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ASYMPTOTIC SOLUTIONS OF THE DIRICHLET PROBLEM FOR THE HEAT EQUATION WITH IMPULSES

АСИМПТОТИЧНІ РОЗВ'ЯЗКИ ЗАДАЧІ ДІРІХЛЕ ДЛЯ РІВНЯННЯ ТЕПЛОПРОВІДНОСТІ З ІМПУЛЬСНОЮ ДІЄЮ

We propose an algorithm for the construction of asymptotic expansions for solutions of the Dirichlet problem for the heat equation with impulses.

Запропоновано алгоритм побудови асимптотичних розвинень для розв'язків задачі Діріхле для рівняння теплопровідності з імпульсною дією.

1. Introduction. The theory of impulsive differential equations [1] is an important part of the modern theory of differential equations which has many applications in practice. Till now, a lot of different problems connected with impulsive differential equations are studied. It has been found that the solutions of differential equations with impulses can demonstrate very complicated behaviour [1 – 5]. In the present paper, we study the problem of the construction of an asymptotic solution of the Dirichlet problem for the heat equation with impulses.

2. Formulation of the problem. Let us consider a differential heat equation with small parameter $\varepsilon \in (0; \varepsilon_0)$ of the form

$$\frac{\partial u}{\partial t} = \varepsilon^2 \frac{\partial^2 u}{\partial x^2} + f(x, t, u, \varepsilon), \qquad (x, t) \in \Omega = (0; 1) \times (0; +\infty), \tag{1}$$

under initial conditions

$$u(x, 0, \varepsilon) = \varphi(x, \varepsilon), \tag{2}$$

boundary conditions

$$u(0, t, \varepsilon) = 0, \qquad u(1, t, \varepsilon) = 0, \quad t \in [0, +\infty), \tag{3}$$

and impulsive conditions at a fixed moment of time

$$\Delta u(x, t, \varepsilon)\big|_{t=t_i} = u(x, t_i + 0, \varepsilon) - u(x, t_i - 0, \varepsilon) = I_i(x, \varepsilon), \qquad i \in \mathbb{N}. \tag{4}$$

We suppose the fulfillment of the following assumptions:

P_1. Functions $f(x, t, u, \varepsilon)$, $\varphi(x, \varepsilon)$, $I_i(x, \varepsilon)$, $i \in \mathbb{N}$, are infinitely differentiable with respect to their variables and can be represented as regular asymptotic expansions with respect to a small parameter $\varepsilon \in (0; \varepsilon_0)$. In particular,

$$\varphi(x, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \varphi_k(x), \qquad I_i(x, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \mathcal{I}_{ik}(x).$$

P_2. The nonperturbed problem

$$\frac{\partial u}{\partial t} = f(x, t, u, 0), \qquad u(x, 0, 0) = \varphi(x, 0),$$

$$\Delta u(x, t, 0)\big|_{t=t_i} = u(x, t_i + 0, 0) - u(x, t_i - 0, 0) = I_i(x, 0), \qquad i \in \mathbb{N},$$

for any $x \in (0; 1)$ (x is considered as a parameter) possesses a solution $u = \bar{u}_0(x, t)$ defined for any $(x, t) \in [0; 1] \times [0; +T)$ (here, the case $T = +\infty$ is not excluded) which is infinitely differentiable for $t \neq t_i$, $i \in \mathbb{N}$.

P_3. The condition $f'_u(x, t, \bar{u}_0(x, t), 0) \neq 0$ takes place.

P_4. The agreement condition $\varphi_0(0, \varepsilon) = 0$ takes place.

3. Algorithm of asymptotic expansion. We seek a solution of problem (1) – (4) in the form of the asymptotic series

$$u(x, t, \varepsilon) = \bar{u}(x, t, \varepsilon) + Q u(\xi, t, \varepsilon) + Q_* u(\xi_*, t, \varepsilon), \tag{5}$$

where

$$\bar{u}(x, t, \varepsilon) = \bar{u}_0(x, t) + \varepsilon \bar{u}_1(x, t) + \varepsilon^2 \bar{u}_2(x, t) + \ldots$$

is the regular part of asymtotics and

$$Q u(\xi, t, \varepsilon) = Q_0 u(\xi, t) + \varepsilon Q_1 u(\xi, t) + \varepsilon^2 Q_2 u(\xi, t) + \ldots ,$$

$$Q_* u(\xi_*, t, \varepsilon) = Q_{*0} u(\xi_*, t) + \varepsilon Q_{*1} u(\xi_*, t) + \varepsilon^2 Q_{*2} u(\xi_*, t) + \ldots$$

are singular parts of asymptotics.

Here, we denote $\xi = x/\varepsilon$, $\xi_* = (1 - x)/\varepsilon$. The functions $Q_k u(\xi, t)$, $k \geq 0$, are supposed to be defined for $\xi \in [0; \varepsilon^{-1}]$, $t \geq 0$, and, at the same time, the functions $Q_{*k} u(\xi_*, t)$, $k \geq 0$, are supposed to be defined for $\xi_* \in [0; \varepsilon^{-1}]$, $t \geq 0$.

By the usual way, we obtain relations for defining terms of asymptotics (5). The terms of the regular part of asymptotics (5) may be found as solutions of the following problems:

$$\frac{d\bar{u}_0}{dt} = f(x, t, \bar{u}_0, 0),$$

$$\bar{u}_0(x, 0) = \varphi_0(x), \qquad x \in [0; 1],$$

$$\Delta \bar{u}_0(x, t)\big|_{t=t_i} = I_i(x, 0), \qquad i \in \mathbb{N},$$

$$\frac{d\bar{u}_k}{dt} = f_u(x, t, \bar{u}_0, 0)\bar{u}_k + f_k(x, t, \bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{k-1}),$$

$$\bar{u}_k(x, 0) = \varphi_k(x), \qquad x \in [0; 1],$$

$$\Delta \bar{u}_k(x, t)\big|_{t=t_i} = \mathcal{T}_{ik}(x), \qquad i \in \mathbb{N},$$

where functions $f_k(x, t, \bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{k-1})$, $k \geq 1$, are recurrently defined by values of $\bar{u}_0(x, t)$, $\bar{u}_1(x, t), \ldots, \bar{u}_{k-1}(x, t)$. Here, $x \in [0; 1]$ is considered as a parameter.

After definition of the regular part $\bar{u}(x, t, \varepsilon)$ of asymptotics (5), we can find the singular part $Q u(\xi, t, \varepsilon)$ of asymptotic (5) which is defined as solutions of the following boundary-value problems:

$$\begin{gathered}
\frac{\partial Q_0 u}{\partial t} = \frac{\partial^2 Q_0 u}{\partial \xi^2} + f(0, t, \bar{u}_0(0, t) + Q_0 u, 0) - f(0, t, \bar{u}_0(0, t), 0), \\
Q_0 u(\xi, 0) = 0, \qquad \xi \in [0; \infty], \\
Q_0 u(0, t) = -\bar{u}_0(0, t), \qquad t \in [0; \infty], \\
\lim_{\xi \to \infty} Q_0 u(\xi, t) = 0 \quad \text{for any} \quad t \in [0; \infty];
\end{gathered} \tag{6}$$

$$\frac{\partial Q_k u}{\partial t} = \frac{\partial^2 Q_k u}{\partial \xi^2} + f_u(0, t, \bar{u}_0(0, t), 0) Q_k u + Q_k f(\xi, t),$$
$$Q_k u(0, \xi) = 0, \quad \xi \in [0; \infty],$$
$$Q_k u(0, t) = -\bar{u}_0(0, t), \quad t \in [0; \infty], \tag{7}$$
$$\lim_{\xi \to \infty} Q_k u(\xi, t) = 0 \quad \text{for any} \quad t \in [0; \infty],$$

where the functions $Q_k f(\xi, t)$, $k \geq 1$, are recurrently defined by the standard procedure by values of the regular part $\bar{u}_0(x, t)$, $\bar{u}_1(x, t), \ldots, \bar{u}_{k-1}(x, t)$, $x = \varepsilon\xi$, of asymptotics (5).

We can now proceed to the calculation of the singular part $Q_* u(\xi, t, \varepsilon)$ of asymptotics (5) which is defined as solutions of the following boundary-value problems:

$$\frac{\partial Q_{*0} u}{\partial t} = \frac{\partial^2 Q_{*0} u}{\partial \xi_*^2} + f\big(1, t, \bar{u}_0(1, t) + Q_0 u + Q_{*0} u, 0\big) - f\big(1, t, \bar{u}_0(1, t) + Q_0 u, 0\big),$$
$$Q_{*0} u(\xi_*, 0) = 0, \quad \xi_* \in [0; \infty],$$
$$Q_{*0} u(0, t) = -\bar{u}_0(1, t), \quad t \in [0; \infty], \tag{8}$$
$$\lim_{\xi_* \to \infty} Q_{*0} u(\xi_*, t) = 0 \quad \text{for any} \quad t \in [0; \infty];$$

$$\frac{\partial Q_{*k} u}{\partial t} = \frac{\partial^2 Q_{*k} u}{\partial \xi_*^2} + f_u(1, t, \bar{u}_0(1, t), 0) Q_{*k} u + Q_{*k} f(\xi_*, t),$$
$$Q_{*k} u(0, \xi_*) = 0, \quad \xi_* \in [0; \infty],$$
$$Q_{*k} u(0, t) = -\bar{u}_0(1, t), \quad t \in [0; \infty], \tag{9}$$
$$\lim_{\xi_* \to \infty} Q_{*k} u(\xi_*, t) = 0 \quad \text{for any} \quad t \in [0; \infty],$$

where the functions $Q_{*k} f(\xi_*, t)$, $k \geq 1$, are recurrently defined by the standard procedure by values of the regular part $\bar{u}_0(x, t)$, $\bar{u}_1(x, t), \ldots, \bar{u}_{k-1}(x, t)$ and the singular part $Q_0 u(\xi, t)$, $Q_{*0} u(\xi_*, t)$, $Q_1 u(\xi, t)$, $Q_{*1} u(\xi_*, t), \ldots, Q_{k-1} u(\xi, t)$, $Q_{*k-1} u(\xi_*, t)$, $x = 1 + \varepsilon\xi_*$, of asymptotics (5).

4. Main result. The following statements are true:

Lemma 1. *Additionally to assumptions* $\mathrm{P}_1 - \mathrm{P}_4$, *let us assume the fulfillment of the following conditions*:

1) *there exist positive values* C_0 *and* γ_0 *such that a solution* $Q_0 u(\xi, t)$ *of problem* (6) *satisfies the inequality*

$$\left| Q_0 u(\xi, t) \right| \leq C_0 e^{-\gamma_0 \xi};$$

2) *the derivative* $f_0\big(0, t, \bar{u}_0(0, t), 0\big)$ *is negative for all* $t \in [0; T)$.

Then, for any $k \in \mathbb{N}$, *there exist solutions* $Q_k u(\xi, t)$ *of problem* (7) *such that inequalities*

$$\left| Q_k u(\xi, t) \right| \leq C_k e^{-\gamma_k \xi}$$

are true for some $C_k > 0$, $\gamma_k > 0$.

Lemma 2. *Additionally to assumptions* $P_1 - P_4$, *let us assume the fulfillment of the following conditions*:

1) *there exist positive values* C_{*0} *and* γ_{*0} *such that a solution* $Q_{*0}u(\xi_*, t)$ *of problem* (8) *satisfies the inequality*

$$\left| Q_{*0}u(\xi_*, t) \right| \le C_{*0} e^{-\gamma_{*0}\xi_*};$$

2) *the derivative* $f_u(1, t, \bar{u}_0(1, t), 0)$ *is negative for all* $t \in [0; T)$.

Then, for any $k \in \mathbb{N}$, *there exist solutions* $Q_{*k}u(\xi_*, t)$ *of problem* (9) *such that inequalities*

$$\left| Q_{*k}u(\xi_*, t) \right| \le C_{*k} e^{-\gamma_{*k}\xi_*}$$

are true for some $C_{*k} > 0$, $\gamma_{*k} > 0$.

Theorem. *Let the conditions of Lemmas* 1, 2 *be fulfilled. Then the series* (5) *is an asymptotic solution of problem* (1) – (4), *i.e.,*

$$\max_{\Omega_1} \left| u(x, t, \varepsilon) - u_N(x, t, \varepsilon) \right| = O(\varepsilon^{N+1}),$$

where $\Omega_1 \subset (0; 1) \times (0; T)$ *is a compact set and a functions* $u_N(x, t, \varepsilon)$ *is defined with the relation*

$$u_N(x, t, \varepsilon) = \sum_{k=0}^{N} \varepsilon^k [\bar{u}_k(x, t) + Q_k u(\varepsilon x, t) + Q_{*k} u(1 + \varepsilon x, t)].$$

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