quite different.

The main tool in our further considerations is the following lemma:

Lemma 1. *For a metric compactum X, the map $\operatorname{supp}: FX \to \exp X$ is measurable of the class 1 (i.e., the preimage $\operatorname{supp}^{-1}(X) \subset FX$ of any open set $U \subset \exp X$ is an F_σ-subset in FX)*

The proof follows from Theorem 1 ([2], § 43, VII) stating that every lower semi-continuous map $\Phi : Y \to \exp X$ is measurable of the class 1, and from [4], where it is proved that the map $\operatorname{supp} : FX \to \exp X$ is lower semi-continuous.

Proposition 1. *If the space X is locally compact, then $FX \in \mathcal{A}_1$. If $X \in \mathcal{M}_1$, then $FX \in \mathcal{M}_2$. If F is a functor with continuous support, then $X \in \mathcal{M}_1$ if and only if $FX \in \mathcal{M}_1$.*

Proof. The first and the second statements are rather trivial. The third one follows from Lemma 1 and Theorem 1 ([2], § 42, II) stating that $X \in \mathcal{M}_1$ if and only if $\exp(X) \in \mathcal{M}_1$.

Theorem 3. *Assume that* $\operatorname{supp}(F(\mathbb{Q})) = \exp(\mathbb{Q})$. *Then* $X \notin \mathcal{M}_1$ *if and only if* FX *is not Borelian. Moreover,* X *is a coanalytic non-complete-metrizable set (i.e.* $X \in \mathcal{P}_2 \setminus \mathcal{M}_1$*) if and only if* $FX \in \mathcal{P}_2 \setminus \mathcal{P}_1$.

Proof. Let us show first that if $X \in \mathcal{P}_2 \setminus \mathcal{M}_1$, then $FX \in \mathcal{P}_2 \setminus \mathcal{P}_1$. By [10, 11], there exists a closed subset $Y \subset X$ homeomorphic to the space of rationales $\mathbb{Q}$. Then FY is closed in FX. The set FY is not analytic. Indeed, by assumption, $\operatorname{supp}(FY) = \exp Y$. Since the space $\exp(Y) \cong \exp(\mathbb{Q})$ is not analytic (see Corollary 3 ([2], § 42, II) and the map $\operatorname{supp}: FY \to \exp Y$ is measurable, by Remark ([2], § 38, III), the space FY is not analytic as well. Therefore, the set $FX \in \mathcal{P}_2$ consisting of non-analytic closed set FY is not analytic.

If $X \notin \mathcal{M}_1$, then $FX \notin \mathcal{P}_0$. Indeed, assume, on the contrary, that $FX \in \mathcal{P}_0$. Then X is Borelian and, consequently, $FX \in \mathcal{P}_2 \setminus \mathcal{P}_1$. This contradicts to $FX \in \mathcal{P}_0$. The proof is complete.

Note that, since $\mathcal{P}_{2n+1} \subset \mathcal{P}_{2n+4}$ for $n \geq 0$, we have $X \in \mathcal{P}_{2n+1}$, $n \geq 0$, implies $X \in \mathcal{P}_{2n+4}$. On the other hand, if F is a functor with finite support, then $X \in \mathcal{P}_n$ if and only if $FX \in \mathcal{P}_n$ for all $n \geq 0$.

Question. Do, for every normal functor F and $X \in \mathcal{P}_n$, $n \geq 2$, we have $FX \in \mathcal{P}_n$?

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