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ASYMPTOTIC BEHAVIOR OF EIGENVALUES AND EIGENFUNCTIONS OF THE FOURIER PROBLEM IN A THICK MULTILEVEL JUNCTION

АСИМПТОТИЧНА ПОВЕДІНКА ВЛАСНИХ ЗНАЧЕНЬ ТА ВЛАСНИХ ФУНКЦІЙ ЗАДАЧІ ФУР'Є В ГУСТОМУ БАГАТОРІВНЕВОМУ З'ЄДНАННІ

A spectral boundary-value problem is considered in a plane thick two-level junction Ω_ε, which is the union of a domain Ω_0 and a large number $2N$ of thin rods with thickness of order $\varepsilon = \mathcal{O}(N^{-1})$. The thin rods are divided into two levels depending on their length. In addition, the thin rods from each level are ε-periodically alternated. The Fourier conditions are given on the lateral boundaries of the thin rods. The asymptotic behavior of the eigenvalues and eigenfunctions is investigated as $\varepsilon \to 0$, i.e., when the number of the thin rods infinitely increases and their thickness tends to zero. The Hausdorff convergence of the spectrum is proved as $\varepsilon \to 0$, the leading terms of asymptotics are constructed and the corresponding asymptotic estimates are justified for the eigenvalues and eigenfunctions.

Розглядається спектральна крайова задача у плоскому дворівневому з'єднанні Ω_ε, яке є об'єднанням області Ω_0 та великого числа $2N$ тонких стержнів товщиною порядку $\varepsilon = \mathcal{O}(N^{-1})$. Тонкі стержні розділено на два рівні в залежності від їх довжини. Крім того, тонкі стержні з кожного рівня ε-періодично чергуються. На вертикальних сторонах тонких стержнів задано крайові умови Фур'є. Вивчено асимптотичну поведінку власних значень та власних функцій при $\varepsilon \to 0$, тобто коли число тонких стержнів необмежено зростає, а їх товщина прямує до нуля. Доведено хаусдорфову збіжність спектра при $\varepsilon \to 0$, побудовано перші члени асимптотики та обґрунтовано відповідні асимптотичні оцінки для власних значень та власних функцій.

1. Introduction and statement of the problem. As has been stated in [1], multiscale modeling and computation is a rapidly evolving area of research that will have a fundamental impact on computational science and applied mathematics. This is connected with the prospect of development of more efficient methods that should be symbiosis of a new class of numerical and analytical modeling techniques. There is a long history in mathematics for the study of multiscale problems. One class of multiscale problems is boundary-value problems in perturbed domains. There are many kinds of the domain perturbations and we need different asymptotic methods to study boundary-value problems in perturbed domains (see, e.g., [2 – 11] and references there).

Perturbed spectral boundary-value problems deserve special attention, since the asymptotic behaviour of the spectrum is highly sensitive to the perturbation and it is unexpected (see, e.g., [12]). If the perturbation is smooth and in some sense small, then with the help of a family of diffeomorphisms we can reduce a perturbed spectral problem to investigation of behaviour of the spectrum of operators defined in some fixed domain. But there are many problems with singular perturbed domains and it is not possible to use above-mentioned approach. The extensive review of such problems was presented in [13].

In this paper a new kind of perturbed domains, namely, *thick multilevel junctions* is considered. Boundary-value problems in thick one-level junctions (thick junctions) are very intensively investigated in the last time. As was shown in the papers [10, 14],

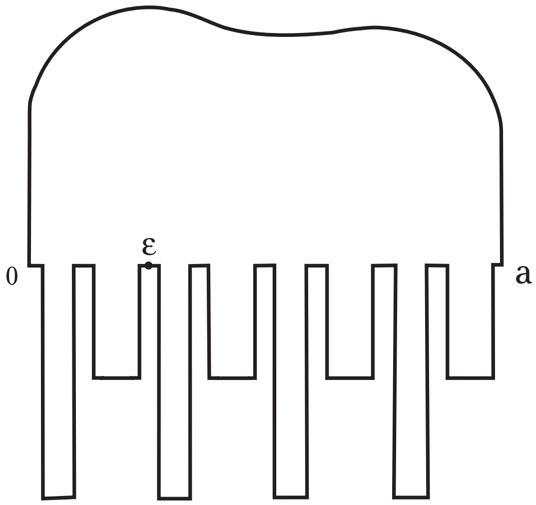

Fig. 1. The thick two-level junction Ω_ε.

such problems lose the coercitivity and compactness as $\varepsilon \to 0$. This creates special difficulties in the asymptotic investigation. In [13, 15 – 22], classification of thick one-level junctions was given and basic results were obtained both for boundary-value and spectral problems in thick junctions of different types. It was shown that qualitative properties of solutions essentially depend on the junction type and on the conditions given on the boundaries of the attached thin domains. A survey of results obtained in this direction is presented in [13, 15 – 22]. Here we mention only the pioneer papers [7, 23, 24], where the asymptotic behaviour of Green's function of the Neumann problem for the Helmholtz equation in unbounded thick junctions was studied.

1.1. Statement of the problem. Let $a,\ d_1,\ d_2,\ b_1,\ b_2,\ h_1,\ h_2$ be positive real numbers and let $d_1 \geq d_2$, $0 < b_1 < b_2 < 1$, $0 < b_1 - h_1/2$, $b_1 + h_1/2 < b_2 - h_2/2$, $b_2 + h_2/2 < 1$. The last restrictions mean that the intervals $I_{h_1}(b_1) := \big(b_1 - h_1/2, b_1 + h_1/2\big)$ and $I_{h_2}(b_2) := (b_2 - h_2/2, b_2 + h_2/2)$ belong to $(0,1)$ and don't intersect. Let us divide the segment $I_0 := [0,a]$ on N equal segments $[\varepsilon j, \varepsilon(j+1)]$, $j = 0, \ldots, N-1$. Here N is a large integer, therefore, the value $\varepsilon = a/N$ is a small discrete parameter.

A model plane thick two-level junction Ω_ε (Fig. 1) consists of the junction's body

$$\Omega_0 = \Big\{x \in \mathbb{R}^2 : 0 < x_1 < a, \quad 0 < x_2 < \gamma(x_1)\Big\},$$

where $\gamma \in C^1([0,a])$, $\gamma(0) = \gamma(a)$, $\min_{[0,a]} \gamma > 0$, and a large number of the thin rods

$$G_j^{(1)}(\varepsilon) = \Big\{x \in \mathbb{R}^2 : |x_1 - \varepsilon(j + b_1)| < \frac{\varepsilon h_1}{2},\ x_2 \in (-d_1, 0]\Big\},$$

$$G_j^{(2)}(\varepsilon) = \Big\{x \in \mathbb{R}^2 : |x_1 - \varepsilon(j + b_2)| < \frac{\varepsilon h_2}{2},\ x_2 \in (-d_2, 0]\Big\},$$

$$j = 0, 1, \ldots, N-1,$$

i.e., $\Omega_\varepsilon = \Omega_0 \cup G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}$, where $G_\varepsilon^{(1)} = \cup_{j=0}^{N-1} G_j^{(1)}(\varepsilon)$, $G_\varepsilon^{(2)} = \cup_{j=0}^{N-1} G_j^{(2)}(\varepsilon)$. We see that the number of the thin rods is equal to $2N$ and they are divided into two levels $G_\varepsilon^{(1)}$ and $G_\varepsilon^{(2)}$ depending on their length. The parameter ε characterizes the distance between the neighboring thin rods and their thickness. The thickness of the rods from the first level is equal to εh_1 and it is equal to εh_2 for the rods from the second one. These thin rods from each level are ε-periodically alternated along the segment $I_0 = \{x : x_1 \in [0, a],\, x_2 = 0\}$ (the joint zone of this thick two-level junction).

Denote by $\Upsilon_j^{(i,\pm)}(\varepsilon)$ the lateral sides of the thin rod $G_j^{(i)}(\varepsilon)$, the signs "+" or "−" indicate the right or left side respectively; the base of $G_j^{(i)}(\varepsilon)$ will be denoted by $\Theta_j^{(i)}(\varepsilon)$. Also we introduce the following notations:

$$\Upsilon_\varepsilon^{(i,\pm)} := \cup_{j=0}^{N-1} \Upsilon_j^{(i,\pm)}(\varepsilon), \qquad \Theta_\varepsilon^{(i)} := \cup_{j=0}^{N-1} \Theta_j^{(i)}(\varepsilon), \quad i = 1, 2.$$

In Ω_ε we consider the following spectral problem:

$$\begin{gathered} -\Delta_x\, u(\varepsilon, x) = \lambda(\varepsilon) u(\varepsilon, x), \quad x \in \Omega_\varepsilon, \\ \partial_\nu u(\varepsilon, x) = -\varepsilon k_1 u(\varepsilon, x), \quad x \in \Upsilon_\varepsilon^{(1,\pm)}, \\ \partial_\nu u(\varepsilon, x) = -\varepsilon k_2 u(\varepsilon, x), \quad x \in \Upsilon_\varepsilon^{(2,\pm)}, \\ \partial_{x_1}^p u(\varepsilon, 0, x_2) = \partial_{x_1}^p u(\varepsilon, a, x_2), \quad x_2 \in [0, \gamma(0)], \quad p = 0, 1, \\ \partial_\nu u(\varepsilon, x) = 0, \quad x \in \Gamma_\varepsilon. \end{gathered} \tag{1}$$

Here $\partial_\nu = \dfrac{\partial}{\partial \nu}$ is the outward normal derivative; $\partial_{x_1} = \dfrac{\partial}{\partial x_1}$; the constants k_1 and k_2 are positive; $\Gamma_\varepsilon = \Theta_\varepsilon^{(1)} \cup \Theta_\varepsilon^{(2)} \cup (I_0 \cap \partial\Omega_\varepsilon) \cup \Gamma_\gamma$, where $\Gamma_\gamma = \{x : x_2 = \gamma(x_1), x_1 \in I_0\}$.

It is well known that for each fixed $\varepsilon > 0$ there is a sequence of eigenvalues of problem (1)

$$0 < \lambda_1(\varepsilon) \le \lambda_2(\varepsilon) \le \ldots \le \lambda_n(\varepsilon) \le \ldots \to +\infty \quad \text{as} \quad n \to \infty, \tag{2}$$

and a sequence of the corresponding eigenfunctions $\{u_n(\varepsilon, \cdot) : \ n \in \mathbb{N}\}$ can be orthonormalized by the following way:

$$(u_n, u_m)_{\Omega_\varepsilon} = \delta_{n,m}, \quad \{n, m\} \in \mathbb{N}, \tag{3}$$

where $(\cdot,\cdot)_\Upsilon$ is the scalar product in $L^2(\Upsilon)$, and $\delta_{n,m}$ is the Kronecker delta.

Our aim is to describe the asymptotic behavior of eigenvalues $\{\lambda_n(\varepsilon) : n \in \mathbb{N}\}$ and eigenfunctions $\{u_n(\varepsilon, \cdot) : n \in \mathbb{N}\}$ as $\varepsilon \to 0$ $(N \to +\infty)$, to find other limiting points of the spectrum of problem (1), and to describe corresponding eigenfunctions.

1.2. Features of the investigation. As was showed in [13, 15 – 22], the corresponding limit problem for a boundary-value problem in a thick one-level junction is derived from the limit problems for each domain forming the thick junction with the help of the solutions to junction-layer problems around the joint zone. However, the junction-layer solutions behave as powers (or logarithm) at infinity and do not decrease exponentially. Therefore, they influence directly the leading terms of the asymptotics. The model problems describing the junction-layer phenomenon are posed in unbounded domains having outlets to infinity. The principal terms of the inner expansion is nontrivial solutions to the corresponding homogeneous junction-layer problem. In the case of a thick one-level junction such a solution is identically defined. But for a thick p-level junction, dimension of the kernel of the corresponding homogeneous junction-layer problem is equal to $p+1$ and the problem is how to define the principal terms of the inner expansion. This fact very complicates the construction of the asymptotic approximation for the solutions. We should modify the view of the inner expansion and consider outer expansions in each thin domains from each level. Matching these asymptotic expansions, we deduce the nonstandard limiting spectral boundary-value problem (41) in an anisotropic Sobolev vector-space.

In this paper we consider the Fourier conditions $\partial_\nu u_\varepsilon = -\varepsilon k_i u_\varepsilon$ on the lateral boundaries $\Upsilon_\varepsilon^{(i,\pm)}$, $i=1,2$, of the thin rods. At first sight it seems that there is no difference between these Fourier condition and the homogeneous Neumann conditions since the terms $k_i u_\varepsilon$, $i=1,2$, are multiplied by the factor ε. But this is quite false. As was mentioned above the boundary conditions on the boundaries of the attached thin domains of thick junctions have essentially influence on the asymptotic behaviour of the solutions. For problem (1) this leads to the appearance of special coefficients in the differential operator of the limit problem.

The Fourier conditions or the nonhomogeneous Neumann conditions make the process of homogenization and approximation more complicated. For this the method of the integral identities was proposed in [20, 22].

For the first time a boundary-value problem in a plane thick multilevel junction was considered in [25], where some results for problem (1) were announced. Then the development of rigorous asymptotic methods for boundary-value problems in thick multilevel junctions of different types have been continued in [26 – 29].

2. Auxiliary inequalities. In the subspace $\mathcal{H}_\varepsilon := \{u \in H^1(\Omega_\varepsilon) : u(0,x_2) = u(a,x_2),\ x_2 \in [0,\gamma(0)]\}$ we introduce a new norm $\|\cdot\|_{\varepsilon,k_1,k_2}$ that is generated by the following scalar product:

$$\langle u,v\rangle_{\varepsilon,k_1,k_2} = \int\limits_{\Omega_\varepsilon} \nabla u\cdot\nabla v dx + \varepsilon k_1 \int\limits_{\Upsilon_\varepsilon^{(1,\pm)}} uv dx_2 + \varepsilon k_2 \int\limits_{\Upsilon_\varepsilon^{(2,\pm)}} uv dx_2.$$

Lemma 1. *For ε small enough, the usual norm $\|\cdot\|_{H^1(\Omega_\varepsilon)}$ in the Sobolev space $H^1(\Omega_\varepsilon)$ and the norm $\|v\|_{\varepsilon,k_1,k_2}$ are uniformly equivalent, i.e., there exist constants $C_1>0$, $C_2>0$ and ε_0 such that for all values $\varepsilon\in(0,\varepsilon_0)$ and any function $v\in\mathcal{H}_\varepsilon$ the following inequalities hold*:

$$C_1\|v\|_{H^1(\Omega_\varepsilon)} \le \|v\|_{\varepsilon,k_1,k_2} \le C_2\|v\|_{H^1(\Omega_\varepsilon)}. \tag{4}$$

***Remark* 1.** Here and further all constants $\{c_i, C_i\}$ in asymptotic inequalities are independent of the parameter ε.

Proof. It follows from the assumptions made for the numbers b_1,b_2,h_1,h_2 that there exists a such number δ_0 that $b_1+h_1/2<\delta_0<b_2-h_2/2$. Defined the following function:

$$Y(t) = \begin{cases} -t+b_1, & t\in[0,\delta_0), \\ \\ -t+b_2, & t\in[\delta_0,1), \end{cases} \tag{5}$$

and then periodically extend it into $\mathbb{R}$. Integrating by parts in the integral

$$\varepsilon \int\limits_{G_\varepsilon^{(i)}} Y(x_1/\varepsilon)\partial_{x_1} v\, dx, \quad i=1,2,$$

we get the identity

$$\varepsilon 2^{-1} h_i \int\limits_{\Upsilon_\varepsilon^{(i,\pm)}} v dx_2 = \int\limits_{G_\varepsilon^{(i)}} v dx - \varepsilon \int\limits_{G_\varepsilon^{(i)}} Y\left(\frac{x_1}{\varepsilon}\right) \partial_{x_1} v dx \quad \forall v \in \mathcal{H}_\varepsilon, \quad i = 1, 2. \tag{6}$$

Since $\max_{\mathbb{R}} |Y| \le 1$, it follows from (6) that

$$\|\sqrt{\varepsilon} v\|_{L^2\left(\Upsilon_\varepsilon^{(1,\pm)} \cup \Upsilon_\varepsilon^{(2,\pm)}\right)} \le C_2 \|v\|_{H^1\left(G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}\right)}$$

for any $v \in \mathcal{H}_\varepsilon$. Therefore, the right inequality in (4) holds.

Using (6), we obtain

$$\|v\|^2_{H^1(\Omega_\varepsilon)} = \int\limits_{\Omega_\varepsilon} |\nabla v|^2 dx + \int\limits_{\Omega_0} v^2 dx +$$

$$+\varepsilon 2^{-1} \sum_{i=1}^{2} h_i \int\limits_{\Upsilon_\varepsilon^{(i,\pm)}} v^2 dx_2 + \varepsilon \int\limits_{G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}} Y\left(\frac{x_1}{\varepsilon}\right) 2v \partial_{x_1} v dx \le$$

$$\le c_3 \|v\|^2_{\varepsilon, k_1, k_2} + \int\limits_{\Omega_0} v^2 dx + \varepsilon \int\limits_{G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}} v^2 dx,$$

whence

$$\|v\|^2_{H^1(\Omega_\varepsilon)} \le c_4 \left(\|v\|^2_{\varepsilon, k_1, k_2} + \int\limits_{\Omega_0} v^2 dx \right). \tag{7}$$

Now let us show that there exists a positive constant c_5 such that for ε small enough

$$\int\limits_{\Omega_0} v^2 dx \le c_5 \|v\|^2_{\varepsilon, k_1, k_2} \quad \forall\, v \in \mathcal{H}_\varepsilon. \tag{8}$$

We argue by contradiction. Then there exist sequences $\{\varepsilon_m : m \in \mathbb{N}\}$ and $\{v_m\} \subset \mathcal{H}_{\varepsilon_m}$ such that $\lim_{m \to 0} \varepsilon_m = 0$,

$$\int\limits_{\Omega_0} v_m^2 dx = 1, \tag{9}$$

$$\int\limits_{\Omega_{\varepsilon_m}} |\nabla v_m|^2 dx + \varepsilon_m \sum_{i=1}^{2} k_i \int\limits_{\Upsilon_{\varepsilon_m}^{(i)}} v_m^2 dx_2 < \frac{1}{m}. \tag{10}$$

Since the sequence $\{v_m\}$ is bounded in $H^1(\Omega_0)$, we may assume without loss of generality that it is a Cauchy sequence in $L^2(\Omega_0)$. From inequality (10) it follows that $\{v_m\}$ is a Cauchy sequence also in $H^1(\Omega_0) : \|v_m - v_n\|^2_{H^1(\Omega_0)} \le \|v_m - v_n\|^2_{L_2(\Omega_0)} + + \frac{1}{m} + \frac{1}{n}$. Hence, $\{v_m\}$ converges to some element $v_0 \in H^1(\Omega_0)$. Obviously, $v_0 \equiv \text{const}$ in $H^1(\Omega_0)$. Due to (9), $v_0 = |\Omega_0|^{-1/2}$, where $|\Omega_0|$ denotes the measure of the domain Ω_0. Then, the sequence of the traces of $\{v_m\}$ converges to v_0 in $L^2(\partial\Omega_0)$ as well and it is easy to verify that

$$\int\limits_{I_0(\varepsilon_m)} v_m^2(x_1,0)dx_1 = \sum_{i=1}^{2} \int\limits_{I_0} \chi_i(x_1/\varepsilon_m) v_m^2(x_1,0)dx_1 \to$$

$$\to \sum_{i=1}^{2} h_i \int\limits_{I_0} v_0^2(x_1,0)dx_1 = (h_1+h_2)\,|\Omega_0|^{-1} a \neq 0 \quad \text{as} \quad m\to\infty, \tag{11}$$

where $I_0(\varepsilon) := I_0 \cap \Omega_\varepsilon$ and $\chi_i(\cdot)$ is 1-periodic function such that

$$\chi_i(t) = \begin{cases} 1, & t \in [b_i - h_i/2, b_i + h_i/2]\,, \\ \\ 0, & t \in [0,1] \setminus [b_i - h_i/2, b_i + h_i/2]\,, \end{cases} \qquad i = 1,2. \tag{12}$$

Obviously, that $\chi_i(x_1/\varepsilon) \to \int_0^1 \chi_i(t)dt = h_i$ weakly in $L^2(0,a)$ as $\varepsilon \to 0$.

On the other hand, from (6) and (10) it follows that $\|v_m\|^2_{H^1\left(G^{(1)}_{\varepsilon_m} \cup G^{(2)}_{\varepsilon_m}\right)} \le \dfrac{c_6}{m}$ and, therefore, $\displaystyle\int_{I_0(\varepsilon_m)} v_m^2(x_1,0)dx_1 \le \frac{c_7}{m}$, where the constants c_6, c_7 are independent of m. This means that

$$\int\limits_{I_0(\varepsilon_m)} v_m^2(x_1,0)dx_1 \to 0 \quad \text{as} \;\; m \to \infty. \tag{13}$$

However (13) is at variance with (11). This contradiction establishes estimate (8).

Thus, by virtue of (8) and (7), we obtain the left inequality in (4).

The lemma is proved.

Definition 1. *A number $\lambda(\varepsilon)$ is called an eigenvalue of problem* (1) *if there exists a function $u(\varepsilon,\cdot) \in \mathcal{H}_\varepsilon \setminus \{0\}$ such that for all functions $\varphi \in \mathcal{H}_\varepsilon$ the following integral identity:*

$$\langle u, \varphi\rangle_{\varepsilon,k_1,k_2} = \lambda(\varepsilon)(u,\varphi)_{\Omega_\varepsilon} \tag{14}$$

holds. The function $u(\varepsilon,\cdot)$ is called the eigenfunction that corresponds to $\lambda(\varepsilon)$.

Define the operator $A_\varepsilon : \mathcal{H}_\varepsilon \longmapsto \mathcal{H}_\varepsilon$ by the following equality

$$\langle A_\varepsilon u, v\rangle_{\varepsilon,k_1,k_2} = (u,v)_{\Omega_\varepsilon} \quad \forall u,v \in \mathcal{H}_\varepsilon. \tag{15}$$

It is easy to verify that A_ε is self-adjoint, positive, compact, and the spectral problem (1) is equivalent to the spectral problem $A_\varepsilon u = \lambda^{-1}(\varepsilon)u \quad$ in $\mathcal{H}_\varepsilon$. Due to Lemma 1, there exist positive constants C_1 and ε_0 such that for all $\varepsilon \in (0,\varepsilon_0)$ $\|A_\varepsilon\| \le C_1$. Therefore,

$$C_1^{-1} \le \lambda_n(\varepsilon) \quad \forall\, n \in \mathbb{N}. \tag{16}$$

Denote by D_i the rectangle $\{x : x_1 \in (0,a),\, x_2 \in (-d_i, 0)\}$ which is filled up by the thin rods $G_j^{(i)}(\varepsilon)$, $j = 0,1,\ldots,N-1$, in the limit passage as $\varepsilon \to 0$ $(N \to +\infty)$; $i = 1,2$. Let $\mathcal{L}_n(\widetilde{\phi}_1,\ldots,\widetilde{\phi}_n)$ be the n-dimensional subspace of $\mathcal{H}_\varepsilon$ that is spanned on n linearly independent functions $\widetilde{\phi}_k$, $k = 1,\ldots,n$, such that $\widetilde{\phi}_k = 0$ in $\Omega_0 \cup G_\varepsilon^{(1)}$ and $\widetilde{\phi}_k = \phi_k$ in $G_\varepsilon^{(2)}$, where $\phi_1,\ldots,\phi_n$ are orthonormal in $L^2(D_2)$ eigenfunctions of a mixed boundary-value problem for the Laplace operator in the rectangle D_2 with the Neumann conditions on the vertical sides and the Dirichlet conditions on the horizontal ones. Denote by $\{\mu_n\}$ the corresponding eigenvalues of this problem. By virtue of the minimax principle for eigenvalues and Lemma 1, we have

$$\lambda_n(\varepsilon) = \min_{E\in\mathbf{E}_n} \max_{v\in E,\, v\neq 0} \frac{\|v\|^2_{\varepsilon,k_1,k_2}}{\|v\|^2_{\Omega_\varepsilon}} \leq$$

$$\leq C_2^2 \min_{E\in\mathbf{E}_n} \max_{v\in E,\, v\neq 0} \left(\frac{\displaystyle\int_{\Omega_\varepsilon} |\nabla v|^2\, dx}{\displaystyle\int_{\Omega_\varepsilon} v^2 dx_2} + 1 \right) \leq$$

$$\leq C_3 \max_{0\neq v\in\mathcal{L}_n} \left(\frac{\displaystyle\int_{\Omega_\varepsilon} |\nabla v|^2\, dx}{\displaystyle\int_{\Omega_\varepsilon} v^2 dx_2} + 1 \right) = C_3 \left(\mu_n \max_{0\neq v\in\mathcal{L}_n} \frac{\displaystyle\int_{D_2} v^2 dx}{\displaystyle\int_{G^{(2)}_\varepsilon} v^2 dx_2} + 1 \right).$$

Here $\mathbf{E_n}$ is a set of all subspaces of $\mathcal{H}_\varepsilon$ with dimension n. By the same arguments as we have proved (8), we can show that for ε small enough

$$\max_{0\neq v\in\mathcal{L}_n} \frac{\displaystyle\int_{D_2} v^2 dx}{\displaystyle\int_{G^{(2)}(\varepsilon)} v^2 dx_2} \leq C_4.$$

Thus, for any fixed $n\in\mathbb{N}$ there exists a constant $C_1(n)$ such that for ε small enough, we have

$$\lambda_n(\varepsilon) \leq C_1(n). \tag{17}$$

From (3), (14), Lemma 1 and (17) it follows that

$$\|u_n(\varepsilon,\cdot)\|_{H^1(\Omega_\varepsilon)} \leq C_2(n). \tag{18}$$

3. Formal asymptotics of the solution on the thin rods. ***3.1. Outer expansions.*** Because of (16) – (18), we seek the leading terms for $\lambda_n(\varepsilon)$ in the form

$$\lambda(\varepsilon) \approx \mu_0 + \varepsilon\mu_1 + \ldots, \tag{19}$$

and for the corresponding eigenfunction $u_n(\varepsilon,\cdot)$, restricted to Ω_0, in the form

$$u(\varepsilon,x) \approx v_0^+(x) + \sum_{k=1}^{\infty} \varepsilon^k v_k^+(x,\varepsilon), \tag{20}$$

and, restricted to each thin rod $G_j^{(i)}(\varepsilon)$, in the form

$$u(\varepsilon,x) \approx v_0^{i,-}(x) + \sum_{k=1}^{\infty} \varepsilon^k v_k^{i,-}(x,\xi_1 - j), \qquad \xi_1 = \varepsilon^{-1}x_1, \quad j = 0,\ldots,N-1, \quad i = 1,2. \tag{21}$$

Hereafter the index n is omitted. The expansions (20) and (21) are usually called *outer expansions*. Substituting series (20) and (19) in the equation of problem (1), in the boundary conditions on $\partial\Omega_0$ and collecting coefficients of the same powers of ε, we get the following relations for function v_0^+ and number μ_0 :

$$-\Delta_x\ v_0^+(x) = \mu_0 v_0^+(x), \quad x \in \Omega_0,$$

$$\partial_{x_1}^p v_0^+(0, x_2) = \partial_{x_1}^p v_0^+(a, x_2), \quad x_2 \in [0, \gamma(0)], \quad p = 0, 1, \tag{22}$$

$$\partial_\nu v_0^+(x) = 0, \quad x \in \Gamma_\gamma.$$

Now we find limiting relations in the rectangle $D_i,\ i = 1, 2$. Assuming for the moment that the functions $v_k^{i,-}$ in (21) are smooth, we write their Taylor series with respect to the x_1 at the point $x_1 = \varepsilon(j + b_i)$ and pass to the "fast" variable $\xi_1 = \varepsilon^{-1} x_1$. Then (20) takes the form

$$u(\varepsilon, x) \approx v_0^{i,-}\left(\varepsilon(j + b_i), x_2\right) + \sum_{k=1}^{+\infty} \varepsilon^k V_k^{i,j}(\xi_1, x_2), \quad x \in G_j^{(i)}(\varepsilon), \tag{23}$$

where

$$V_k^{i,j}(\xi_1, x_2) = v_k^{i,-}\left(\varepsilon(j + b_i), x_2, \xi_1 - j\right) +$$

$$+ \sum_{m=1}^{k} \frac{(\xi_1 - j - b_i)^m}{m!} \frac{\partial^m v_{k-m}^{i,-}}{\partial x_1^m}\left(\varepsilon(j + b_i), x_2, \xi_1 - j\right). \tag{24}$$

Let us substitute μ_0 and (23) into (1) instead of $\lambda(\varepsilon)$ and $u(\varepsilon, \cdot)$ respectively. Since the Laplace operator takes the form $\Delta_x = \varepsilon^{-2} \dfrac{\partial^2}{\partial \xi_1^2} + \dfrac{\partial^2}{\partial x_2^2}$, the collection of coefficients of the same power of ε gives us one dimensional boundary value problems with respect to ξ_1.

The first problem is the following:

$$\partial_{\xi_1 \xi_1}^2 V_1^{i,j}(\xi_1, x_2) = 0, \quad \xi_2 \in I_{h_i}(b_i), \qquad \partial_{\xi_1} V_1^{i,j}(b_i \pm h_i/2, x_2) = 0, \tag{25}$$

where $\partial_{\xi_1} = \dfrac{\partial}{\partial \xi_1}$, $\partial_{\xi_1 \xi_1}^2 = \dfrac{\partial^2}{\partial \xi_1^2}$. From (25) it follows that function $V_1^{i,j}$ doesn't depend on ξ_1. We restrict ourselves to the leading term of the asymptotics and set $V_1^{i,j} \equiv 0$. Then, due to (24), we have

$$v_1^{i,-}\big(\varepsilon(j + b_i), x_2, \xi_1 - j\big) = -\partial_{x_1} v_0^{i,-}\big(\varepsilon(j + b_i), x_2\big)\big(\xi_1 - j - b_i\big).$$

The problem for the function $V_2^{i,j}$ is as follows:

$$-\partial_{\xi_1 \xi_1}^2 V_2^{i,j}(\xi_1, x_2) =$$

$$= \partial_{x_2 x_2}^2 v_0^{i,-}(\varepsilon(j + b_i), x_2) + \mu_0 v_0^{i,-}(\varepsilon(j + b_i), x_2), \quad \xi_1 \in I_{h_i}(b_i), \tag{26}$$

$$\partial_{\xi_1} V_2^{i,j}\left(b_i \pm h_i/2, x_2\right) = \pm k_i v_0^{i,-}\left(\varepsilon(j + b_i), x_2\right).$$

The solvability condition for problem (26) is given by the differential equation

$$-h_i \partial_{x_2 x_2}^2 v_0^{i,-}\left(\varepsilon(j + b_i), x_2\right) + 2k_i v_0^{i,-}\left(\varepsilon(j + b_i), x_2\right) = h_i \mu_0 v_0^{i,-}\left(\varepsilon(j + b_i), x_2\right). \tag{27}$$

Due to the Neumann conditions for the eigenfunction $u(\varepsilon, \cdot)$ on the bases $\Theta^{(i)}(\varepsilon)$, we must require from $v_0^{i,-}$ to satisfy the following condition:

$$\partial_{x_2} v_0^{i,-}\left(\varepsilon(j + b_i), -d_i\right) = 0. \tag{28}$$

To find conditions in points of the joint zone I_0, we use the method of matched asymptotic expansions for the outer expansions (20), (21) and an inner expansion that is constructed in the following subsection.

3.2. Inner expansion. In a neighborhood of the joint zone I_0 we introduce the "rapid" coordinates $\xi = (\xi_1, \xi_2)$, where $\xi_1 = \varepsilon^{-1}x_1$ and $\xi_2 = \varepsilon^{-1}x_2$. The Laplace operator takes the following form $\varepsilon^{-2}\Delta_\xi$ in the coordinates ξ. We seek the leading terms of the inner expansion in a neighborhood of the joint zone I_0 in the form

$$u_\varepsilon(x) \approx v_0^+(x_1,0) + \varepsilon\Big(Z_1\,(x/\varepsilon)\,\partial_{x_1}v_0^+(x_1,0) + Z_2\,(x/\varepsilon)\,\partial_{x_2}v_0^+(x_1,0)\Big) + \ldots, \quad (29)$$

where functions $Z_1(\xi)$ and $Z_2(\xi)$, $\xi \in \Pi$, are 1-periodic with respect to ξ_1. Here Π is the union of semiinfinite strips $\Pi^+ = (0,1)\times(0,+\infty)$, $\Pi^-_{h_1} = I_{h_1}(b_1)\times(-\infty,0]$ and $\Pi^-_{h_2} = I_{h_2}(b_2)\times(-\infty,0]$. Substituting (29) in the differential equation of problem (1) and in the corresponding boundary conditions, collecting the coefficients of the same power of ε, we arrive junction-layer problems for the functions Z_1 and Z_2 :

$$\begin{gathered}
-\Delta_\xi\, Z_i(\xi) = 0, \quad \xi \in \Pi, \\
\partial_{\xi_2} Z_i(\xi_1, 0) = 0, \quad \xi_1 \in (0,1) \setminus \left(I_{h_1}(b_1) \cup I_{h_2}(b_2)\right), \\
\partial_{\xi_1} Z_i(\xi) = -\delta_{1i}, \quad \xi \in \Big(\partial\Pi^-_{h_1} \setminus I_{h_1}(b_1)\Big) \cup \Big(\partial\Pi^-_{h_2} \setminus I_{h_2}(b_2)\Big), \\
\partial^p_{\xi_1} Z_i(0,\xi_2) = \partial^p_{\xi_1} Z_i(1,\xi_2), \quad \xi_2 > 0, \quad p = 0,1.
\end{gathered} \quad (30)$$

The main asymptotic relations for the functions $\{Z_i\}$ can be obtained from general results about the asymptotic behaviour of solutions to elliptic problems in domains with different exits to infinity [6, 30, 31]. The proofs simplify substantially if the polynomial property of the corresponding sesquilinear forms is employed [32]. However, for the domain Π, we can define more exactly the asymptotic relations and detect other properties of the junction-layer solutions Z_1, Z_2 similarly as in the papers [16, 17].

Statement 1. *There exist two solutions* Ξ_1, $\Xi_2 \in H^1_{\sharp,\mathrm{loc}}(\Pi)$ *to the homogeneous problem* (30) $(i = 2)$, *which have the following differentiable asymptotics*:

$$\Xi_1(\xi) = \begin{cases} \xi_2 + \mathcal{O}(\exp(-2\pi\xi_2)), & \xi_2 \to +\infty,\ \xi \in \Pi^+, \\ h_1^{-1}\xi_2 + \alpha_1^{(1)} + \mathcal{O}(\exp(\pi h_1^{-1}\xi_2)), & \xi_2 \to -\infty,\ \xi \in \Pi^-_{h_1}, \\ \alpha_1^{(2)} + \mathcal{O}(\exp(\pi h_2^{-1}\xi_2)), & \xi_2 \to -\infty,\ \xi \in \Pi^-_{h_2}, \end{cases} \quad (31)$$

$$\Xi_2(\xi) = \begin{cases} \xi_2 + \mathcal{O}(\exp(-2\pi\xi_2)), & \xi_2 \to +\infty,\ \xi \in \Pi^+, \\ \alpha_2^{(1)} + \mathcal{O}(\exp(\pi h_1^{-1}\xi_2)), & \xi_2 \to -\infty,\ \xi \in \Pi^-_{h_1}, \\ h_2^{-1}\xi_2 + \alpha_2^{(2)} + \mathcal{O}(\exp(\pi h_2^{-1}\xi_2)), & \xi_2 \to -\infty,\ \xi \in \Pi^-_{h_2}. \end{cases} \quad (32)$$

Any other solution to the homogeneous problem (30)*, which has polynomial grow at infinity, can be presented as a linear combination* $\beta_0 + \beta_1\Xi_1 + \beta_2\Xi_2$.

The solution Z_1 *to problem* (30) *at* $i = 1$ *has the following asymptotics*:

$$Z_1(\xi) = \begin{cases} \mathcal{O}(\exp(-2\pi\xi_2)), & \xi_2 \to +\infty,\ \xi \in \Pi^+, \\ -\xi_1 + b_1 + \alpha_3^{(1)} + \mathcal{O}(\exp(\pi h_1^{-1}\xi_2)), & \xi_2 \to -\infty,\ \xi \in \Pi_{h_1}^-, \\ -\xi_1 + b_2 + \alpha_3^{(2)} + \mathcal{O}(\exp(\pi h_2^{-1}\xi_2)), & \xi_2 \to -\infty, \quad \xi \in \Pi_{h_2}^-. \end{cases} \tag{33}$$

Here $H^1_{\sharp,\mathrm{loc}}(\Pi) = \{u : \Pi \to \mathbb{R} \mid u(0,\xi_2) = u(1,\xi_2)$ *for any* $\xi_2 > 0,\ u \in H^1(\Pi_R)$ *for any* $R > 0\}$, *where* $\Pi_R = \Pi \cap \{\xi : -R < \xi_2 < R\}$; $\alpha_1^{(i)}, \alpha_2^{(i)}, \alpha_3^{(i)}$, $i = 1, 2$, *are some fixed constants.*

Now we verify the matching conditions for the outer expansions (20), (21) and the inner expansion (29), namely, the leading terms of the asymptotics of the outer expansions as $x_2 \to \pm 0$ must coincide with the leading terms of the inner expansion as $\xi_2 \to \pm\infty$ respectively. Near the point $(\varepsilon(j + b_i), 0) \in I_0$ the function v_0^+ has the following asymptotics:

$$v_0^+(\varepsilon(j+b_i),0) + \varepsilon\, \xi_2 \partial_{x_2} v_0^+(\varepsilon(j+b_i),0) + \mathcal{O}(\varepsilon^2\xi_2^2), \quad x_2 \to 0+0.$$

We see that the matching condition is satisfied for the expansions (20) and (29) if $Z_2 = \beta_1\Xi_1 + (1-\beta_1)\Xi_2$.

The asymptotics of (21) is equal to

$$\begin{gathered} v_0^{i,-}(\varepsilon(j+b_i),0) + \\ +\varepsilon\Big((-\xi_1 + b_i + j)\, \partial_{x_1} v_0^{i,-}(\varepsilon(j+b_i),0) + \xi_2 \partial_{x_2} v_0^{i,-}(\varepsilon(j+b_i),0) \Big) + \ldots \\ \text{as} \quad x_2 \to 0-0, \quad x \in G_j^{(i)}(\varepsilon), \quad i = 1, 2. \end{gathered} \tag{34}$$

The asymptotics of (29) is equal to

$$\begin{gathered} v_0^+(\varepsilon(j+b_1),0) + \varepsilon\Big(\Big(-\xi_1 + j + b_1 + \alpha_3^{(1)}\Big)\, \partial_{x_1} v_0^+(\varepsilon(j+b_1),0) + \\ + \Big\{ \beta_1 \Big(h_1^{-1}\xi_2 + \alpha_1^{(1)} \Big) + (1-\beta_1)\, \alpha_2^{(1)} \Big\}\, \partial_{x_2} v_0^+(\varepsilon(j+b_1),0) \Big) + \ldots \\ \text{as} \quad \xi_2 \to -\infty, \quad \xi \in \Pi_{h_1}^-, \end{gathered} \tag{35}$$

and it is equal to

$$\begin{gathered} v_0^+(\varepsilon(j+b_2),0) + \varepsilon\Big(\Big(-\xi_1 + j + b_2 + \alpha_3^{(2)}\Big)\, \partial_{x_1} v_0^+(\varepsilon(j+b_2),0) + \\ + \Big\{ (1-\beta_1) \Big(h_1^{-1}\xi_2 + \alpha_2^{(2)} \Big) + \beta_1 \alpha_1^{(2)} \Big\}\, \partial_{x_2} v_0^+(\varepsilon(j+b_2),0) \Big) + \ldots \\ \text{as} \quad \xi_2 \to -\infty, \quad \xi \in \Pi_{h_2}^-. \end{gathered} \tag{36}$$

Comparing the first terms of (34), (35), and (36), we get

$$v_0^+(\varepsilon(j+b_i),0) = v_0^{i,-}(\varepsilon(j+b_i),0), \quad j = 0, 1, \ldots, N-1, \quad i = 1, 2. \tag{37}$$

Comparing the second terms of (34) and (35), and (34) and (36), we find

$$\beta_1 \partial_{x_2} v_0^+(\varepsilon(j+b_1),0) = h_1 \partial_{x_2} v_0^{1,-}(\varepsilon(j+b_1),0), \tag{38}$$

$$(1-\beta_1)\,\partial_{x_2} v_0^{+}(\varepsilon(j+b_2),0) = h_2 \partial_{x_2} v_0^{2,-}(\varepsilon(j+b_2),0), \quad j=0,1,\ldots,N-1. \quad (39)$$

Since the segments $\{x : x_1 = \varepsilon(j+b_i),\ x_2 \in [-d_i, 0]\}$, $j = 0,1,\ldots,N-1$, fill out the rectangle $\overline{D}_i$ in the limit passage as $\varepsilon \to 0$ $(N \to +\infty)$ both for $i=1$ and for $i=2$, we can spread the equation (27) into rectangle $D_1 = I_0 \times (-d_1, 0)$ for $i=1$ and into rectangle D_2 for $i=2$. On the basis of the same arguments, we spread the relations (28), (37), (38), and (39) into all interval I_0. From the limiting relations (38) and (39) it follows that

$$\partial_{x_2} v_0^{+}(x_1,0) = h_1 \partial_{x_2} v_0^{1,-}(x_1,0) + h_2 \partial_{x_2} v_0^{2,-}(x_1,0), \quad x_1 \in I_0.$$

Now define the following vector function:

$$\mathbf{v}_0(x) = \begin{cases} v_0^{+}(x), & x \in \Omega_0, \\ v_0^{1,-}(x), & x \in D_1, \\ v_0^{2,-}(x), & x \in D_2. \end{cases} \quad (40)$$

As follows from the foregoing the components of this function must satisfy the relations

$$\begin{gathered} -\Delta_x\, v_0^{+}(x) = \mu_0 v_0^{+}(x), \quad x \in \Omega_0, \\ \partial_{x_1}^{p} v_0^{+}(0,x_2) = \partial_{x_1}^{p} v_0^{+}(a,x_2), \quad p=0,1, \quad x_2 \in [0,\gamma(0)], \\ \partial_\nu v_0^{+}(x) = 0, \quad x \in \Gamma_\gamma, \\ -h_1 \partial^2_{x_2x_2} v_0^{1,-}(x) + 2k_1 v_0^{1,-}(x) = h_1 \mu_0 v_0^{1,-}(x), \quad x \in D_1, \\ \partial_{x_2} v_0^{1,-}(x_1,-d_1) = 0, \quad x_1 \in I_0, \\ -h_2 \partial^2_{x_2x_2} v_0^{2,-}(x) + 2k_2 v_0^{2,-}(x) = h_2 \mu_0 v_0^{2,-}(x), \quad x \in D_2, \\ \partial_{x_2} v_0^{2,-}(x_1,-d_2) = 0, \quad x_1 \in I_0, \\ v_0^{+}(x_1,0) = v_0^{i,-}(x_1,0), \quad i=1,2, \quad x_1 \in I_0, \\ h_1 \partial_{x_2} v_0^{1,-}(x_1,0) + h_2 \partial_{x_2} v_0^{2,-}(x_1,0) = \partial_{x_2} v_0^{+}(x_1,0), \quad x_1 \in I_0. \end{gathered} \quad (41)$$

These relations form the spectral limiting problem for problem (1); here μ_0 is the spectral parameter. Let us investigate its spectrum.

4. The resulting limit problem and its spectrum. Denote by $\mathcal{V}_0$ the vector-space $L^2(\Omega_0) \times L^2(D_1) \times L^2(D_2)$ with the following scalar product:

$$(\mathbf{u},\mathbf{v})_{\mathcal{V}_0} = \int\limits_{\Omega_0} u_0 v_0\, dx + \sum_{i=1}^{2} h_i \int\limits_{D_i} u_i v_i\, dx,$$

where $\mathbf{u} = (u_0, u_1, u_2)$ and $\mathbf{v} = (v_0, v_1, v_2)$ belong to $\mathcal{V}_0$. Also we define the Hilbert space $\mathcal{H}_0 = \{\mathbf{u} \in \mathcal{V}_0 : u_0 \in H^1(\Omega_0),\ u_0(0,x_2) = u_0(a,x_2)$ for $x_2 \in (0,\gamma(0))$; $\exists\, \partial_{x_2} u_1 \in L^2(D_1)$; $\exists\, \partial_{x_2} u_2 \in L^2(D_2)$; $u_0(x_1,0) = u_1(x_1,0) = u_2(x_1,0)$ for $x_1 \in I_0\}$ with the following scalar product:

$$(\mathbf{u},\mathbf{v})_{\mathcal{H}_0} = \int\limits_{\Omega_0} \nabla u_0 \cdot \nabla v_0 dx + \sum_{i=1}^{2} \int\limits_{D_i} \Big(h_i \partial_{x_2} u_i \partial_{x_2} v_i + 2k_i u_i v_i \Big) dx.$$

Obviously, $\mathcal{H}_0$ continuously embeds in $\mathcal{V}_0$. If we define the operator $A_0 : \mathcal{H}_0 \longmapsto \mathcal{H}_0$ by the following equality:

$$(A_0\mathbf{u},\mathbf{v})_{\mathcal{H}_0} = (\mathbf{u},\mathbf{v})_{\mathcal{V}_0} \quad \forall\, \mathbf{u},\mathbf{v} \in \mathcal{H}_0, \tag{42}$$

then problem (41) is equivalent to the spectral problem $A_0\mathbf{v}_0 = \mu_0^{-1}\mathbf{v}_0$ in $\mathcal{H}_0$. It is easy to verify that A_0 is self-adjoint, positive, continuous, noncompact and $0 \notin \sigma(A_0)$. Thus $\sigma(A_0) \subset (c_0, +\infty)$, where c_0 is some positive constant.

Next we assume that $c_0 \geq \max\left(\dfrac{2k_1}{h_1}, \dfrac{2k_2}{h_2}\right)$; the other cases we will be discussed in Remark 2. Solving the ordinary differential equations of problem (41) in the rectangles D_1 and D_2 with regard of the first conjugation condition in the joint zone I_0 and the Neumann conditions on the opposite sides of these rectangles, we get

$$v_0^{i,-}(x) = \frac{v_0^+(x_1,0)}{\cos\left(d_i\sqrt{\mu_0 - 2k_i h_i^{-1}}\right)} \cos\left(\sqrt{\mu_0 - 2k_i h_i^{-1}}(x_2 + d_i)\right), \quad i = 1,2. \tag{43}$$

Substituting these relations into the second conjugation condition, we obtain the following spectral problem:

$$-\Delta_x\, v_0^+(x) = \mu_0 v_0^+(x), \quad x \in \Omega_0,$$

$$\partial_{x_1}^p v_0^+(0,x_2) = \partial_{x_1}^p v_0^+(a,x_2), \quad x_2 \in [0,\gamma(0)], \quad p = 0,1,$$

$$\partial_\nu v_0^+(x) = 0, \quad x \in \Gamma_\gamma, \tag{44}$$

$$\partial_{x_2} v_0^+(x_1,0) =$$
$$= -v_0^+(x_1,0)\sum_{i=1}^{2} h_i\sqrt{\mu_0 - 2k_i h_i^{-1}} \tan\left(d_i\sqrt{\mu_0 - 2k_i h_i^{-1}}\right), \quad x_1 \in I_0,$$

with the spectral parameter μ_0 occurring both in the differential equation and in the boundary condition on I_0, where it enters in a nonlinear way. Problem (44) is called *the resulting problem* for problem (1).

Multiplying the differential equation of problem (44) with an arbitrary function $\psi \in H^1_{\sharp,x_1}(\Omega_0) = \{u \in H^1(\Omega_0) :\ u \text{ is 1-periodic with respect to } x_1\}$ and integrating by parts in Ω_0, we reduce the nonlinear spectral problem (44) to the spectral problem

$$L(\mu)v_0^+ = 0 \quad \text{in} \quad H^1_{\sharp,x_1}(\Omega_0), \quad \mu \in [c_0, +\infty),$$

for the following operator-function:

$$L(\mu) := (\mu + 1)A_1 + \sum_{i=1}^{2} h_i\sqrt{\mu - \frac{2k_i}{h_i}} \tan\left(d_i\sqrt{\mu - \frac{2k_i}{h_i}}\right) A_2 - \mathbb{I}, \tag{45}$$

where $\mathbb{I}$ is the identity operator in $H^1_{\sharp,x_1}(\Omega_0)$; A_1, A_2 are self-adjoint, compact operators in $H^1_{\sharp,x_1}(\Omega_0)$ such that for all $\varphi, \psi \in H^1_{\sharp,x_1}(\Omega_0)$

$$(A_1\varphi,\psi)_{H^1(\Omega_0)} = \int\limits_{\Omega_0} \varphi(x)\psi(x)dx,$$

$$(A_2\varphi, \psi)_{H^1(\Omega_0)} = \int_0^a \varphi(x_1, 0)\ \psi(x_1, 0)dx_1.$$

Theorems on existence and concentration of the spectrum for such self-adjoint discontinuous operator-functions and minimax principles for the eigenvalues were proved in [33, 34]. From these results it follows the following theorem.

Theorem 1. *The spectrum of L consists of normal eigenvalues and points $\{P_m : m \in \mathbb{N}\}$ of the essential spectrum, which are poles of the functions*

$$\tan\left(d_i\sqrt{\mu - 2k_i h_i^{-1}}\right), \quad i = 1, 2, \quad \mu \in (c_0, +\infty).$$

These points divide the eigenvalues into the sequences

$$c_0 < \mu_1^{(1)} \le \ldots \le \mu_n^{(1)} \le \ldots \to P_1,$$

$$P_{m-1} < \mu_1^{(m)} \le \ldots \le \mu_n^{(m)} \le \ldots \to P_m \quad as \quad n \to \infty.$$

We recall that an eigenvalue is called normal eigenvalue if it has finite multiplicity and the corresponding eigenvectors have no Jordan chain.

***Remark* 2.** Consider for example the case $\dfrac{2k_1}{h_1} \le c_0 < \dfrac{2k_2}{h_2}$. Then $v_0^{1,-}$ is represented by (43) and

$$v_0^{2,-}(x) = \frac{v_0^+(x_1, 0)}{\cosh\left(d_i\sqrt{2k_2h_2^{-1} - \mu_0}\right)} \cosh\left(\sqrt{2k_2h_2^{-1} - \mu_0}(x_2 + d_2))\right).$$

Using these representations, we similarly as before reduce problem (41) to the nonlinear spectral problem for the following operator-function:

$$L(\mu) := (\mu + 1)A_1 + \left(h_1\sqrt{\mu - \frac{2k_1}{h_1}}\tan\Big(d_1\sqrt{\mu - \frac{2k_1}{h_1}}\Big) + \right.$$

$$\left. + h_2\sqrt{\frac{2k_2}{h_2} - \mu}\tanh\Big(d_2\sqrt{\frac{2k_2}{h_2} - \mu}\Big)\right)\ A_2 - \mathbb{I}, \quad \mu \in \left(c_0, \frac{2k_2}{h_2}\right).$$

It follows from [33, 34] that the spectrum of L on $\left(c_0, \dfrac{2k_2}{h_2}\right)$ consists of normal eigenvalues and points of the essential spectrum, which are poles of function $\tan\left(d_1\sqrt{\mu - 2k_1h_1^{-1}}\right)$ on $\left(c_0, \dfrac{2k_2}{h_2}\right)$. In addition, the points of the essential spectrum are left accumulation points of the normal eigenvalues. Thus, in fact, Theorem 1 describes structure of the spectrum of problem (41) in all cases.

5. Asymptotic approximations. ***5.1. The case of the discrete spectrum.*** Let μ_0 be an eigenvalue of the limiting problem (41) and $\mathbf{v}_0 = (v_0^+, v_0^{1,-}, v_0^{2,-})$ is the corresponding eigenfunction, i.e., v_0^+ is the eigenfunction of problem (44) and $v_0^{i,-}$, $i = 1, 2$, are defined by (43). With the help of $\mathbf{v}_0$ and the junction-layer solutions Z_1, Ξ_1, Ξ_2 (see Section 3), we define the leading terms in (20), (21), and (29). Then matching these expansions, we construct an asymptotic approximation R_ε belonging to $\mathcal{H}_\varepsilon$. It is equal to

$$R^+_\varepsilon(x) := v^+_0(x) + \varepsilon\chi_0(x_2)\mathcal{N}^+\left(\frac{x}{\varepsilon}, x_1\right), \quad x \in \Omega_0, \tag{46}$$

$$R^{i,-}_\varepsilon := v^{i,-}_0(x) + \varepsilon\left(Y_1\left(\frac{x_1}{\varepsilon}\right)\partial_{x_1}v^{i,-}_0(x) + \chi_0(x_2)\mathcal{N}^-\left(\frac{x}{\varepsilon}, x_1\right)\right), \tag{47}$$

$$x \in G^{(i)}_\varepsilon, \quad i = 1, 2.$$

Here

$$\mathcal{N}^+(\xi, x_1) = Z_1(\xi)\partial_{x_1}v^+_0(x_1, 0) + \big(\beta_1\Xi_1(\xi) + (1-\beta_1)\big(\Xi_2(\xi) - \xi_2\big)\big)\,\partial_{x_2}v^+_0(x_1, 0),$$

$$\mathcal{N}^-(\xi, x_1) = \left(Z_1(\xi) - Y_1(\xi_1)\right)\partial_{x_1}v^+_0(x_1, 0) +$$

$$+\big(\beta_1\Xi_1(\xi) + (1-\beta_1)\big(\Xi_2(\xi) - Y_2(\xi_2)\big)\big)\partial_{x_2}v^+_0(x_1, 0),$$

where $\xi = x/\varepsilon$, Y_1 and Y_2 are 1-periodic functions with respect to ξ_1 and on the corresponding cells of periodicity they are equal to

$$Y_1(\xi_1) = \begin{cases} -\xi_1 + b_1 + \alpha^{(1)}_3, & \xi_1 \in [0, \delta_0), \\ -\xi_1 + b_2 + \alpha^{(2)}_3, & \xi_1 \in [\delta_0, 1), \end{cases}$$

$$Y_2(\xi_2) = \begin{cases} \beta_1(h_1^{-1}\xi_2 + \alpha^{(1)}_1) + (1-\beta_1)\alpha^{(1)}_2, & \xi \in \Pi^-_{h_1}, \\ \beta_1\alpha^{(2)}_1 + (1-\beta_1)(h_2^{-1}\xi_2 + \alpha^{(2)}_2), & \xi \in \Pi^-_{h_2}, \end{cases}$$

the number β_1 is defined from relation (38), (39) and it is equal to

$$\beta_1 = \frac{h_1\sqrt{\mu_0 - \dfrac{2k_1}{h_1}}\ \tan\left(d_1\sqrt{\mu_0 - \dfrac{2k_1}{h_1}}\right)}{\sum_{i=1}^2 h_i\sqrt{\mu_0 - \dfrac{2k_i}{h_i}}\ \tan\left(d_i\sqrt{\mu_0 - \dfrac{2k_i}{h_i}}\right)},$$

the function χ_0 is a smooth cut-off function such that $\chi_0(x_2) = 1$ for $|x_2| \le \alpha_0/2$ and $\chi_0(x_2) = 0$ for $|x_2| \ge \alpha_0$, where $0 < \alpha_0 < 2^{-1}\min\{d_1, d_2, \min_{[0,a]}\gamma(x)\}$.

5.1.1. Discrepancies in the domain Ω_0. Taking into account the properties of the functions Z_1, Ξ_1, Ξ_2 and v^+_0, we conclude that R^+_ε is a-periodic with respect to x_1, $\partial_\nu R^+_\varepsilon = 0$ on Γ_γ, and $\partial_{x_2}R^+_\varepsilon(x_1, 0) = 0$ for any $x_1 \in I_0 \setminus I_0(\varepsilon)$. Thus R^+_ε satisfies all boundary conditions for problem (1) on $\partial\Omega_0 \cap \partial\Omega_\varepsilon$. Putting R^+_ε and μ_0 in the equation of problem (1), we get

$$-\Delta_x R^+_\varepsilon - \mu_0 R^+_\varepsilon =$$

$$= \Big(-\chi_0'\partial_{\xi_2}\mathcal{N}^+(\xi, x_1) - \chi_0\partial^2_{x_1\xi_1}\mathcal{N}^+(\xi, x_1) - \varepsilon\partial_{x_2}\left(\chi_0'\mathcal{N}^+(x/\varepsilon, x_1)\right) -$$

$$-\varepsilon\chi_0\partial_{x_1}\left(\left(\partial_{x_1}\mathcal{N}^+(\xi, x_1)\right)|_{\xi=x/\varepsilon}\right) - \varepsilon\mu_0\chi_0\mathcal{N}^+(\xi, x_1)\Big)\Big|_{\xi=x/\varepsilon}, \quad x \in \Omega_0. \tag{48}$$

Further, the arguments of functions involved in calculations are indicated only if their absence may cause confusion. We multiply the identity (48) by a test function $\psi \in \mathcal{H}_\varepsilon$ and integrate by parts in Ω_0 :

$$-\int\limits_{I_0(\varepsilon)} \partial_{x_2} R_\varepsilon^+(x_1,0)\psi dx_1 + \int\limits_{\Omega_0} \nabla_x R_\varepsilon^+ \cdot \nabla_x \psi dx - \mu_0 \int\limits_{\Omega_0} R_\varepsilon^+ \psi dx = \sum_{i=1}^{5} I_i^+(\varepsilon,\psi), \quad (49)$$

where

$$I_1^+(\varepsilon,\psi) = -\int\limits_{\Omega_0} \chi_0' \left(\partial_{\xi_2}\mathcal{N}^+(\xi,x_1)\right)\Big|_{\xi=\frac{x}{\varepsilon}} \psi dx,$$

$$I_2^+(\varepsilon,\psi) = -\int\limits_{\Omega_0} \chi_0 \left(\partial^2_{x_1\xi_1}\mathcal{N}^+(\xi,x_1)\right)\Big|_{\xi=\frac{x}{\varepsilon}} \psi dx,$$

$$I_3^+(\varepsilon,\psi) = \varepsilon \int\limits_{\Omega_0} \chi_0' \mathcal{N}^+\left(\frac{x}{\varepsilon},x_1\right)\partial_{x_2}\psi dx,$$

$$I_4^+(\varepsilon,\psi) = \varepsilon \int\limits_{\Omega_0} \chi_0 \left(\partial_{x_1}\mathcal{N}^+(\xi,x_1)\right)\Big|_{\xi=\frac{x}{\varepsilon}} \partial_{x_1}\psi dx,$$

$$I_5^+(\varepsilon,\psi) = -\varepsilon\mu_0 \int\limits_{\Omega_0} \chi_0(x_2)\mathcal{N}^+(\xi,x_1)\Big|_{\xi=\frac{x}{\varepsilon}} \psi dx.$$

5.1.2. Discrepancies in the thin rods. It is easy to calculate that $\partial_{x_2} R_\varepsilon^{i,-}(x_1, -d_i) = 0$,

$$\partial_{x_2} R_\varepsilon^{i,-}(x_1,0) = \varepsilon Y_1\left(\frac{x_1}{\varepsilon}\right)\partial^2_{x_2x_1} v_0^{i,-}(x_1,0) + \partial_{x_2}R_\varepsilon^+(x_1,0), \quad x_1 \in I_0 \cap G_\varepsilon^{(i)}, \quad (50)$$

$$\partial_\nu R_\varepsilon^{i,-}(x) = \pm\varepsilon\left(Y_1\left(\frac{x_1}{\varepsilon}\right)\partial^2_{x_1x_1} v_0^{i,-}(x) + \chi_0(x_2)\left(\partial_{x_1}\mathcal{N}^-(\xi,x_1)\right)\Big|_{\xi=\frac{x}{\varepsilon}}\right), \quad (51)$$

$$x \in \Upsilon_\varepsilon^{(i,\pm)}, \quad i=1,2.$$

Putting $R_\varepsilon^{i,-}$ and μ_0 in the differential equation of problem (1), we obtain

$$-\Delta_x R_\varepsilon^{i,-}(x) - \mu_0 R_\varepsilon^{i,-}(x) =$$

$$= -\chi_0'(x_2)\left(\partial_{\xi_2}\mathcal{N}^-(\xi,x_1)\right)|_{\xi=\frac{x}{\varepsilon}} - \chi_0(x_2)\left(\partial^2_{x_1\xi_1}\mathcal{N}^-(\xi,x_1)\right)|_{\xi=\frac{x}{\varepsilon}} -$$

$$-\varepsilon\partial_{x_2}\left(\chi_0'(x_2)\mathcal{N}^-\left(\frac{x}{\varepsilon},x_1\right)\right) - \varepsilon\chi_0\partial_{x_1}\left(\left(\partial_{x_1}\mathcal{N}^-(\xi,x_1)\right)|_{\xi=\frac{x}{\varepsilon}}\right) -$$

$$-\mathrm{div}\left(Y_1\left(\frac{x_1}{\varepsilon}\right)\nabla_x\left(\partial_{x_1}v_0^{i,-}\right)\right) - \varepsilon\mu_0\left(Y_1\left(\frac{x_1}{\varepsilon}\right)\partial_{x_1}v_0^{i,-}(x) + \chi_0\mathcal{N}^-\left(\frac{x}{\varepsilon},x_1\right)\right) -$$

$$-2k_i h_i^{-1} v_0^{i,-}(x), \quad x \in G_\varepsilon^{(i)}, \quad i=1,2. \quad (52)$$

Using (6) and taking into account the boundary values of $\partial_\nu R_\varepsilon^{i,-}$ (see (60), (51)), we multiply (52) by a test function $\psi \in \mathcal{H}_\varepsilon$ and integrate by parts in $G_\varepsilon^{(i)}$, $i=1,2$. This yields

$$\int\limits_{I_0(\varepsilon)} \partial_{x_2} R_\varepsilon^{+}(x_1,0)\psi dx_1 + \int\limits_{G_\varepsilon^{(i)}} \nabla_x R_\varepsilon^{i,-} \cdot \nabla_x \psi dx +$$

$$+\varepsilon k_i \int\limits_{\Upsilon_\varepsilon^{(i)}} R_\varepsilon^{i,-} \psi dx_2 - \mu_0 \int\limits_{G_\varepsilon^{(i)}} R_\varepsilon^{i,-}(x)\psi dx =$$

$$= I_1^{i,-}(\varepsilon,\psi) + \ldots + I_7^{i,-}(\varepsilon,\psi), \tag{53}$$

where

$$I_1^{i,-} = -\int\limits_{G_\varepsilon^{(i)}} \chi_0' \left(\partial_{\xi_2} \mathcal{N}^-(\xi,x_1)\right)|_{\xi=\frac{x}{\varepsilon}} \psi dx,$$

$$I_2^{i,-} = -\int\limits_{G_\varepsilon^{(i)}} \chi_0 \left(\partial^2_{x_1\xi_1} \mathcal{N}^-(\xi,x_1)\right)|_{\xi=\frac{x}{\varepsilon}} \psi dx,$$

$$I_3^{i,-} = \varepsilon \int\limits_{G_\varepsilon^{(i)}} \chi_0' \mathcal{N}^- \left(\frac{x}{\varepsilon}, x_1\right) \partial_{x_2} \psi dx,$$

$$I_4^{i,-} = \varepsilon \int\limits_{G_\varepsilon^{(i)}} \chi_0 \Big(\partial_{x_1} \mathcal{N}^-(\xi,x_1)\Big)\Big|_{\xi=\frac{x}{\varepsilon}} \partial_{x_1} \psi dx,$$

$$I_5^{i,-}(\varepsilon,\psi) = -\varepsilon\mu_0 \int\limits_{G_\varepsilon^{(i)}} \left(Y_1\left(\frac{x_1}{\varepsilon}\right)\partial_{x_1} v_0^{i,-}(x) + \chi_0 \mathcal{N}^-\left(\frac{x}{\varepsilon}, x_1\right)\right)\psi dx,$$

$$I_6^{i,-}(\varepsilon,\psi) = \varepsilon \int\limits_{G_\varepsilon^{(i)}} Y_1\left(\frac{x_1}{\varepsilon}\right) \nabla_x \left(\partial_{x_1} v_0^{i,-}\right) \cdot \nabla_x \psi dx,$$

$$I_7^{i,-}(\varepsilon,\psi) = k_i\varepsilon \int\limits_{\Upsilon_\varepsilon^{(i)}} R_\varepsilon^{i,-} \psi dx_2 - k_i \varepsilon \int\limits_{\Upsilon_\varepsilon^{(i,\pm)}} v_0^{i,-} \psi dx_2 -$$

$$-2k_i h_i^{-1} \varepsilon \int\limits_{G_\varepsilon^{(i)}} Y\left(\frac{x_1}{\varepsilon}\right) \partial_{x_1}(v_0^{i,-}\psi) dx.$$

Summing (49) and (53), we see that the function R_ε constructed by formulas (46) and (47) satisfies the following integral identity:

$$\int\limits_{\Omega_\varepsilon} \nabla_x R_\varepsilon \cdot \nabla_x \psi dx + \varepsilon \sum_{i=1}^{2} k_i \int\limits_{\Upsilon^{(i)}(\varepsilon)} R_\varepsilon \psi dx_2 - \mu_0 \int\limits_{\Omega_\varepsilon} R_\varepsilon \psi dx = F_\varepsilon(\psi) \quad \forall \psi \in \mathcal{H}_\varepsilon, \tag{54}$$

where

$$F_\varepsilon(\psi) = I_1^{\pm}(\varepsilon,\psi) + \ldots + I_5^{\pm}(\varepsilon,\psi) + I_6^{-}(\varepsilon,\psi) + I_7^{-}(\varepsilon,\psi),$$

$$I_j^{\pm}(\varepsilon,\psi) = I_j^{+}(\varepsilon,\psi) + I_j^{-}(\varepsilon,\psi),$$

$$I_j^{-}(\varepsilon,\psi) = I_j^{1,-}(\varepsilon,\psi) + I_j^{2,-}(\varepsilon,\psi), \quad j = 1,\ldots,7.$$

Using (6), Lemma 1 and doing similar calculations as in the paper [16], we can show that for any positive fixed number δ and for any $\psi \in \mathcal{H}_\varepsilon$ the following inequality $|F_\varepsilon(\psi)| \leq \leq c(\delta)\varepsilon^{1-\delta}\|\psi\|_{\mathcal{H}_\varepsilon}$ holds. Then with the help of the definition of operator A_ε and the Riesz theorem, we deduce from (54) that for any $\delta > 0$

$$\|R_\varepsilon - \mu_0 A_\varepsilon R_\varepsilon\|_{\mathcal{H}_\varepsilon} \leq c(\delta)\varepsilon^{1-\delta}. \tag{55}$$

5.2. The case of the essential spectrum. Let $\mu_0 \in \sigma_{\text{ess}}(A_0)$, i.e., μ_0 coincides with one of the numbers $\{P_m : m \in \mathbb{N}\}$ $\Bigl($they are poles of the functions $\tan\Bigl(d_i\sqrt{\mu - 2k_i h_i^{-1}}\Bigr)$, $\mu \in (c_0, +\infty)$, $i = 1,2$; see Theorem 1$\Bigr)$. For definiteness we assume that $i = 1$. Then we choose the following approximation function:

$$W_\varepsilon(x) =$$

$$= \begin{cases} \sqrt{\dfrac{2}{\varepsilon(h_1+k_1)d_1(\mu_0 - 2k_1h_1^{-1})}}\cos\sqrt{\mu_0 - 2k_1h_1^{-1}}(x_2+d_1), & x \in G_{j_0}^{(1)}(\varepsilon), \\ 0, & x \in \Omega_\varepsilon \backslash G_{j_0}^{(1)}(\varepsilon), \end{cases} \tag{56}$$

where $G_{j_0}^{(1)}(\varepsilon)$ is an arbitrary rod from the first level. It is easy to verify that $\|W_\varepsilon\|_{\mathcal{H}_\varepsilon} = 1$.

Substituting the function W_ε and the number μ_0 in problem (1) instead of $u(\varepsilon,\cdot)$ and $\lambda(\varepsilon)$ respectively, we find residuals and deduce that there exist constants $c > 0$ and ε_0 such that for any values $\varepsilon \in (0,\varepsilon_0)$ the following inequality is satisfied:

$$\|W_\varepsilon - \mu_0 A_\varepsilon W_\varepsilon\|_{\mathcal{H}_\varepsilon} \leq c\,\varepsilon^{\frac{1}{4}}. \tag{57}$$

6. Justification and asymptotic estimates. To justify the constructed asymptotic approximations we use the scheme proposed in [13], where an abstract scheme of investigation of the asymptotic behaviour of eigenvalues and eigenvectors of some family of abstract operators $\{A_\varepsilon : \varepsilon > 0\}$ acting in different spaces was proposed. This scheme generalizes the procedure of justification of the asymptotic behaviour of eigenvalues and eigenvectors of boundary value problems in perturbed domains.

In our case this is the family of the operators $\{A_\varepsilon : \varepsilon > 0\}$ acting in the spaces $\{\mathcal{H}_\varepsilon : \varepsilon > 0\}$ and they are defined by (15). Recall that operator A_ε corresponds to problem (1) and operator $A_0 : \mathcal{H}_0 \longmapsto \mathcal{H}_0$, which is defined by (42) corresponds to the limiting problem (41).

Then we should define special coupling operators P_ε and S_ε. For better understanding, we write the diagram

$$\begin{array}{ccc} \mathcal{H}_\varepsilon & \subset\subset & \mathcal{V}_\varepsilon \\ P_\varepsilon\big\downarrow & & \big\uparrow S_\varepsilon \\ \mathcal{Z}_0 \subset \mathcal{H}_0 & \subset & \mathcal{V}_0 \end{array}$$

in which the imbedding $\mathcal{H} \subset \mathcal{V}$ means that the space $\mathcal{H}$ is densely and only continuously embedded into $\mathcal{V}$, but the imbedding $\mathcal{H} \subset\subset \mathcal{V}$ is compact in addition. Here $\mathcal{Z}_0 =$

$= \{\mathbf{u} = (u_0, u_1, u_2) \in \mathcal{V}_0 : u_0 \in H^1(\Omega_0),\ u_0(0, x_2) = u_0(a, x_2)$ for $x_2 \in (0, \gamma(0))$; $u_1 \in H^1(D_1)$; $u_2 \in H^1(D_2)$; $u_0(x_1, 0) = u_1(x_1, 0) = u_2(x_1, 0)$ for $x_1 \in I_0\}$ is a Hilbert space with the scalar product $(\mathbf{u}, \mathbf{v})_{\mathcal{Z}_0} = (u_0, v_0)_{H^1(\Omega_0)} + (u_1, v_1)_{H^1(D_1)} + $ $+ (u_1, v_1)_{H^1(D_1)}$. Obviously, that $\mathcal{Z}_0 \subset\subset \mathcal{V}_0$.

The operator $S_\varepsilon : \mathcal{V}_0 \mapsto \mathcal{V}_\varepsilon$ assigns to any vector-function $\mathbf{v} = (v_0, v_1, v_2)$ from $\mathcal{V}_0$ a function $S_\varepsilon \mathbf{v}$, which is equal to v_0 in Ω_0 and to $v_i|_{G_\varepsilon^{(i)}}, i = 1, 2$, where $v_i|_{G_\varepsilon^{(i)}}$ is the restriction of v_i on $G_\varepsilon^{(i)}$. It is easy to verify that operator S_ε is uniformly bounded with respect to ε. Thus the condition (C1) in the scheme [13] is satisfied.

The operator P_ε from condition (C2) is associated with special extension operator $\mathbf{P}_\varepsilon = \left(\mathbf{P}_\varepsilon^{(1)}, \mathbf{P}_\varepsilon^{(2)}\right)$, where $\mathbf{P}_\varepsilon^{(1)} : H^1(\Omega_0 \cup G^{(1)}(\varepsilon)) \mapsto H^1(\Omega_1)$ and $\mathbf{P}_\varepsilon^{(2)} : H^1(\Omega_0 \cup$ $\cup\, G^{(2)}(\varepsilon)) \mapsto H^1(\Omega_2)$, where Ω_i is the interior of $\overline{\Omega_0} \cup \overline{D_i}$, $i = 1, 2$. The operators $\mathbf{P}_\varepsilon^{(1)}$ and $\mathbf{P}_\varepsilon^{(2)}$ can be constructed similarly as in [16] (see also [26]). Thus operator $\mathbf{P}_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{Z}_0$ every u from $\mathcal{H}_\varepsilon$ puts in the correspondence a vector-function $\mathbf{u} =$ $= \left(u|_{\Omega_0}, \mathbf{P}_\varepsilon^{(1)} u|_{D_1}, \mathbf{P}_\varepsilon^{(2)} u|_{D_2}\right)$ from $\mathcal{Z}_0$. Despite the fact that the norm of this operator takes an infinitely large value as $\varepsilon \to 0$, the norm of its restriction to an arbitrary finite combination of eigenfunctions of problem (1) is uniformly bounded with respect to ε, i.e., the following statement is true: $\forall n \in \mathbb{N}\ \exists c > 0\ \exists \varepsilon_0 > 0\ \forall \varepsilon \in (0, \varepsilon_0) : \|\mathbf{P}_\varepsilon u_n(\varepsilon, \cdot)\|_{\mathcal{Z}_0} \leq$ $\leq c(n) \|u_n(\varepsilon, \cdot)\|_{\mathcal{H}_\varepsilon}$. Furthermore, this operator is also uniformly bounded on sequences from condition (C2) (the proof of this fact is analogous to the corresponding part of the proof of Theorem 5.4 [18]).

Conditions (C5) and (C6), in fact, have been verified in the previous section. The result of the action of the operator R_ε from the condition (C5) is the construction of the approximation function R_ε (see (46) and (47)) on the basis of an eigenfunction of the limit spectral problem (41). In addition, this approximation function satisfies the estimate (55), which coincides with similar estimate from condition (C5). The estimate (57) coincides with similar estimate from condition (C6). To verify conditions (C3) and (C4) we prove the following theorem.

Theorem 2. *Let* $\{\lambda(\varepsilon) : \ \varepsilon > 0\}$ *be a sequence of eigenvalues of problem* (1) *such that* $\lim\limits_{\varepsilon \to 0} \lambda(\varepsilon) = \Lambda$ *and* $\dfrac{1}{\Lambda} \notin \sigma_{\text{ess}}(A_0)$; *let* $\{u^\varepsilon\}$ *be the corresponding sequence of eigenfunctions such that* $\|u^\varepsilon\|_{L^2(\Omega_\varepsilon)} = 1$ *for any value* ε *and* $\mathbf{P}_\varepsilon u^\varepsilon \to \mathbf{u}^* =$ $= \left(u_0^+, u_0^{1,-}, u_0^{2,-}\right)$ *weakly in* $\mathcal{Z}_0$ *as* $\varepsilon \to 0$.

Then Λ *is the eigenvalue of the limiting problem* (41) *and* $\mathbf{u}^*$ *is the corresponding eigenfunction.*

Proof. Using operator $\mathbf{P}_\varepsilon$ and the functions χ_1 and χ_2 defined in (12), we can rewrite the equality $(u^\varepsilon, u^\varepsilon)_{\Omega_\varepsilon} = 1$ in the following form:

$$1 = \int\limits_{\Omega_0} (u^\varepsilon)^2\, dx + \int\limits_{D_1} \chi_1(x_1/\varepsilon) \left(\mathbf{P}_\varepsilon^{(1)} u^\varepsilon\right)^2 dx + \int\limits_{D_2} \chi_2(x_1/\varepsilon) \left(\mathbf{P}_\varepsilon^{(2)} u^\varepsilon\right)^2 dx.$$

Passing to the limit in this relation as $\varepsilon \to 0$, we obtain $1 = \|\mathbf{u}^*\|^2_{\mathcal{V}_0}$, whence $\mathbf{u}^* \neq 0$.

With the help of the identity (6), the extension operators $\mathbf{P}_\varepsilon^{(i)}$ and the functions χ_i, $i = 1, 2$, we rewrite identity (14) in the following way:

$$\int_{\Omega_0} \nabla u^\varepsilon \cdot \nabla \varphi_0 dx +$$

$$+ \sum_{i=1}^{2} \left(\int_{D_i} \chi_i(x_1/\varepsilon) \nabla \left(\mathbf{P}_\varepsilon^{(i)} u^\varepsilon \right) \cdot \nabla \varphi_i dx + \frac{2k_i}{h_i} \int_{D_i} \chi_i(x_1/\varepsilon) \mathbf{P}_\varepsilon^{(i)} u^\varepsilon \varphi_i dx \right) -$$

$$- 2\varepsilon \sum_{i=1}^{2} \frac{k_i}{h_i} \int_{G_\varepsilon^{(i)}} Y\left(\frac{x_1}{\varepsilon}\right) \partial_{x_1} \left(u^\varepsilon \varphi_i\right) dx =$$

$$= \lambda(\varepsilon) \left(\int_{\Omega_0} u^*(x)\varphi(x) dx + \sum_{i=1}^{2} \int_{D_i} \chi_i(x_1/\varepsilon) \left(\mathbf{P}_\varepsilon^{(1)} u^\varepsilon \right)(x) \varphi_i(x) dx \right) \tag{58}$$

$$\forall\, (\varphi_0, \varphi_1, \varphi_2) \in \mathcal{Z}_0.$$

Obviously, that the last summand in the left-hand side of (58) vanishes as $\varepsilon \to 0$. Now, passing to the limit in (58) and taking the theorem conditions into account, we obtain

$$\int_{\Omega_0} \nabla u_0^+ \cdot \nabla \varphi_0 dx + \sum_{i=1}^{2} \left(\int_{D_i} \sum_{j=1}^{2} \sigma_j^{(i)}(x) \partial_{x_j} \varphi_i(x) dx + 2k_i \int_{D_i} u_0^{i,-} \varphi_i dx \right) =$$

$$= \Lambda \left(\int_{\Omega_0} u_0(x)\varphi_0(x) dx + \sum_{i=1}^{2} h_i \int_{D_i} u_0^{i,-}(x) \varphi_i(x) dx \right) \tag{59}$$

$$\forall\, (\varphi_0, \varphi_1, \varphi_2) \in \mathcal{Z}_0,$$

where $\sigma_j^{(i)}$ is the weak limit of the sequence $\chi_i\left(\frac{x_1}{\varepsilon}\right) \partial_{x_j} \left(\mathbf{P}_\varepsilon^{(i)} u^\varepsilon\right)$, $j = 1, 2,$ $i = 1, 2.$ Next we should find these limits.

In order to determine $\sigma_1^{(i)}$, $i = 1, 2$, we consider the integral identity (14) with the following test functions :

$$\psi_1(x) = \varepsilon \begin{cases} 0, & x \in \Omega_0 \cup G_\varepsilon^{(2)}, \\ Y(x_1/\varepsilon)\phi_1(x), & x \in G_\varepsilon^{(1)}, \end{cases}$$

$$\psi_2(x) = \varepsilon \begin{cases} 0, & x \in \Omega_0 \cup G_\varepsilon^{(1)}, \\ Y(x_1/\varepsilon)\phi_2(x), & x \in G_\varepsilon^{(2)}, \end{cases}$$

where ϕ_1 and ϕ_2 are arbitrary functions from $C_0^\infty(D_1)$ and $C_0^\infty(D_2)$ respectively. It is obvious that ψ_1 and ψ_2 belong to $\mathcal{H}_\varepsilon$. As a result, we get

$$\int_{D_1} \chi_1\left(\frac{x_1}{\varepsilon}\right) \partial_{x_1} \mathbf{P}_\varepsilon^{(1)}(u^\varepsilon) \phi_1 dx = \mathcal{O}(\varepsilon),$$

$$\int\limits_{D_2} \chi_\varepsilon^{(2)}(x)\partial_{x_1}\mathbf{P}_\varepsilon^{(2)}(u_\varepsilon)\phi_2 dx = \mathcal{O}(\varepsilon), \quad \varepsilon \to 0,$$

whence $\sigma_1^{(1)} \equiv 0$ and $\sigma_1^{(2)} \equiv 0$.

Next let us define $\sigma_2^{(i)}, \ i = 1, 2$. Take any function $\phi \in C_0^\infty(D_i)$ and pass to the limit in the following relation:

$$\int\limits_{D_i} \chi_i(x_1/\varepsilon)\partial_{x_2}\left(\mathbf{P}_\varepsilon^{(i)} u^\varepsilon\right)\phi(x)dx = -\int\limits_{D_i} \chi_i(x_1/\varepsilon)\left(\mathbf{P}_\varepsilon^{(i)} u^\varepsilon\right)\partial_{x_2}\phi dx. \qquad (60)$$

As a result, we get that $\sigma_2^{(i)}(x) = h_i \partial_{x_2} u_0^{i,-}(x)$, $x \in D_i$, $i = 1, 2$.

Thus, we obtain that $\mathbf{u}^*$ satisfies the following identity $(\mathbf{u}^*, \mathbf{v})_{\mathcal{H}_0} = \Lambda\,(\mathbf{u}^*, \mathbf{v})_{\mathcal{V}_0}$ for any vector-function $\mathbf{v} = (\varphi_0, \varphi_1, \varphi_2) \in \mathcal{Z}_0$. This identity is the corresponding integral identity for the spectral limiting problem (41) (see (42)). This means that Λ is the eigenvalue of problem (41) and $\mathbf{u}^*$ is the corresponding eigenfunction.

The theorem is proved.

Thus, all conditions (C1) – (C6) of the scheme from [13] are satisfied for problems (1) and (41). Applying this scheme, we get the following theorems.

Theorem 3 (the Hausdorff convergence). *Only the points of the spectrum of problem* (41) *are accumulation points for the spectrum of problem* (1) *as* $\varepsilon \to 0$.

The eigenvalues $\{\lambda_n(\varepsilon)\}$ at fixed indices n, are usually called *low eigenvalues* (see [21]); the corresponding eigenfunctions are called *low frequency oscillations*.

Definition 2 [21]. *The value* $\mathcal{T} := \sup_{n\in\mathbf{N}} \overline{\lim}_{\varepsilon\to 0}\lambda_n(\varepsilon)$ *is called the threshold of the low eigenvalues of problem* (1).

Theorem 4 (low-frequency convergence). *Let* $\{\lambda_n(\varepsilon) : n \in \mathbb{N}_0\}$ *be the ordered sequence* (2) *of eigenvalues of problem* (1), *let* $\{u_n(\varepsilon, \cdot) : n \in \mathbb{N}\}$ *be the corresponding sequence of eigenfunction orthonormalized by condition* (3), *and let* $c_0 < \mu_1^{(1)} \le \ldots \ldots \le \mu_n^{(1)} \le \ldots \to P_1$ *be the first series of eigenvalues of the limiting problem* (41) (*see Theorem* 1).

Then the threshold of the low eigenvalues of problem (1) *is equal to* P_1, *and for any* $n \in \mathbb{N}$ $\lambda_n(\varepsilon) \to \mu_n^{(1)}$ *as* $\varepsilon \to 0$. *There exists a subsequence of the sequence* $\{\varepsilon\}$ (*again denoted by* $\{\varepsilon\}$) *such that* $\mathbf{P}_\varepsilon u_n(\varepsilon, \cdot) \to \mathbf{v}_n^{(0)}$ *weakly in* $\mathcal{Z}_0$ *as* $\varepsilon \to 0$, *where* $\{\mathbf{v}_n^{(0)}\}$ *are the corresponding eigenfunctions of the limiting problem* (41) *that satisfy the condition* $\left(\mathbf{v}_n^{(0)}, \mathbf{v}_m^{(0)}\right)_{\mathcal{V}_0} = \delta_{n,m}$.

Theorem 5. *Let* $\mu_n^{(1)} = \mu_{n+1}^{(1)} = \ldots = \mu_{n+r-1}^{(1)}$ *be an* r*-multiple eigenvalue of problem* (41) *from the first series* (*see Theorem* 1) *and let* $\mathbf{v}_n^{(1)}, \ldots, \mathbf{v}_{n+r-1}^{(1)}$ *be the corresponding eigenfunction orthonormalized in* $\mathcal{V}_0$.

Then for any $\delta > 0$ *and* $i \in \{0, 1, \ldots, r-1\}$, *there exist* $\varepsilon_0 > 0, C_i > 0$, *and* $\{\alpha_{ik}(\varepsilon), \, k = 0, 1, \ldots, r-1\} \subset \mathbb{R}$, *such that for any* $\varepsilon \in (0, \varepsilon_0)$:

$$\left\| R_\varepsilon^{(n+i)} - \sum_{k=0}^{r-1} \alpha_{ik}(\varepsilon) u_{n+k}(\varepsilon, \cdot) \right\|_{H^1(\Omega_\varepsilon)} \le C_i(n, \delta)\, \varepsilon^{1-\delta},$$

$$0 < c_1 < \sum_{k=0}^{r-1} (\alpha_{ik}(\varepsilon))^2 < c_2,$$

where $\{R_\varepsilon^{(n+i)}\}$ *is approximation function defined by* (46) *and* (47) *with the help of* $\mathbf{v}_{n+i}^{(1)}$.

For any $\delta > 0$ *and* $n \in \mathbb{N}$ *and sufficiently small* ε, *we have* $|\lambda_n(\varepsilon) - \mu_n^{(1)}| \leq \leq c_0(n,\delta)\varepsilon^{1-\delta}$.

Theorem 6. *Let* $\mu_n^{(m)} = \mu_{n+1}^{(m)} = \ldots = \mu_{n+r-1}^{(m)}$ *be an* r*-multiple eigenvalue of problem* (41) *from the* m*-th series* (*see Theorem* 1) *and* $\mathbf{v}_n^{(m)}, \ldots, \mathbf{v}_{n+r-1}^{(m)}$ *be the corresponding eigenfunction orthonormalized in* $\mathcal{V}_0$.

Then, for any $\delta > 0$, *there exist* $\varepsilon_{n,m} > 0$ *and* $c > 0$ *such that for all value of the parameter* $\varepsilon \in (0, \varepsilon_{n,m})$ *in the interval* $I_{n,m}(\varepsilon) = \left(\mu_n^{(m)} - c\varepsilon^{1-\delta},\, \mu_n^{(m)} + c\varepsilon^{1-\delta}\right)$ *contains exactly* r *eigenvalues of problem* (41).

For the approximation function $R_\varepsilon^{n+i,m}$, $i = 0, 1, \ldots, r-1$, *constructed by* (46) *and* (47) *on the basis of* $\mathbf{v}_{n+i}^{(m)}$, *the following asymptotic estimate is true:*

$$\left\| \frac{R_\varepsilon^{n+i,m}}{\|R_\varepsilon^{n+i,m}\|_{\mathcal{H}_\varepsilon}} - \widetilde{U}_i(\varepsilon,\cdot) \right\|_{\mathcal{H}_\varepsilon} \leq c(n,m,\delta)\varepsilon^{1-\delta}, \qquad \|\widetilde{U}_i(\varepsilon,\cdot)\|_{\mathcal{H}_\varepsilon} = 1,$$

where $\widetilde{U}_i(\varepsilon,\cdot)$ *is a linear combination of the eigenfunctions of problem* (1) *that correspond to the eigenvalues from the interval* $I_{n,m}(\varepsilon)$.

Theorem 7. *Let* μ_0 *coincides with one of the points of the essential spectrum* $\{P_m : m \in \mathbb{N}\}$ *of the limiting problem* (41).

Then there exist $c_0 > 0$ *and* $\varepsilon_0 > 0$ *such that for all values of the parameter* $\varepsilon \in \in (0, \varepsilon_0)$, *the interval* $\left(\dfrac{1}{\mu_0} - c_0\varepsilon^{\frac{1}{4}},\, \dfrac{1}{\mu_0} + c_0\varepsilon^{\frac{1}{4}}\right)$ *contains finitely many eigenvalues of the operator* A_ε.

There exists a finite linear combination $\widetilde{U}_\varepsilon$ $(\|\widetilde{U}_\varepsilon\|_\varepsilon = 1)$ *of the eigenfunction* $u_{k(\varepsilon)+i}^\varepsilon$, $i = \overline{0, p(\varepsilon)}$, *that correspond, respectively, to the eigenvalues* $\left(\lambda_{k(\varepsilon)+i}(\varepsilon)\right)^{-1}$ *of the operator* A_ε *from the segment* $\left[\dfrac{1}{\mu_0} - c_0\varepsilon^{\frac{1}{8}},\, \dfrac{1}{\mu_0} + c_0\varepsilon^{\frac{1}{8}}\right]$, *such that* $\left\|W_\varepsilon - \widetilde{U}_\varepsilon\right\|_{\mathcal{H}_\varepsilon} \leq \leq 2\varepsilon^{\frac{1}{8}}$, *where* W_ε *is defined by* (56).

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