ON PROPER ACCRETIVE EXTENSIONS OF POSITIVE LINEAR RELATIONS

PRO ВЛАСНІ АКРЕТИВНІ РОЗШИРЕНИЯ ДОДАТНИХ ЛІНІЙНИХ ВІДНОШЕНЬ

A linear relation $\tilde{S}$ is called a proper extension of a symmetric linear relation $S$ if $S \subseteq \tilde{S} \subseteq S^*$. As is well known, an arbitrary dissipative extension of a symmetric linear relation is proper.

In the paper criteria for accretive extension of a given positive symmetric linear relation to be proper are established.

Лінійне відносіння $\tilde{S}$ називається властивим розширенням симетричного лінійного відношення $S$, якщо $S \subseteq \tilde{S} \subseteq S^*$. Як відомо, довільне диссипативне розширення симетричного лінійного відношення є властивим.

Одержані критерії того, що акретивне розширення даного додатного відношення є властивим.

1. Introduction. Let $H$ be a complex Hilbert space and let $H^2 = H \oplus H$ be the set of all pairs $\langle u, u' \rangle$, $u, u' \in H$, with the inner product

$$\langle \langle u, u' \rangle, \langle v, v' \rangle \rangle = (u, v) + (u', v'), \quad \langle u, u' \rangle, \langle v, v' \rangle \in H^2.$$  

As is well known [1], a closed subspace $S \subseteq H^2$ is a linear relation (l.r.) or a multivalued linear operator. If $T$ is a closed linear operator in $H$, then its graph

$$\text{Gr}(T) = \{\langle u, Tu \rangle, u \in \mathcal{D}(T)\}$$

is an l.r.

Basic concepts connected with l.r. can be found in [1]. In particular, $\mathcal{D}(S) = \{u \in H : \langle u, u' \rangle \in S \text{ for some } u' \in H\}$ is the domain of $S$. $S(u) = \{u' \in H : \langle u, u' \rangle \in S\}$, the subspace $S^* = H^2 \ominus JS$, where $J\langle x, x' \rangle = \langle -x', x \rangle$ for all $\langle x, x' \rangle \in H^2$, is called the adjoint of $S$.

A l.r. $S$ will be called

a) symmetric if $S \subseteq S^*$;

b) selfadjoint if $S = S^*$;

c) positive if $(S(u), u) \geq 0$ for all $u \in \mathcal{D}(S)$;

d) dissipative if $\text{Re} (S(u), u) \geq 0$ for all $u \in \mathcal{D}(S)$;

e) accretive if $\text{Re} (S(u), u) \geq 0$ for all $u \in \mathcal{D}(S)$;

f) $\alpha$-sectorial if $S$ is accretive and $|\text{Im} (S(u), u)| \leq \text{tg} \alpha \text{Re} (S(u), u)$ for all $u \in \mathcal{D}(S)$, where $\alpha \in [0, \pi/2)$;

g) $m$-accretive if both $S$ and $S^*$ are accretive;

h) $m - \alpha$-sectorial if $S$ is $m$-accretive and $\alpha$-sectorial.

A l.r. $\tilde{S}$ will be called a proper extension of a symmetric l.r. $S$ if $S \subseteq \tilde{S} \subseteq S^*$.

It is well known [2] that an arbitrary dissipative extension of a symmetric l.r. is proper.

In this paper, criteria for an accretive or $\alpha$-sectorial extension of a positive l.r. to be proper are established.

Assume that $S$ is a positive l.r., the sesquilinear form $(S(u), v)$, $u, v \in \mathcal{D}(S)$, has the closure [1, 3] defined on a certain linear $\mathcal{D}[S] \supseteq \mathcal{D}(S)$. Its values are denoted by $S[u, v]$, $u, v \in \mathcal{D}[S]$, and $S[u] = S[u, u]$.

Let $S_F$ and $S_N$ be the Friedrichs and von Neumann positive selfadjoint extensions of $S$ [1]. For an arbitrary positive selfadjoint extension $\tilde{S}$ of $S$, we have $\mathcal{D}[S] = \mathcal{D}[\tilde{S}]$.
\[ \mathcal{D}[S_F] \subseteq \mathcal{D}[\hat{S}] \subseteq \mathcal{D}[S_N]. \ \hat{S}[u] = S_F[u] = S[u] \text{ for all } u \in \mathcal{D}[S]. \ \text{Thus, } S_N[u] \leq \hat{S}[u] \text{ for all } u \in \mathcal{D}[\hat{S}] \] [4].

Assume that \( \omega[u] \) is a positive functional on the linear space \( \mathcal{D} \) and \( \mathcal{D}_0 \subset \mathcal{D} \). For \( u \in \mathcal{D} \), we set

\[ \langle \omega[u] \rangle_{\mathcal{D}_0} = \inf \{ \omega[u-u_0] : u_0 \in \mathcal{D}_0 \}. \]

If \( \Theta \) is an \( \alpha \)-sectorial l.r., then the quadratic forms

\[ \text{Re}(\Theta(u), u) \quad \text{Re}[(1 \pm i \cot \alpha)(\Theta(u), u)] = \text{Re}(\Theta(u), u) \pm \cot \alpha \text{Im}(\Theta(u), u) \]

are positive on \( \mathcal{D}(\Theta) \).

We will prove the following theorems:

**Theorem 1.** Let \( S \) be a positive l.r. and let \( \hat{S} \) be an accretive extension of \( S \). The following statements are equivalent: 1) \( \hat{S} \subset S^* \); 2) \( \mathcal{D}[\hat{S}] \subseteq \mathcal{D}[S_N] \) and \( \text{Re}(\hat{S}(v), v) \geq S_N[v] \) for all \( v \in \mathcal{D}(\hat{S}) \); 3) \( |(S(u), v)|^2 \leq (S(u), u)\text{Re}(\hat{S}(v), v) \) for all \( u \in \mathcal{D}(S), \ v \in \mathcal{D}(\hat{S}) \).

**Theorem 2.** Let \( S \) be a positive l.r. and let \( \Theta \subset S^* \) be \( m \)-accretive. The following statements are equivalent: 1) \( \Theta \supset S \); 2) \( \mathcal{D}(\Theta) \subseteq \mathcal{D}[S_N] \) and \( \text{Re}(\Theta(v), v) \geq S_N[v] \) for all \( v \in \mathcal{D}(\Theta) \); 3) \( |(S(u), v)|^2 \leq (S(u), u)\text{Re}(\Theta(v), v) \) for all \( u \in \mathcal{D}(S), \ v \in \mathcal{D}(\Theta) \).

**Theorem 3.** Suppose that \( S \) is a positive l.r. and \( \Theta \) is an \( \alpha \)-sectorial extension of \( S \). The following statements are equivalent:

1) \( \Theta \subset S^* \);
2) \( \langle \text{Re}[(1-i\cot \alpha)(\Theta(v), v)] \rangle_{\mathcal{D}(S)} + \langle \text{Re}[(1+i\cot \alpha)(\Theta(v), v)] \rangle_{\mathcal{D}(S)} = \)
   \[ 2\langle \text{Re}(\Theta(v), v) \rangle_{\mathcal{D}(S)} \text{ for all } v \in \mathcal{D}(\Theta) ; \]
3) the sesquilinear form \( \omega[u, v] = (\Theta(u), v) - S_N[u, v] \)

is \( \alpha \)-sectorial on \( \mathcal{D}(\Theta) \).

**2. Preliminaries.** (A) Let \( S \) be a l.r. and let

\[ \mu(S) = \{ (u+u', u-u') : (u, u') \in S \} \]

be a fractional-linear transformation (f.-l.t.).

It possesses the properties \( \mu(\mu(S)) = S, \ \mu(S^*) = (\mu(S))^*, \ \mu(S_1) \subseteq \mu(S_2) \) if \( S_1 \subset S_2 \).

One can easily check that \( S \) is accretive (positive) if and only if \( \mu(S) = \text{Gr}(T) \), where \( T \) is a contraction (Hermitian contraction) and \( S \) is \( m \)-accretive (positive self-adjoint) if and only if \( T \) is defined on \( H \) (selfadjoint contraction).

**B** Assume that \( A \) is an Hermitian contraction defined on the subspace \( \mathcal{D}(A) \subset H \). M. G. Krein in [5] described the set of all selfadjoint contractive (sc) extensions of \( A \) as the operator segment \( [A_{\mu}, A_M] \) where \( A_{\mu} \) and \( A_M \) are the so-called hard and soft sc-extensions of \( A \), i.e., unique sc-extensions possessing the properties: for all \( f \in H \)

\[ \inf \{ \| (I + A_{\mu})(f-f_0) \| : f_0 \in \mathcal{D}(A) \} = 0, \]

\[ \inf \{ \| (I - A_M)(f-f_0) \| : f_0 \in \mathcal{D}(A) \} = 0. \]  

A linear operator \( T \) defined on \( H \) is called a quasiselfadjoint contractive (qsc) extension of a Hermitian contraction \( A \) if

ISSN 0041-6053. Український математичний журнал, 1995, т. 47, № 6
ON PROPER ACCRETIVE EXTENSIONS OF POSITIVE LINEAR RELATIONS

\[ T \supseteq A, \quad T^* \supseteq A, \quad \| T \| \leq 1. \]

In [6, 7], it was obtained that the formula

\[ T = (A_M + A_\mu)/2 + (A_M - A_\mu)^{1/2}X(A_M - A_\mu)^{1/2}/2 \]  

establishes a bijective correspondence between the set of all qsc-extensions of \( A \) and the set of all contractions \( X \) in the space \( \mathcal{H}_0 = (A_M - A_\mu)H \) and, if \( A_\mu = A_M \), then \( A \) has a unique qsc-extension (the symbol \( B^{1/2} \) denotes the positive square root of the positive selfadjoint operator \( B \)).

Let

\[ H = H_0 \otimes \mathcal{D}(A), \quad H_0 = \overline{(I + A_M)H}, \quad \mathcal{M} = \{ \varphi \in H_0 : (I + A_M)^{1/2} \varphi \in \mathcal{H} \} \]

and let \( (I + A_M)H \) be the “shorted operator” [5, 8]. Then, for all \( f \in H \),

\[ ((I + A_M)H f, f) = \inf \{ ((I + A_M)(f - \varphi), f - \varphi), \varphi \in \mathcal{D}(A) \} = \]

\[ = ((I + A_M)^{1/2} P_\mathcal{M} (I + A_M)^{1/2} f, f), \]

where \( P_\mathcal{M} \) is the orthogonal projection onto \( \mathcal{M} \). From (1), \( (I + A_M)H = A_M - A_\mu \).

Consequently, \( (A_M - A_\mu)^{1/2} = UP_\mathcal{M} (I + A_M)^{1/2} \), where \( U \) is the unitary operator from \( \mathcal{M} \) onto \( H_0 \).

Hence, (3) implies the following descriptions of qsc-extensions:

\[ T = A_M + (I + A_M)^{1/2}(Y - I)P_\mathcal{M} (I + A_M)^{1/2}/2, \]

where \( Y \) is an arbitrary contraction in \( \mathcal{M} \).

C) Let \( S \) be a positive lr. Then \( \mu(S) = \text{Gr}(A) \), where \( A \) is an Hermitian contraction, \( \mathcal{D}(A) = (S + I)\mathcal{D}(S) \). In [5, 1], it was established that the following equalities hold:

\[ \mu(S_F) = \text{Gr}(A_\mu), \quad \mu(S_N) = \text{Gr}(A_M). \]

Put \( A_M^0 = A_M |H_0^0 \). \( (I + A_M^0)^{-1/2} \) to be the inverse of \( (I + A_M)^{1/2} \) in \( H_0 \). Since \( S_N = \{(I + A_M^0)f, (I - A_M)H, f \in H \} \), we get, for \( v = (I + A_M)H \),

\[ (S_N(v), v) = ((I - A_M^0)f, (I + A_M)H) = -\| (I + A_M)H f \|^2 + 2 \| (I + A_M^0)^{-1/2} v \|^2 = \]

\[ = -\| v \|^2 + 2 \| (I + A_M^0)^{-1/2} v \|^2. \]

Therefore, \( \mathcal{D}[S_N] = (I + A_M)^{1/2}H = (I + A_M^0)^{1/2}H_0 \) and

\[ S_N[v] = -\| v \|^2 + 2 \| (I + A_M^0)^{-1/2} v \|^2 \]

for all \( v \in \mathcal{D}[S_N] \).

D) Assume that \( \tilde{S} \) is an accretive lr. Then \( \mu(\tilde{S}) = \text{Gr}(\tilde{T}) \), where \( \tilde{T} \) is a contraction, \( \mathcal{D}(\tilde{T}) = (\tilde{S} + I)\mathcal{D}(\tilde{S}) \), and

\[ \tilde{S} = \{ ((I + \tilde{T})f, (I - \tilde{T})f), f \in \mathcal{D}(\tilde{T}) \}. \]

Hence, for \( v = (I + \tilde{T})f, f \in \mathcal{D}(\tilde{T}) \), we have

\[ (\tilde{S}(v), v) = -\| v \|^2 + 2 (f, (I + \tilde{T})f). \]

E) Let \( \Theta \) be a densely defined \( \alpha \)-sectorial operator.
In accordance with [3], the Fredrichs $m - \alpha$-sectorial extension of $\Theta_F$ is the operator associated with the closure of the sesquilinear form $(\Theta u, v)$, $u, v \in \mathcal{D}(\Theta)$, $\mathcal{D}[\Theta] = \mathcal{D}[\Theta_F]$.

If $\Theta$ is an $\alpha$-sectorial l.r., then

$$\Theta = \text{Gr}(\Theta) \oplus \langle 0, \Theta(0) \rangle,$$

where $\Theta$ is an $\alpha$-sectorial closed operator (the operator part of $\Theta$). Put $\tilde{\Theta}_0 = \overline{\mathcal{D}(\Theta)}$, $\pi_0$ to be the orthogonal projection onto $\tilde{\Theta}_0$. $\Theta_0 = \pi_0 \Theta$. Let $\Theta_{0F}$ be the Friedrichs extension of $\Theta_0$ in $\tilde{\Theta}_0$. Put

$$\Theta_F = \text{Gr}(\Theta_{0F}) \oplus \langle 0, \tilde{\Theta}_0^\perp \rangle,$$

where $\tilde{\Theta}_0^\perp = H \ominus \tilde{\Theta}_0$.

Clearly, $\Theta(0) \subseteq \tilde{\Theta}_0^\perp$ and $\Theta_F$ is an $m - \alpha$-sectorial extension of $\Theta$. We will call $\Theta_F$ the Friedrichs extension of $\Theta$. It readily follows from the definition that

$$\mathcal{D}[\Theta] = \mathcal{D}[\Theta_F]$$

$\Theta[u, v] = \Theta_F[u, v] = \Theta_{0F}[u, v]$, $u, v \in \mathcal{D}[\Theta]$.

F) Let $\tilde{S}$ be an $m - \alpha$-sectorial l.r. Then

$$\tilde{S} = \text{Gr}(\tilde{S}) \oplus \langle 0, \tilde{S}(0) \rangle,$$

where $\tilde{S}$ is an $m - \alpha$-sectorial operator in the subspace $\tilde{\Theta} = \overline{\mathcal{D}(\tilde{S})}$.

In accordance with [3], the operator $\tilde{S}$ has the representation

$$\tilde{S} = \tilde{S}^{1/2} \tilde{R} \tilde{G} \tilde{S}^{1/2},$$

where $\tilde{S}_R$ is the positive selfadjoint operator associated with the positive form $b[u, v] = \tilde{G}[u, v] + \overline{\tilde{S}_0[u, v]}$ in $\tilde{\mathcal{D}}(\tilde{S}_R)$. $G = G^*$. $\|G\| \leq \tan \alpha$ is an operator in the subspace $\overline{R(\tilde{S}_R^{1/2})}$, and

$$\mathcal{D}[\tilde{S}] = \mathcal{D}[\tilde{S}] = \mathcal{D} \left( \tilde{S}_R^{1/2} \right).$$

G) Let $S_N$ be the von Neumann extension of the positive l.r. $S$. Passing to the operator part $S_N$, and using the relation established in [4], one can prove that, for all $v \in \mathcal{D}(S_N)$,

$$\sup \{|(S(u), v)|^2 / (S(u), u), u \in \mathcal{D}(S)\} = \|S_N^{1/2} v\|^2,$$

and $\mathcal{D}[S_N] = \mathcal{D}[S_N^{1/2}]$ consists of all vectors $v$, for which the left-hand side of (8) is finite.

H) Let $\tilde{S}$ be an $m - \alpha$-sectorial extension of the positive l.r. $S$ and let $\tilde{S}$ be the operator part of $\tilde{S}$. Using (7) for $v \in \mathcal{D}(S_R^{1/2})$, $u \in \mathcal{D}(S)$, we get

$$(S(u), v) = (\tilde{S}(u), v) = (\tilde{S}_R^{1/2}(I + i\tilde{G}) \tilde{S}_R^{1/2} u, v) = (\tilde{S}_R^{1/2} u, (I - i\tilde{G}) \tilde{S}_R^{1/2} v).$$

Denote by $\tilde{\pi}$ the orthogonal projection onto the subspace $\overline{S_R^{1/2} \mathcal{D}(S)}$. Taking into account the above relation, we obtain, for all $v \in \mathcal{D}(\tilde{S})$,

$$\sup \{|(S(u), v)|^2 / (S(u), u), u \in \mathcal{D}(S)\} = \|\tilde{\pi} (I - i\tilde{G}) \tilde{S}_R^{1/2} v\|^2.$$

Therefore, $\mathcal{D}[\tilde{S}] \subseteq \mathcal{D}[S_N]$ and, from (8), (9),

ISSN 0041-6053. Укр. мат. журн., 1995, т. 47, № 6
\[\| S_N^{1/2} v \|^2 = \| \tilde{\mathcal{N}} (I - iG) S_R^{1/2} v \|^2 \quad \text{for all } v \in \mathcal{D} [\tilde{S}] . \]  

(10)

I) The following lemma will be used in the proof of the Theorem 3:

**Lemma.** Suppose that \( F \) is a selfadjoint contraction in \( H \), \( \mathfrak{C} \) is a subspace in \( H \), and \( P_{\mathfrak{C}} \) is the orthogonal projection onto \( \mathfrak{C} \). The following statements are equivalent:

1) \( \mathfrak{C} \) reduces \( F \);

2) \((I - F)_{\mathfrak{C}} + (I + F)_{\mathfrak{C}} = 2P_{\mathfrak{C}}\), where \((I \pm F)_{\mathfrak{C}}\) are "shorted operators".

**Proof.** Put \( \mathfrak{C}^\perp = H \ominus \mathfrak{C}, \mathfrak{D} = H \ominus (I \pm F)^{1/2} \mathfrak{C}^\perp, P_\pm \) to be orthogonal projections onto \( \mathfrak{D} \). By the definition \([5, 8]\), \(((I \pm F)_{\mathfrak{C}} f, f) = \inf \{((I \pm F)(f - \varphi), f - \varphi)\}, \varphi \in \mathfrak{C}^\perp\} = \| P_\pm (I \pm F)^{1/2} f \| \) for all \( f \in H_+ \).

1) \( \Rightarrow \) 2). If \( F \mathfrak{C} \subseteq \mathfrak{C} \), then \( F \mathfrak{C}^\perp \subseteq \mathfrak{C}^\perp \). Therefore,

\[\| P_\pm (I \pm F)^{1/2} f \|^2 = \| (I \pm F)^{1/2} P_{\mathfrak{C}} f \|^2, \]

\[((I + F)_{\mathfrak{C}} f, f) + ((I - F)_{\mathfrak{C}} f, f) = \| (I + F)^{1/2} P_{\mathfrak{C}} f \|^2 + \| (I - F)^{1/2} P_{\mathfrak{C}} f \|^2 = 2\| P_{\mathfrak{C}} f \|^2, \quad f \in H. \]

2) \( \Rightarrow \) 1). For all \( f \in H \), we have

\[\| P_+(I + F)^{1/2} f \|^2 + \| P_-(I - F)^{1/2} f \|^2 = 2\| P_{\mathfrak{C}} f \|^2. \]  

(11)

Substituting \( f \in \mathfrak{C} \) in (11), we obtain

\[2\| f \|^2 = \| P_+(I + F)^{1/2} f \|^2 + \| P_-(I - F)^{1/2} f \|^2 \leq \| (I + F)^{1/2} f \|^2 + \| (I - F)^{1/2} f \|^2 = 2\| f \|^2. \]

Consequently, \( P_+(I \pm F)^{1/2} f = (I \pm F)^{1/2} f \) for all \( f \in \mathfrak{C} \). Hence, \( F \mathfrak{C} \subseteq \mathfrak{C} \). Q.E.D.

3. **Proof of Theorem 1.** Suppose that \( A \) and \( \tilde{T} \) are f.-l.t. of \( S \) and \( \tilde{\mathcal{S}} \) respectively. Then \( \tilde{T} \) is a contraction extension of the Hermitian contraction \( A \). Denote by \( \tilde{H} \) the domain of \( \tilde{T} \) and let \( \tilde{P} \) be the orthogonal projection onto \( \tilde{H} \).

1) \( \Rightarrow \) 2). For all \( f \in \tilde{H} \) and \( \varphi \in \mathcal{D} (A) \), we have the equality \((\tilde{T} f, \varphi) = (f, A \varphi)\). Therefore \([9]\), there exists a contractive extension \( T \) of \( \tilde{T} \) on \( H \) such that \( T^* \supset A \). This means that \( T \) is a \( \mathcal{S}\)-contractive extension of \( A \). Put \( \Theta = \mu (\operatorname{Gr} (T)) \). Then \( \Theta \) is an \( m \)-accretive proper extension of \( S \), \( \mathcal{D} (\Theta) = (I + T) H, \Theta \supseteq \tilde{\mathcal{S}} \). From (4),

\[I + T = (I + A_M^{1/2})(I + 1/2(Y - I)P_{\mathcal{M}})(I + A_M)^{1/2}, \]

where \( Y \) is the contraction in \( \mathcal{M} \).

From (5) and (6), for \( v = (I + \tilde{T}) f = (I + T) f, f \in \tilde{H} \), we get

\[\operatorname{Re} \left( \tilde{\mathcal{N}} (v, v) - S_N [v] \right) = 2 \operatorname{Re} \left( (I + T) f, f \right) - 2 \| (I + A_M^0)^{-1/2} (I + T) f \|^2 = \]

\[= 2 \operatorname{Re} \left( (I + 1/2(Y - I)P_{\mathcal{M}})(I + A_M^{1/2}, (I + A_M)^{1/2} f \right) - 2 \| (I + 1/2(Y - I)P_{\mathcal{M}})(I + A_M^{1/2} f \|^2 = \]

\[1/2 \left( \| P_{\mathcal{M}} (I + A_M^{1/2} f \|^2 - \| Y P_{\mathcal{M}} (I + A_M^{1/2} f \|^2 \right) > 0. \]

2) \( \Rightarrow \) 1). Relations (5) and (6) imply \((I + \tilde{T}) \tilde{H} \subseteq (I + A_M)^{1/2} H, \)
\[ \text{Re} \left( f, (I + \tilde{T}) f \right) \geq \| (I + A^0_M)^{-1/2}(I + \tilde{T}) f \|^2, \quad f \in \tilde{H}. \] (12)

Put \( Q = \tilde{P} \tilde{T} \). Then \( Q \) is a contraction in \( \tilde{H} \), \( Q_R = (Q + Q^*)/2 \) is a selfadjoint contraction in \( \tilde{H} \).

Inequality (12) can be rewritten as
\[ \| (I + Q_R)^{1/2} f \|^2 \geq \| (I + A^0_M)^{-1/2}(I + \tilde{T}) f \|^2, \quad f \in \tilde{H}. \]

Hence, \( (I + A^0_M)^{-1/2}(I + \tilde{T}) f = W(I + Q_R)^{1/2} f, f \in \tilde{H} \), where \( W \) is a contraction.

Furthermore, we have, for all \( f \in \tilde{H} \),
\[ \| (I + Q_R)^{1/2} f \|^2 = \text{Re} \left( f, (I + \tilde{T}) f \right) = \text{Re} \left( f, (I + A^0_M)^{1/2} W(I + Q_R)^{1/2} f \right) \leq \| (I + A_M)^{1/2} f \| \| (I + Q_R)^{1/2} f \|. \]

This implies \( \| (I + Q_R)^{1/2} f \|^2 \leq \| (I + A^0_M)^{1/2} f \|^2, f \in \tilde{H} \). or \( Q_R \leq \tilde{P} A_M \tilde{P} \). For \( \varphi \in \mathcal{D}(A) \), \( (Q_R \varphi, \varphi) = \text{Re} \left( \tilde{T} \varphi, \varphi \right) = (A \varphi, \varphi) = (\tilde{P} A_M \varphi, \varphi) \). Hence, \( Q_R \mathcal{A}(A) = \tilde{P} A \). Consequently, \( Q^* \mathcal{D}(A) = Q \mathcal{D}(A) = \tilde{P} A \). i.e., \( Q \) is a qsc-extension in \( \tilde{H} \) of the Hermitian contraction \( \tilde{P} A \). This yields \( \text{Gr}(\tilde{T}) \subset \text{Gr}(A)^* \) and \( \tilde{S} \subset S^* \).

2) \( \iff 3) \) is an immediate consequence of (8). Q.E.D.

**4. Proof of Theorem 2.** 1) \( \Rightarrow 2) \) is a consequence of Theorem 1.

2) \( \Rightarrow 1) \). Let \( T \) be a f.-l.t. of \( \Theta \). Then \( T \) is a contraction defined on \( H \) and \( T^* \supset A \).

For \( T_R = (T + T^*)/2 \), using (5) and (6), we get
\[ \| (I + T_R)^{1/2} f \|^2 \geq \| (I + A^0_M)^{-1/2}(I + T) f \|^2, f \in H. \]

As before, this inequality implies \( T_R \leq A_M \) and, in view of \( T^* \supset A \), we have \( T \mathcal{D}(A) = A \). Thus, \( \Theta \supset S \).

2) \( \iff 3) \) is a corollary of (8). Q.E.D.

**5. Proof of Theorem 3.** Consider the Friedrichs extension \( \tilde{S} \) of \( \Theta \). Then \( \tilde{S} \) is \( m - \alpha \)-sectorial. \( \mathcal{D}[\tilde{S}] = \mathcal{D}[\Theta] \subseteq \mathcal{D}[S_N] \), and, for the operator part \( \tilde{S} \),
\[ \tilde{S} = \tilde{S}_R^{1/2}(I + i\tilde{G}) \tilde{S}_R^{1/2}, \quad \tilde{G} = \tilde{G}^*, \quad \| \tilde{G} \| \leq \tan \alpha. \]

Put \( \tilde{P}_R = \mathcal{R}(\tilde{S}_R^{1/2}) \cap \tilde{S}_R^{1/2} \mathcal{D}(S), \tilde{\pi}, P_R \) to be orthogonal projection onto \( \mathcal{R}_\perp \) and \( \mathcal{R} \) respectively.

A direct consequences of the definitions are the following relations for all \( v \in \mathcal{D}(\Theta) \):
\[ \langle \text{Re} \left[ (I + i \cot \alpha)(\Theta(v), v) \right], \tilde{S}_R^{1/2} v \rangle_{\mathcal{D}(S)} = \langle (I + i \cot \alpha) \tilde{G} \tilde{S}_R^{1/2} v, \tilde{S}_R^{1/2} v \rangle, \] (13)
\[ \langle \text{Re}(\Theta(v), v) \rangle_{\mathcal{D}(S)} = \| P_R \tilde{S}_R^{1/2} v \|^2. \] (14)

Besides, for \( u \in \mathcal{D}(S) \),
\[ \| \tilde{S}_R^{1/2} u \|^2 = \text{Re} (\tilde{S}(u), u) = (\tilde{S}(u), u) = ((I + i \tilde{G}) \tilde{S}_R^{1/2} u, \tilde{S}_R^{1/2} u). \]

Hence,
\[ \tilde{\pi} \tilde{G} \tilde{\pi} = 0. \] (15)

1) \( \Rightarrow 2) \). Since \( \Theta \) is accretive, from Theorem 1.
Re (Θ(v), v) ≥ || S^{1/2}_N v ||^2 \quad \text{for all} \quad v \in \mathcal{D}(Θ). \quad (16)

Since \mathcal{D}(Θ) is the core of the sesquilinear form Θ[u, v] = \tilde{S}[u, v], (16) implies that

Re \tilde{S}[v] ≥ || S^{1/2}_N v ||^2 \quad \text{for all} \quad v \in \mathcal{D}[Θ].

It follows from relation (10) that

|| \tilde{S}_{\mathcal{R}}^{1/2} v ||^2 ≥ || \tilde{\pi} (I - i \tilde{G}) \tilde{S}_{\mathcal{R}}^{1/2} v ||^2. \quad v \in \mathcal{D}[Θ].

Therefore, the bounded selfadjoint operator

L = I - (I + i \tilde{G}) \tilde{\pi} (I - i \tilde{G})

acting in the subspace \mathcal{R}(\tilde{S}_{\mathcal{R}}^{1/2}) is positive. Using (15) for \varphi = \tilde{S}_{\mathcal{R}}^{1/2} u, u \in \mathcal{D}(S), we have \langle L \varphi, \varphi \rangle = || \varphi ||^2 - || \varphi ||^2 = 0. Consequently, L \tilde{\pi} = 0. This yields \tilde{G} \tilde{\pi} = = 0. Thus, \tilde{G} \mathcal{K} \subseteq \mathcal{K}.

Now the lemma implies

\[(I - \cotg \alpha \tilde{G}) \mathcal{K} + (I + \cotg \alpha \tilde{G}) \mathcal{K} = 2P_{\mathcal{K}}.\]

Hence, in view of (13) and (14), for \( v \in \mathcal{D}(Θ), \)

\[
\langle \text{Re} \left[(1 - i \cotg \alpha)(Θ(v), v)\right], \mathcal{D}(S) \rangle + \langle \text{Re} \left[(1 + i \cotg \alpha)(Θ(v), v)\right], \mathcal{D}(S) \rangle =
\]

\[
= 2 \langle \text{Re} (Θ(v), v), \mathcal{D}(S) \rangle.
\]

(17)

2) ⇒ 1). Let (17) be true for all \( v \in \mathcal{D}(Θ). \) Since \mathcal{D}(Θ) is the core of \( \tilde{S}_{\mathcal{R}}^{1/2}, \) we have from (13), (14), and the lemma that \( \tilde{G} \mathcal{K} \subseteq \mathcal{K}. \)

Taking (15) into account, we get \( \tilde{G} \tilde{\pi} = \tilde{\pi} \tilde{G} = 0. \) Hence, from (10), for \( v \in \mathcal{D}(Θ), \)

\[
S_N[v] = || S_N^{1/2} v ||^2 = || \tilde{\pi} (I - i \tilde{G}) \tilde{S}_{\mathcal{R}}^{1/2} v ||^2 = || \tilde{\pi} \tilde{S}_{\mathcal{R}}^{1/2} v ||^2 ≤ \text{Re} (Θ(v), v).
\]

In accordance with Theorem 1, \( Θ \subseteq S^*. \)

3) ⇒ 1). If \( \omega \) is an \( α \)-sectorial form, then

\[
\text{Re} (Θ(v), v) ≥ S_N[v] \quad \text{for all} \quad v \in \mathcal{D}(Θ).
\]

Furthermore, we apply Theorem 1.

1) ⇒ 3). Since \( Θ \) is a proper accretive extension of \( S, \) from Theorem 1, \( \mathcal{D}(Θ) \subseteq \mathcal{D}(S_N) \). For all \( u \in \mathcal{D}(S_N) \) and \( u_0 \in \mathcal{D}(S). \) we have

\[S_N[u, u_0] = (u, S(u_0)).\]

Hence, it is easy to check that, for all \( u \in \mathcal{D}(Θ) \) and \( u_0 \in \mathcal{D}(S), \) we have

\[
\omega [u - u_0] = \omega [u]. \quad (18)
\]

An immediate consequence of (8) is the relation

\[
\inf \left\{ S_N[u - u_0], u_0 \in \mathcal{D}(S) \right\} = 0 \quad \text{for all} \quad u \in \mathcal{D}(S_N).
\]

Therefore, for given \( ε > 0 \) and \( u \in \mathcal{D}(Θ), \) one can find \( u_0 \in \mathcal{D}(S) \) such that \( S_N[u - u_0] < ε. \) Taking (18) into account, we obtain

\[
\text{Re} \, \omega [u] ± \cotg α \text{Im} \, \omega [u] =
\]

ISSN 0041-6053. Укр. мат. журн., 1995, т. 47, № 6