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ON INTERACTION OF AN ELASTIC WALL WITH A POISEUILLE TYPE FLOW

ПРО ВЗАЄМОДІЮ ПРУЖНОЇ СТІНКИ З ПОТОКОМ ПУАЗЕЙЛІВСЬКОГО ТИПУ

We study dynamics of a coupled system consisting of the 3D Navier – Stokes equations which is linearized near a certain Poiseuille type flow in an (unbounded) domain and a classical (possibly nonlinear) elastic plate equation for transversal displacement on a flexible flat part of the boundary. We first show that this problem generates an evolution semigroup S_t on an appropriate phase space. Then under some conditions concerning the underlying (Poiseuille type) flow we prove the existence of a compact finite-dimensional global attractor for this semigroup and also show that S_t is an exponentially stable C_0-semigroup of linear operators in the fully linear case. Since we do not assume any kind of mechanical damping in the plate component, this means that dissipation of the energy in the fluid flow due to viscosity is sufficient to stabilize the system.

Вивчається динаміка зв'язної системи, що складається з тривимірних рівнянь Нав'є – Стокса, які лінеаризовані в околі деякої течії пуазейлівського типу в (необмеженій) області, та класичного (можливо, нелінійного) рівняння для поперечного відхилення пружної пластини на гнучкій частині межі. Показано, що задача породжує еволюційну півгрупу S_t у придатному фазовому просторі. При деяких умовах щодо основної течії встановлено існування компактного скінченновимірного глобального атрактора цієї півгрупи, а також показано, що S_t є експоненціально стійкою C_0-півгрупою лінійних операторів у повністю лінійному випадку. Оскільки не припускається наявність механічного демпфірування у пластині, отримані результати означають, що дисипації енергії в потоці рідини через в'язкість достатньо для стабілізації системи.

1. Introduction. Let $\mathcal{O} \subset \mathbb{R}^3$ be a (possibly unbounded) domain with a sufficiently smooth boundary $\partial\mathcal{O}$. We assume that $\partial\mathcal{O} = \overline{\Omega} \cup \overline{S}$, where $\Omega \cap S = \varnothing$,

$$\Omega \subset \left\{ x = (x_1; x_2; 0) \colon x' \equiv (x_1; x_2) \in \mathbb{R}^2 \right\}$$

is bounded in $\mathbb{R}^2$ and has the smooth contour $\Gamma = \partial\Omega$. We refer to Assumption 2.1 below for further hypotheses concerning the domain $\mathcal{O}$.

Let $a_0(x) = \left(a_0^1(x); a_0^2(x); a_0^3(x) \right)$ be a smooth bounded field defined on $\overline{\mathcal{O}}$ such that $\operatorname{div} a_0 = 0$, $(n, a_0) = 0$ on $\partial\mathcal{O}$ (n is the exterior normal to $\partial\mathcal{O}$, $n = (0; 0; 1)$ on Ω) and $A = A(x)$ be a bounded measurable 3×3 matrix, $x \in \overline{\mathcal{O}}$. We introduce a linear first order operator L_0 of the form

$$L_0 v = (a_0, \nabla) v + A v \tag{1}$$

and consider the following *linear* Navier – Stokes equations in $\mathcal{O}$ for the fluid velocity field $v = v(x,t) = \left(v^1(x,t); v^2(x,t); v^3(x,t) \right)$ and for the pressure $p(x,t)$:

$$v_t - \nu \Delta v + L_0 v + \nabla p = G_f(t) \qquad \text{in} \quad \mathcal{O} \times (0, +\infty), \tag{2}$$

$$\operatorname{div} v = 0 \qquad \text{in} \quad \mathcal{O} \times (0, +\infty), \tag{3}$$

where $\nu > 0$ is the dynamical viscosity, $G_f(t)$ is a volume force (which may depend on t). We supplement (2) and (3) with the (non-slip) boundary conditions imposed on the velocity field $v = v(x,t)$:

$$v = 0 \quad \text{on} \quad S, \qquad v \equiv (v^1; v^2; v^3) = (0; 0; u_t) \quad \text{on} \quad \Omega. \tag{4}$$

Here, as in [14], $u = u(x,t)$ is the transversal displacement of the plate occupying Ω and satisfying the following equation:

$$u_{tt} + \Delta^2 u + \mathcal{F}(u) = G_{pl}(t) + p|_\Omega \qquad \text{in} \quad \Omega \times (0,\infty), \tag{5}$$

where $\mathcal{F}(u)$ is a nonlinear (feedback) force (see Assumption 4.1 below), p is the pressure from (2), $G_{pl}(t)$ is a given external (non-autonomous) load. We refer to [14] and to the references therein for some discussion of this plate model and for an explanation of the structure of the force exerted by the fluid on the plate.

We impose clamped boundary conditions on the plate deflection

$$u|_{\partial\Omega} = \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0 \tag{6}$$

and supply (2)–(6) with initial data of the form

$$v(0) = v_0, \qquad u(0) = u_0, \qquad u_t(0) = u_1. \tag{7}$$

If we assume that the velocity field v decays sufficiently fast as $|x| \to +\infty$ and $x \in \overline{\mathcal{O}}$, then (3) and (4) imply the following compatibility condition:

$$\int\limits_\Omega u_t(x',t)dx' = 0 \qquad \text{for all} \quad t \geq 0, \tag{8}$$

which can be interpreted as preservation of the volume of the fluid.

Below (see Definitions 3.1 and 4.1) we define a solution to (2)–(8) as a pair $(v;u)$ satisfying some variational type relation. If the pair $(v;u)$ is already determined, then (at least formally) we can find ∇p in $\mathcal{O}$ and the trace of p on Ω from (2) and (5). Thus the pressure p is uniquely defined by $(v;u)$.

The main example which we have in mind is the *Poiseuille* flow (see, e.g., [6] for some details). In this case we deal with the domain

$$\mathcal{O} = \big\{(x_1; x_2; x_3)\colon \ (x_2; x_3) \in \mathcal{B} \subset \mathbb{R}^2, \ \ x_1 \in \mathbb{R}\big\}, \tag{9}$$

where $\mathcal{B}$ is a domain in $\mathbb{R}^2$, and the Poiseuille velocity field has the form $a_0 = (a(x_2; x_3); 0; 0)$, where $a(x_2; x_3)$ solves the elliptic problem

$$\nu \Delta a = -k \qquad \text{in} \quad \mathcal{B}, \qquad a = 0 \qquad \text{on} \quad \partial\mathcal{B}, \tag{10}$$

where k is a positive parameter. Linearization of the nonlinear Navier–Stokes equations around the flow a_0 gives us the model with

$$L_0 v = (a_0, \nabla)v + (v, \nabla)a_0. \tag{11}$$

There are two important special cases of the choice of $\mathcal{B}$ in (9): (i) $\mathcal{B}$ is a bounded domain in $\mathbb{R}^2$ (the Poiseuille flow in a cylindrical tube) and (ii) a flow between two parallel planes. In the latter case

$$\mathcal{B} = \big\{(x_2; x_3)\colon x_2 \in \mathbb{R}, \ \ x_3 \in (-h, 0)\big\}, \qquad a(x_2; x_3) = -\frac{kx_3}{2\nu}(h + x_3). \tag{12}$$

Another possibility included in the framework above is the *Oseen* modification of (2) (see, e.g., [6]). In this case $L_0 v = U \partial_{x_1} v$, where U is the parameter which has the sense of the speed of the unperturbed flow moving along the x_1-axis. This corresponds to the case when $a_0 = (U; 0; 0)$ and $A(x) \equiv 0$ in (1). We can also consider the situation when $a_0 \equiv 0$ and $A(x) \not\equiv 0$ in (1). In this case we note that if $A(x)$ is a symmetric strictly positive matrix (e.g., $A(x) = \sigma I$, $\sigma > 0$), then $L_0 v = A(x) v$ can be interpreted as a drag/friction term which models the resistance offered by the fluid against its flow (see, e.g., [29] for some discussion).

Thus, our general model includes the case of interaction of the Poiseuille (or Oseen type) flow (with a possible drag/friction) in the domain $\mathcal{O}$ bounded by the (solid) wall S and a horizontal boundary Ω on which a thin (nonlinear) elastic plate is placed. The motion of the fluid is described by the 3D Navier–Stokes equations *linearized* around the Poiseuille (or Oseen) flow $a_0(x)$. To describe deformations of the plate we consider a generalized plate model which accounts only for transversal displacements and covers a general large deflection Karman type model and can be also applied to nonlinear Berger and Kirchhoff plates (see the discussion in [14] and also in Section 4). Since we deal with *linearized* fluid equations the interaction model considered assumes that large deflections of the plate produce small effect on the corresponding underlying flow.

We note that the mathematical studies of the problem of fluid–structure interaction in the case of viscous fluids and elastic plates/bodies have a long history. We refer to [5, 8, 14, 20–24, 30] and the references therein for the case of plates/membranes. The case of moving elastic bodies [17] and the case of elastic bodies with the fixed interface [1, 2, 4, 19] were studied; see also the literature cited in these references. We also mention the recent short survey [16] and the paper [15] which deals with dynamical issues for a model taking into account both transversal and longitudinal deformations. All these sources deals with the case of bounded reservoirs $\mathcal{O}$.

In this paper our main point of interest is well-posedness and long-time dynamics of solutions to the coupled problem in (2)–(7) for the velocity v and the displacement u in the case of *unbounded* domains $\mathcal{O}$.

In our argument we use the ideas and methods developed in our previous paper [14]. Our main difficulties in comparison with [14] are related to the facts that (i) we deal with the (possibly) *unbounded* domain $\mathcal{O}$ (hence, we loose some compactness properties of the fluid velocity variable and cannot use eigenfunctions of the Stokes operator) and (ii) the fluid equation (2) is perturbed by nonconservative and nondissipative term (hence, we can loose the energy monotonicity and need some additional argument for non-monotone parts). To overcome these difficulties we are enforced to use a general basis in the fluid component and a specially constructed extension operator Ext of functions on Ω into solenoidal functions on $\mathcal{O}$.

The paper is organized as follows. In Section 2 we introduce Sobolev type spaces we need and provide with some results concerning the extension operator Ext. In Section 3 we prove Theorem 3.1 on well-posedness in the case of linear model and study stability properties of solutions in Theorem 3.2. Section 4 is devoted to nonlinear problem. We prove here that the problem generates a dynamical system (see Theorem 4.1) which, under some additional conditions, possesses a compact finite-dimensional global attractor (Theorem 4.2).

2. Preliminaries. Let D be a sufficiently smooth domain in $\mathbb{R}^d$ and $H^s(D)$ be the Sobolev space of order $s \in \mathbb{R}$ on D which we define (see [35]) as restriction (in the sense of distributions) of the space $H^s(\mathbb{R}^d)$ (introduced via Fourier transform). We define the norm in $H^s(D)$ by the relation

$$\|u\|_{s,D}^2 = \inf\left\{\|w\|_{s,\mathbb{R}^d}^2 \colon w \in H^s(\mathbb{R}^d), \ \ w = u \ \text{ on } \ D\right\}.$$

We also use the notation $\|\cdot\|_D = \|\cdot\|_{0,D}$ for the corresponding L_2 norm and, similarly, $(\cdot,\cdot)_D$ for the L_2 inner product. We denote by $H_0^s(D)$ the closure of $C_0^\infty(D)$ in $H^s(D)$ (with respect to $\|\cdot\|_{s,D}$) and introduce the spaces

$$H_*^s(D) := \left\{f\big|_D \colon f \in H^s(\mathbb{R}^d), \ \text{supp}\, f \subset \overline{D}\right\}, \quad s \in \mathbb{R},$$

to describe boundary traces on $\Omega \subset \partial\mathcal{O}$. Since the extension by zero of elements from $H_*^s(D)$ give us elements of $H^s(\mathbb{R}^d)$, these spaces $H_*^s(D)$ can be treated not only as functional classes defined on D (and contained in $H^s(D)$) but also as (closed) subspaces of $H^s(\mathbb{R}^d)$. We endow the classes $H_*^s(D)$ with the induced norms $\|f\|_{s,D}^* = \|f\|_{s,\mathbb{R}^d}$ for $f \in H_*^s(D)$. It is clear that

$$\|f\|_{s,D} \le \|f\|_{s,D}^*, \quad f \in H_*^s(D).$$

It is known (see [35], Theorem 4.3.2/1) that $C_0^\infty(D)$ is dense in $H_*^s(D)$ and

$$H_*^s(D) = H_0^s(D) \ \text{ for } \ -1/2 < s < \infty, \ \ s - 1/2 \notin \{0, 1, 2, \ldots\}.$$

The norms $\|\cdot\|_{s,D}^*$ and $\|\cdot\|_{s,D}$ are equivalent for these s. Note that in the notations of [27] the space $H_*^{m+1/2}(D)$ is the same as $H_{00}^{m+1/2}(D)$ for every $m = 0, 1, 2, \ldots$. Below we also use the factor-spaces $H^s(D)/\mathbb{R}$ with the naturally induced norm. To describe fluid velocity fields we first introduce the class $\mathcal{C}_0(\mathcal{O})$ of C^∞ vector-valued solenoidal (i.e., divergence-free) functions $v = (v^1; v^2; v^3)$ on $\mathcal{O}$ which vanish in a neighborhood of $\partial\mathcal{O}$ and also for $|x|$ large enough. Then we denote by $\widetilde{X}$ the closure of $\mathcal{C}_0(\mathcal{O})$ with respect to the L_2-norm and by $\widetilde{V}$ the closure of $\mathcal{C}_0(\mathcal{O})$ with respect to the H^1-norm. One can see that

$$\widetilde{X} = \left\{v = (v^1; v^2; v^3) \in [L_2(\mathcal{O})]^3 \colon \text{div}\, v = 0; \ \gamma_n v \equiv (v, n) = 0 \ \text{ on } \ \partial\mathcal{O}\right\}$$

and

$$\widetilde{V} \subseteq \widetilde{V}^\diamond \equiv \left\{v = (v^1; v^2; v^3) \in [H^1(\mathcal{O})]^3 \colon \text{div}\, v = 0; \ v = 0 \ \text{ on } \ \partial\mathcal{O}\right\}. \tag{13}$$

For some details concerning this type of spaces see, e.g., [25, 34] and [18].

The following (geometry type) hypothesis plays an important role in our further considerations.

Assumption 2.1 (Domain Hypothesis). *We assume that*

(i) *there exists a smooth bounded domain $\mathcal{O}' \subseteq \mathcal{O}$ such that $\overline{\Omega} \subset \partial\mathcal{O}'$;*

(ii) *we have the equality in* (13), *i.e.,* $\widetilde{V} = \widetilde{V}^\diamond$.

The sense of the first requirement in Assumption 2.1 is obvious. As for the second one we refer to [26] for a discussion of conditions on the domain which guarantee the equality $\widetilde{V} = \widetilde{V}^\diamond$ (see also [18] (Sect. 4.3) and the references therein). Here, as examples, we only note that this property holds in the following cases: (i) $\mathcal{O}$ is a smooth domain with the compact boundary; (ii) $\mathcal{O} = \mathbb{R}_-^3 = \{x_3 \le 0\}$; (iii) $\mathcal{O}$ is given by (9) with smooth bounded $\mathcal{B}$ or with $\mathcal{B}$ as in (12) (infinitely long pipes and tubes of possibly varying cross section are also admissible).

We also need the Sobolev spaces consisting of functions with zero average on the domain Ω, namely we consider the subspace

$$\widehat{L}_2(\Omega) = \left\{ u \in L_2(\Omega) \colon \int_\Omega u(x')dx' = 0 \right\}$$

in $L_2(\Omega)$ and also the subspaces $\widehat{H}^s(\Omega) = H^s(\Omega) \cap \widehat{L}_2(\Omega)$ in $H^s(\Omega)$ for $s > 0$ with the standard $H^s(\Omega)$-norm. The notations $\widehat{H}^s_*(\Omega)$ and $\widehat{H}^s_0(\Omega)$ have a similar meaning. We denote by $\widehat{P}$ the projection on $\widehat{H}^2_0(\Omega)$ in $H^2_0(\Omega)$ which is orthogonal with respect to the inner product $(\Delta\cdot, \Delta\cdot)_\Omega$. As it was already mentioned in [14] the subspace $(I - \widehat{P})H^2_0(\Omega)$ consists of functions $u \in H^2_0(\Omega)$ such that $\Delta^2 u = \text{const}$ and thus has dimension one.

In further considerations we need the following assertion concerning extension of functions defined on Ω.

Proposition 2.1. *Let Assumption* 2.1 (i) *be in force. Then there exists a linear bounded operator* $\text{Ext}\colon \widehat{L}_2(\Omega) \mapsto \left[L_2(\mathcal{O}) \right]^3$ *such that*

$$\text{div}\,\text{Ext}[\psi] = 0 \qquad in \quad \mathcal{O}, \qquad (\text{Ext}[\psi], n)\big|_\Omega = \psi, \qquad (\text{Ext}[\psi], n)\big|_S = 0,$$

and

$$\| \text{Ext}[\psi] \|_{\left[H^{1/2-\delta}(\mathcal{O}) \right]^3} \leq C \|\psi\|_\Omega \qquad \forall \delta > 0 \quad \forall \psi \in \widehat{L}_2(\Omega).$$

Moreover,
    if $\psi \in H^s_*(\Omega)$ *for some* $0 < s < 1$, *then* $\text{Ext}[\psi] \in \left[H^{s+1/2}(\mathcal{O}) \right]^3$ *with the estimate*

$$\| \text{Ext}[\psi] \|_{\left[H^{s+1/2}(\mathcal{O}) \right]^3} \leq C \|\psi\|_{H^s_*(\Omega)}, \tag{14}$$

and the relations $\text{Ext}[\psi]\big|_S = (0; 0; 0)$ *and* $\text{Ext}[\psi]\big|_\Omega = (0; 0; \psi)$ *on the boundary of* $\partial\mathcal{O}$;
    there exists a smooth bounded subdomain $\mathcal{O}'$ *in* $\mathcal{O}$ *such that* (i) $\Omega \subset \partial\mathcal{O}'$, (ii) $\text{Ext}[\psi]\big|_{\mathcal{O}\backslash\mathcal{O}'} = 0$, *and* (iii) $\text{Ext}[\psi]\big|_{\mathcal{O}'} \in \left[H^2(\mathcal{O}') \right]^3$ *provided* $\psi \in H^{3/2+\delta}_0(\Omega)$ *for some* $\delta > 0$.

Proof. On a smooth bounded subdomain $\mathcal{O}'$ in $\mathcal{O}$ such that $\Omega \subset \partial\mathcal{O}'$ we consider the following Stokes problem:

$$-\nu\Delta v + \nabla p = 0, \qquad \text{div}\, v = 0 \qquad in \quad \mathcal{O}';$$

$$v = 0 \qquad on \quad \partial\mathcal{O}' \setminus \Omega; \qquad v = (0; 0; \psi) \qquad on \quad \Omega, \tag{15}$$

where $\psi \in \widehat{L}_2(\Omega)$ is given. This type of boundary-value problems in *bounded* domains was studied by many authors (see, e.g., [25, 34] and also the recent monograph [18] and the references therein). To construct an extension operator we need the following properties of solutions to (15) (for some discussion and references concerning the assertion below we refer to [14]).

Proposition 2.2. *Let* $\psi \in H^s_*(\Omega)$ *with* $-1/2 \leq s \leq 3/2$ *and* $\int_\Omega \psi(x')dx' = 0$. *Then problem* (15) *has a unique solution*

$$\{v; p\} \in [H^{s+1/2}(\mathcal{O}')]^3 \times [H^{s-1/2}(\mathcal{O}')/\mathbb{R}]$$

such that

$$\|v\|_{[H^{s+1/2}(\mathcal{O}')]^3} + \|p\|_{H^{s-1/2}(\mathcal{O}')/\mathbb{R}} \leq c_0 \|\psi\|_{H^s_*(\Omega)}.$$

Now we can take a solution v to (15) and define $\mathrm{Ext}[\psi]$ as the zero extension of v on the domain $\mathcal{O}$. One can see that for this operator Ext all statements of Proposition 2.1 are in force.

Remark **2.1.**   We could not find in the literature an appropriate statement of Proposition 2.2 for *unbounded* domains. On the other hand we do not know an extension result in the class solenoidal functions with estimate (14) for some range of the parameter s. This is why we use this way for a construction of the operator Ext. We also note that in the case when $\mathcal{O}$ is bounded we can take $\mathcal{O}' = \mathcal{O}$. In this case Ext is a Green type operator which maps ψ into v according to (15). Exactly this extension operator was used in [14].

Using the extension operator constructed above we introduce the spaces which we need to describe the interaction between fluid and plate.

Let Assumption 2.1 be valid and

$$\mathcal{M}(\mathcal{O}) = \left\{ v = v_0 + \mathrm{Ext}[\psi] \colon v_0 \in \mathcal{C}_0(\mathcal{O}),\ \psi \in \widehat{H}_0^2(\Omega) \right\}.$$

Then we denote by X the closure of $\mathcal{M}(\mathcal{O})$ with respect to the L_2-norm and by V the closure of $\mathcal{M}(\mathcal{O})$ with respect to the H^1-norm. One can see that

$$X = \left\{ v = (v^1; v^2; v^3) \in [L_2(\mathcal{O})]^3 \colon \mathrm{div}\, v = 0;\ \gamma_n v \equiv (v, n) = 0 \text{ on } S \right\}$$

and

$$V = V^\diamond \equiv \left\{ v = (v^1; v^2; v^3) \in [H^1(\mathcal{O})]^3 \ \middle|\ \begin{array}{l} \mathrm{div}\, v = 0,\ v = 0 \text{ on } S, \\ v^1 = v^2 = 0 \text{ on } \Omega. \end{array} \right\}.$$

We equip X with L_2-type norm $\|\cdot\|_{\mathcal{O}}$ and denote by $(\cdot, \cdot)_{\mathcal{O}}$ the corresponding inner product. The space V is endowed with the standard H^1-norm.

In conclusion of this section we mention that in the the case of the Poiseuille flow in the tube or between two planes described above we deal with a domain satisfying the Friedrichs–Póincare property[1]:

$$\exists d_{\mathcal{O}} > 0: \qquad \int_{\mathcal{O}} |v(x)|^2 dx \le d_{\mathcal{O}}^2 \int_{\mathcal{O}} |\nabla v(x)|^2 dx \quad \forall v \in H_0^1(\mathcal{O}). \tag{16}$$

By the localization argument one can show that the inequality in (16) implies a similar property for any $v \in \{g \in H^1(\mathcal{O}) \colon g|_S = 0\}$ and thus

$$\exists c_{\mathcal{O}} > 0: \ \|v\|_{\mathcal{O}} \le c_{\mathcal{O}} \|\nabla v\|_{\mathcal{O}} \quad \forall v \in V, \tag{17}$$

for the Friedrichs–Póincare domains.

3. Linear problem. In this section we consider a linear version of (2)–(8) which is obtained by replacing equation (5) with its linear counterpart. Thus we deal with the following problem:

$$v_t - \nu \Delta v + L_0 v + \nabla p = G_f(t) \qquad \text{and} \qquad \mathrm{div}\, v = 0 \qquad \text{in} \quad \mathcal{O} \times (0, +\infty), \tag{18}$$

$$v = 0 \quad \text{on} \quad S \quad \text{and} \quad v \equiv (v^1; v^2; v^3) = (0; 0; u_t) \qquad \text{on} \quad \Omega, \tag{19}$$

$$u_{tt} + \Delta^2 u = G_{pl}(t) + p|_\Omega \qquad \text{on} \quad \Omega, \tag{20}$$

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[1]This property is valid in the case when the domain $\mathcal{O}$ is bounded at least in one direction.

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial \Omega, \quad \int_\Omega u_t(x', t)dx' = 0 \quad \text{for all} \quad t \geq 0, \tag{21}$$

which we supply with initial data of the form

$$v(0) = v_0, \qquad u(0) = u_0, \qquad u_t(0) = u_1. \tag{22}$$

Similarly to [14] we consider the following class of test functions

$$\mathcal{L}_T = \left\{ \phi \left| \begin{array}{l} \phi \in L_2(0, T; [H^1(\mathcal{O})]^3), \ \phi_t \in L_2(0, T; [L_2(\mathcal{O})]^3), \\ \text{div}\phi = 0, \ \phi|_S = 0, \ \phi|_\Omega = (0; 0; b), \ \phi(T) = 0, \\ b \in L_2(0, T; \widehat{H}_0^2(\Omega)), \ b_t \in L_2(0, T; \widehat{L}_2(\Omega)). \end{array} \right. \right\}$$

and introduce the following definition.

Definition 3.1. *A pair of functions $(v(t); u(t))$ is said to be a weak solution to the problem in* (18)–(22) *on a time interval $[0, T]$ if*

$v \in L_\infty(0, T; X) \bigcap L_2(0, T; V);$

$u \in L_\infty(0, T; H_0^2(\Omega)), \ u_t \in L_\infty(0, T; \widehat{L}_2(\Omega))$ *and* $u(0) = u_0;$

for every $\phi \in \mathcal{L}_T$ *the following equality holds*:

$$- \int_0^T (v, \phi_t)_\mathcal{O} dt + \nu \int_0^T (\nabla v, \nabla \phi)_\mathcal{O} dt + \int_0^T (L_0 v, \phi)_\mathcal{O} dt -$$

$$- \int_0^T (u_t, b_t)_\Omega dt + \int_0^T (\Delta u, \Delta b)_\Omega dt =$$

$$= \int_0^T (G_f, \phi)_\mathcal{O} dt + \int_0^T (G_{pl}, b)_\Omega dt + (v_0, \phi(0))_\mathcal{O} + (u_1, b(0))_\Omega; \tag{23}$$

the compatibility condition $v(t)|_\Omega = (0; 0; u_t(t))$ *holds for almost all t.*

The same argument as in [14] shows that a weak solution $(v(t); u(t))$ satisfies the relation

$$(v(t), \psi)_\mathcal{O} + (u_t(t), \beta)_\Omega = (v_0, \psi)_\mathcal{O} + (u_1, \beta)_\Omega -$$

$$- \int_0^t \left[\nu(\nabla v, \nabla \psi)_\mathcal{O} + (L_0 v, \psi)_\mathcal{O} + (\Delta u, \Delta \beta)_\Omega - (G_f, \psi)_\mathcal{O} - (G_{pl}, \beta)_\Omega \right] d\tau$$

for almost all $t \in [0, T]$ and for all $\psi = (\psi^1; \psi^2; \psi^3) \in W$, where $\beta = \psi^3|_\Omega$ and

$$W = \left\{ \psi \in V \, \middle| \, \psi|_\Omega = (0; 0; \beta), \ \beta \in \widehat{H}_0^2(\Omega) \right\}. \tag{24}$$

It also follows from the compatibility condition and the standard trace theorem that the plate velocity u_t possesses an additional spatial regularity, namely we have that $u_t \in L_2(0, T; H_*^{1/2}(\Omega))$.

Below as phase spaces we use

$$\mathcal{H} = \Big\{ (v_0; u_0; u_1) \in X \times H_0^2(\Omega) \times \widehat{L}_2(\Omega) \colon \ (v_0, n) \equiv v_0^3 = u_1 \ \text{ on } \ \Omega \Big\} \tag{25}$$

and

$$\widehat{\mathcal{H}} = \Big\{ (v_0; u_0; u_1) \in \mathcal{H} \colon u_0 \in \widehat{H}_0^2(\Omega) \Big\} \subset \mathcal{H} \tag{26}$$

with the norm $\|(u_0; u_0; u_1)\|_{\mathcal{H}}^2 = \|v_0\|_{\mathcal{O}}^2 + \|\Delta u_0\|_{\Omega}^2 + \|u_1\|_{\Omega}^2$.

Our main result in this section is the following well-posedness theorem concerning the linear problem.

Theorem 3.1. *Let Assumption 2.1 be in force. Assume that*

$$U_0 = (v_0; u_0; u_1) \in \mathcal{H}, \qquad G_f(t) \in L_2(0, T; V'), \qquad G_{pl}(t) \in L_2(0, T; H^{-1/2}(\Omega)).$$

Then for any interval $[0, T]$ there exists a unique weak solution $(v(t); u(t))$ to (18)–(22) with the initial data U_0. This solution possesses the property

$$U(t; U_0) \equiv U(t) \equiv (v(t); u(t); u_t(t)) \in C(0, T; X \times H_0^2(\Omega) \times \widehat{L}_2(\Omega)),$$

and satisfies the energy balance equality

$$\mathcal{E}_0(v(t), u(t), u_t(t)) + \int\limits_0^t \big[\nu \|\nabla v\|_{\mathcal{O}}^2 + (Av, v)_{\mathcal{O}} \big] d\tau =$$

$$= \mathcal{E}_0(v_0, u_0, u_1) + \int\limits_0^t (G_f, v)_{\mathcal{O}} d\tau + \int\limits_0^t (G_{pl}, u_\tau)_{\Omega} d\tau \tag{27}$$

for every $t > 0$, where the energy functional $\mathcal{E}_0$ is defined by the relation

$$\mathcal{E}_0(v(t), u(t), u_t(t)) = \frac{1}{2} \Big(\|v(t)\|_{\mathcal{O}}^2 + \|u_t(t)\|_{\Omega}^2 + \|\Delta u(t)\|_{\Omega}^2 \Big).$$

If $G_f \equiv 0$ and $G_{pl} \equiv 0$, then Theorem 3.1 implies that the problem in (18)–(22) generates a strongly continuous semigroup. In order to state our result on asymptotic stability of this semigroup we need additional assumptions.

Assumption 3.1 (Stability Hypothesis). *Assume that one of the following conditions is valid:*

either the matrix $A(x)$ in (1) is uniformly strictly positive, i.e.,

$$\exists \sigma > 0 \colon \ (A(x)\xi, \xi)_{\mathbb{R}^3} \geq \sigma |\xi|_{\mathbb{R}^3}^2 \qquad \forall \xi \in \mathbb{R}^3, \quad x \in \overline{\mathcal{O}};$$

or the domain $\mathcal{O}$ satisfies the Friedrichs–Póincare property (16) and

$$\exists \delta > 0 \colon \ (A(x)\xi, \xi)_{\mathbb{R}^3} \geq -\left(\frac{\nu}{c_{\mathcal{O}}^2} - \delta \right) |\xi|_{\mathbb{R}^3}^2 \qquad \forall \xi \in \mathbb{R}^3, \quad x \in \overline{\mathcal{O}}, \tag{28}$$

where $c_{\mathcal{O}}$ is the constant from the Friedrichs–Póincare inequality in (17).

Thus in the case of a general domain $\mathcal{O}$ satisfying Assumption 2.1 to obtain a result on long-time dynamics we need to assume the presence of some additional damping mechanism (drag/friction terms). If the domain satisfies the Friedrichs – Póincare property, then the result can be achieved without any damping (e.g., we can take $A(x) \equiv 0$). Moreover, we note that the condition in (28) is true when $\sup_{x \in \mathcal{O}} |A(x)| < \nu c_{\mathcal{O}}^{-2}$, where $|A(x)|$ is the operator (Euclidian) norm in $\mathbb{R}^3$. In the case of the Poiseuille type flow (see (11)) this means that $|\nabla_{x_1, x_2} a|$ is small enough. Since the profile a can be written in the form $a = k\nu^{-1} a_*$, where a_* solves (10) with $\nu = 1$ and $k = 1$, the latter condition is satisfied when $k\nu^{-2} \leq c(\mathcal{B})$. Here k is the Poiseuille velocity parameter and $c(\mathcal{B})$ is a constant depending on the cross-section $\mathcal{B}$ of the tube $\mathcal{O}$. In the case of the Oseen model we have $a_0 = (U; 0; 0)$ in (11) and thus there are no restrictions on the velocity U of the underlining flow for Friedrichs – Póincare domains.

Theorem 3.2. *In addition to the hypotheses of Theorem* 3.1 *we assume that Assumption* 3.1 *is in force. Then there exist positive constants M and γ such that for every initial data $U_0 = (v_0; u_0; u_1)$ from $\widehat{\mathcal{H}}$ we have*

$$\|U(t)\|_{\mathcal{H}}^2 \leq Me^{-\gamma t}\|U_0\|_{\mathcal{H}}^2 + M \int_0^t e^{-\gamma(t-\tau)} \left[\|G_f(\tau)\|_{V'}^2 + \|G_{pl}(\tau)\|_{-1/2, \Omega}^2 \right] d\tau. \qquad (29)$$

In particular, if $G_f \equiv 0$ and $G_{pl} \equiv 0$, then the C_0-semigroup generated by (18) – (22) *is exponentially stable in $\widehat{\mathcal{H}}$.*

In the case of a general operator L_0 we need to add the term

$$\left(M_1 + M_2 \sup_{x \in \mathcal{O}} |A(x)| \right) \int_0^t e^{-\gamma(t-\tau)} \|v(\tau)\|^2 d\tau$$

in the right-hand side of (29)*. Here $|A(x)|$ denotes the operator (Euclidian) norm in $\mathbb{R}^3$ and $M_1 = 0$ when the domain $\mathcal{O}$ satisfies the Friedrichs – Póincare property in* (16)*.*

***Proof of Theorem* 3.1.** In the case when $L_0 \equiv 0$ and $\mathcal{O}$ is bounded this theorem was proved in [14] (see also [30] for a similar result). We use the same idea as in [14]. The main difficulty which we are faced is that we loose several compactness properties of the model (e.g., we cannot use the basis of eigenfunctions of the Stokes operator).

Step 1. *Existence of an approximate solution.* Let $\{\psi_i\}_{i \in \mathbb{N}}$ be an (orthonormal) basis in the space $\widetilde{V}$ consisting of the smooth finite in $\mathcal{O}$ functions. Denote by $\{\xi_i\}_{i \in \mathbb{N}}$ the basis in $\widehat{H}_0^2(\Omega)$ which consists of eigenfunctions of the following problem:

$$(\Delta \xi_i, \Delta w)_\Omega = \kappa_i(\xi_i, w)_\Omega \qquad \forall w \in \widehat{H}_0^2(\Omega),$$

with the eigenvalues $0 < \kappa_1 \leq \kappa_2 \leq \ldots$ and $\|\xi_i\|_\Omega = 1$. Let $\phi_i = \mathrm{Ext}[\xi_i]$, where the operator Ext is defined in Proposition 2.1. This proposition also yields ϕ_i is H^2 in some vicinity of Ω and thus as in [14] one can conclude that $\partial_{x_3} \phi_i^3 = 0$ on Ω.

We define an approximate solution as a pair of functions

$$v_{n,m}(t) = \sum_{i=1}^m \alpha_i(t)\psi_i + \sum_{j=1}^n \dot{\beta}_j(t)\phi_j, \qquad u_n(t) = \sum_{j=1}^n \beta_j(t)\xi_j + (I - \widehat{P})u_0, \qquad (30)$$

satisfying the relations

$$(\dot{v}_{n,m}(t), \chi)_{\mathcal{O}} + (\ddot{u}_n(t), h)_{\Omega} + \nu(\nabla v_{n,m}(t), \nabla \chi)_{\mathcal{O}} + (\Delta u_n(t), \Delta h)_{\Omega} =$$

$$= -(L_0 v_{n,m}, \chi)_{\mathcal{O}} + (G_f(t), \chi)_{\mathcal{O}} + (G_{pl}(t), h)_{\Omega} \qquad (31)$$

for $t \in [0, T]$ and for every χ and h of the form

$$\chi = \sum_{k=1}^{m'} \chi_k \psi_k + \text{Ext}[h] \quad \text{with} \quad h = \sum_{k=1}^{n'} h_k \xi_k, \qquad (32)$$

where $m' \leq m$ and $n' \leq n$. It is clear that $\chi \in W$ and $\chi|_{\Omega} = (0; 0; h)$. The system in (31) is endowed with the initial data

$$v_{n,m}(0) = \Pi_m(v_0 - \text{Ext}[u_1]) + \text{Ext}[P_n u_1],$$

$$u_n(0) = P_n \widehat{P} u_0 + (I - \widehat{P}) u_0, \qquad \dot{u}_n(0) = P_n u_1,$$

where Π_m is the orthoprojector on $\text{Lin}\{\psi_j \colon j = 1, \ldots, m\}$ in $\widetilde{X}$ and P_n is orthoprojector on $\text{Lin}\{\xi_i \colon i = 1, \ldots, n\}$ in $\widehat{L}_2(\Omega)$. Since $\text{Ext} \colon \widehat{L}_2(\Omega) \mapsto X$, it is clear that

$$(v_{n,m}(0); u_n(0); \dot{u}_n(0)) \to (v_0; u_0; u_1) \qquad \text{strongly in} \quad \mathcal{H} \quad \text{as} \quad m, n \to \infty.$$

As in [14] one can show that (31) can be reduced to some ODE in $\mathbb{R}^{m+n}$ and with given initial data has a unique solution on any time interval $[0, T]$.

It follows from (30) that

$$v_{n,m}(t) = \sum_{i=1}^{m} \alpha_i(t) \psi_i + \text{Ext}[\partial_t u_n(t)].$$

This implies the boundary compatibility condition:

$$v_{n,m}(t) = (0; 0; \partial_t u_n(t)) \quad \text{on} \quad \Omega. \qquad (33)$$

Step 2. *Energy relation and a priori estimate for an approximate solution.* Taking $\chi = v_{n,m}$ and $h = \partial_t u_n(t)$ in (31) we obtain the following energy balance relation for approximate solutions:

$$\mathcal{E}_0(v_{n,m}(t), u_n(t), \partial_t u_n(t)) + \nu \int_0^t \int_{\mathcal{O}} |\nabla v_{n,m}|^2 dx d\tau + \int_0^t (A v_{n,m}, v_{n,m})_{\mathcal{O}} d\tau =$$

$$= \mathcal{E}_0(v_{n,m}(0), u_n(0), \partial_t u_n(0)) + \int_0^t (G_f, v_{n,m})_{\mathcal{O}} d\tau + \int_0^t (G_{pl}, \partial_t u_n)_{\Omega} d\tau. \qquad (34)$$

We use here the structure of L_0 which after simple calculations (see, e.g., Lemma 1.3 [34], Ch. 2) yields the equality $(L_0 v_{n,m}, v_{n,m})_{\mathcal{O}} = (A v_{n,m}, v_{n,m})_{\mathcal{O}}$. The relation in (34) and Gronwall's lemma implies the following a priori estimate:

$$\sup_{t\in[0,T]} \left\{ \|v_{n,m}(t)\|_{\mathcal{O}}^2 + \|\Delta u_n(t)\|_{\Omega}^2 + \|\partial_t u_n(t))\|_{\Omega}^2 \right\} +$$

$$+ \int_0^T \left(\|\nabla v_{n,m}\|_{\mathcal{O}}^2 + \|v_{n,m}\|_{\mathcal{O}}^2 \right) d\tau \le C_T. \tag{35}$$

By the trace theorem from (33) and (35) we also have that

$$\int_0^T \|\partial_t u_n(\tau))\|_{H_*^{1/2}(\Omega)}^2 d\tau = \int_0^T \|v_{n,m}(\tau)\|_{1/2,\partial\mathcal{O}}^2 d\tau \le C_T. \tag{36}$$

Step 3. *Limit transition.* By (35) the sequence $\{(v_{n,m}; u_n; \partial_t u_n)\}$ contains a subsequence such that

$$(v_{n,m}; u_n; \partial_t u_n) \rightharpoonup (v; u; \partial_t u) \qquad *\text{-weakly in} \quad L_\infty(0,T;\mathcal{H}), \tag{37}$$

$$u_n \to u \qquad \text{strongly in} \quad C(0,T;H_0^{2-\epsilon}(\Omega)) \quad \forall \varepsilon > 0, \tag{38}$$

$$v_{n,m} \rightharpoonup v \qquad \text{weakly in} \quad L_2(0,T;V). \tag{39}$$

To obtain (38) we use the Aubin–Dubinsky theorem (see, e.g., [32], Corollary 4). By (36) we can also suppose that

$$\partial_t u_n \rightharpoonup \partial_t u \qquad \text{weakly in} \quad L_2(0,T;H_*^{1/2}(\Omega)), \tag{40}$$

$$v_{n,m} \rightharpoonup v \qquad \text{weakly in} \quad L_2(0,T;H^{1/2}(\partial\mathcal{O})). \tag{41}$$

Applying the same argument as in [14] and using relations (37)–(41) we conclude the proof of the existence of weak solutions which satisfy the corresponding energy balance *inequality*. At this point we use Assumption 2.1(ii) to approximate elements from W by elements of the form (32). We need this to establish (23) for $\phi \in \mathcal{L}_T$.

Step 4. *Uniqueness.* We first consider the case when $L_0 \equiv 0$ and use Lions' idea (see [28]), with the same test function as [14] in the case of a bounded domain. After establishing properties of solutions in this case we consider the term $L_0 v$ as a perturbation.

Let $U^j(t) = (v^j(t); u^j(t); u_t^j(t))$, $j = 1, 2$, be two different solutions to the problem in question with the same initial data and $L_0 \equiv 0$. Then their difference $U(t) = U^1(t) - U^2(t) = (v(t); u(t); u_t(t))$ satisfies the variational equality

$$- \int_0^T (v, \phi_t)_{\mathcal{O}} + \nu \int_0^T (\nabla v, \nabla \phi)_{\mathcal{O}} - \int_0^T (u_t, b_t)_{\Omega} + \int_0^T (\Delta u, \Delta b)_{\Omega} = 0$$

for all $\phi \in \mathcal{L}_T$, $b = (\phi|_{\Omega})^3$. Now for every $0 < s < T$ we take

$$\phi(t) \equiv \phi^s(t) = \begin{cases} -\int_t^s d\tau \int_0^\tau d\zeta v(\zeta), & t < s, \\ 0, & t \ge s, \end{cases}$$

as a test function. The same calculation as in [14] yields the uniqueness in the case $L_0 \equiv 0$.

Step 5. *Continuity with respect to t and the energy equality.* Using the Lions lemma (see [27], Lemma 8.1) by the same argument as in [14] we first prove any weak solution $(v(t); u(t); u_t(t))$ is weakly continuous in $X \times H_0^2(\Omega) \times \widehat{L}_2(\Omega)$.

To prove the energy equality (in the case $L_0 = 0$), we follow the scheme of [28] (Ch. 1), see also [27] (Ch. 3), in the form presented in [14]. Thus as in [14] we can conclude that the solution is strongly continuous in t. Moreover, the energy relation in the case $L_0 = 0$ with $G_f = 0$ and $G_{pl} = 0$ implies that the corresponding solutions generates strongly continuous semigroup.

Step 6. *Case* $L_0 \neq 0$. Using the energy relation for the problem with $L_0 = 0$ and $G_f(t) :=$ $:= G_f(t) - L_0 v(t)$ we can establish the uniqueness of solutions via the Gronwall's type argument and also the smoothness properties in the general case.

Theorem 3.1 is proved.

Proof of Theorem 3.2. To prove the estimate in (29), we construct a Lyapunov function using an idea from [8] (see also [14]). Let

$$V(v_0, u_0, u_1) = \mathcal{E}_0(v_0, u_0, u_1) + \epsilon \Psi(v_0, u_0, u_1),$$

where $\Psi(v_0, u_0, u_1) = (u_0, u_1)_\Omega + (v_0, \mathrm{Ext}[u_0])_\mathcal{O}$ and $\epsilon > 0$ is a small parameter which will be chosen later. We consider these functionals on approximate solutions $(v_{n,m}; u_n)$ for which $\widehat{P}u_0 = u_0$ and thus $\widehat{P}u_n(t) = u_n(t)$ for all $t > 0$. This allow us to substitute in (31) $\mathrm{Ext}[u_n]$ instead of χ and obtain that

$$\frac{d}{dt}\Psi_{n,m}(t) \equiv \frac{d}{dt}\Psi(v_{n,m}(t), u_n(t), \partial_t u_n(t)) =$$

$$= \|\partial_t u_n\|_\Omega^2 + (v_{n,m}, \mathrm{Ext}[\partial_t u_n])_\mathcal{O} - (L_0 v_{n,m}, \mathrm{Ext}[u_n])_\mathcal{O} -$$

$$-\nu(\nabla v_{n,m}, \nabla \mathrm{Ext}[u_n])_\mathcal{O} - \|\Delta u_n\|_\Omega^2 + (G_f, \mathrm{Ext}[u_n])_\mathcal{O} + (G_{pl}, u_n)_\Omega. \tag{42}$$

By Proposition 2.1, using the compatibility condition in (33) and the trace theorem we have that

$$\left|(v_{n,m}, \mathrm{Ext}[\partial_t u_n])_\mathcal{O}\right| \leq C\|v_{n,m}\|_\mathcal{O}\|\partial_t u_n\|_\Omega \leq C\left[\|\nabla v_{n,m}\|_\mathcal{O}^2 + \|v_{n,m}\|_\mathcal{O}^2\right].$$

Similarly, for every $\eta > 0$ we have

$$\left|(\nabla v_{n,m}, \nabla \mathrm{Ext}[u_n])_\mathcal{O}\right| \leq \eta\|\Delta u_n\|_\Omega^2 + C_\eta\left[\|\nabla v_{n,m}\|_\mathcal{O}^2 + \|v_{n,m}\|_\mathcal{O}^2\right]$$

and

$$\left|(G_f, \mathrm{Ext}[u_n])_\mathcal{O} + (G_{pl}, u_n)_\Omega\right| \leq \eta\|\Delta u_n\|_\Omega^2 + C_\eta\left[\|G_f\|_{V'}^2 + \|G_{pl}\|_{-1/2,\Omega}^2\right].$$

It is also clear that

$$\left|(L_0 v_{n,m}, \mathrm{Ext}[u_n])_\mathcal{O}\right| \leq \eta\|\Delta u_n\|_\Omega^2 + C_\eta\left[\|\nabla v_{n,m}\|_\mathcal{O}^2 + \|v_{n,m}\|_\mathcal{O}^2\right].$$

Therefore it follows from (42) that

$$\frac{d}{dt}\Psi_{n,m}(t) \leq -\frac{1}{2}\|\Delta u_n\|_\Omega^2 + C\left[\|\nabla v_{n,m}\|_\mathcal{O}^2 + \|v_{n,m}\|_\mathcal{O}^2\right] + C\left[\|G_f\|_{V'}^2 + \|G_{pl}\|_{-1/2,\Omega}^2\right].$$

Using the energy relation in (34) we also have that

$$\frac{d}{dt}\mathcal{E}_0(v_{n,m}(t), u_n(t), \partial_t u_n(t)) \leq -(\nu - \eta)\|\nabla v_{n,m}\|_{\mathcal{O}}^2 + \eta\|v_{n,m}\|_{\mathcal{O}}^2 +$$

$$+C_\eta\left[\|G_f\|_{V'}^2 + \|G_{pl}\|_{-1/2,\Omega}^2\right] - (Av_{n,m}, v_{n,m})_{\mathcal{O}} \quad \forall \eta > 0.$$

One can see that the function $V_{n,m}(t) \equiv V(v_{n,m}(t), u_n(t), \partial_t u_n(t))$ satisfies the relations

$$a_0\mathcal{E}_0(v_{n,m}(t), u_n(t), \partial_t u_n(t)) \leq V_{n,m}(t) \leq a_1\mathcal{E}_0(v_{n,m}(t), u_n(t), \partial_t u_n(t))$$

for sufficiently small $\varepsilon > 0$. Using the stability hypothesis in Assumption 3.1 we can choose $\eta > 0$ and $\sigma > 0$ such that

$$(\nu - \eta)\|\nabla v_{n,m}\|_{\mathcal{O}}^2 - \eta\|v_{n,m}\|_{\mathcal{O}}^2 + (Av_{n,m}, v_{n,m})_{\mathcal{O}} \geq \sigma\left[\|\nabla v_{n,m}\|_{\mathcal{O}}^2 + \|v_{n,m}\|_{\mathcal{O}}^2\right].$$

Therefore we have that

$$\frac{d}{dt}V_{n,m}(t) + a_2 V_{n,m}(t) \leq a_3\left[\|G_f\|_{V'}^2 + \|G_{pl}\|_{-1/2,\Omega}^2\right]$$

with positive constants a_i. This implies relation (29) for approximate solutions. The limit transition yields (29) for every weak solution.

In the general case we can apply (29) with $L_0 := (a_0, \nabla)v + \mu v$ and $G_f := G_f - Av + \mu v$, where $\mu > 0$ (in the case of the Friedrichs–Póincare domains we can take $\mu = 0$). This implies the desired conclusion and completes the proof of Theorem 3.2.

4. Nonlinear problem. In this section we deal with problem (2)–(8) with a nonlinear feedback force. First we impose the following hypotheses concerning the force $\mathcal{F}(u)$ in the plate equation (5).

Assumption 4.1. *There exists $\epsilon > 0$ such that $\mathcal{F}(u)$ is locally Lipschitz from $H_0^{2-\epsilon}(\Omega)$ into[2] $H^{-1/2}(\Omega)$ in the sense that*

$$\left\|\mathcal{F}(u_1) - \mathcal{F}_2(u_2)\right\|_{-1/2,\Omega} \leq C_R\|u_1 - u_2\|_{2-\varepsilon,\Omega} \tag{43}$$

for any $u_i \in H_0^2(\Omega)$ such that $\|u_i\|_{2,\Omega} \leq R$.

There exists a C^1-functional $\Pi(u)$ on $H_0^2(\Omega)$ such that $\mathcal{F}(u) = \Pi'(u)$, where Π' denotes the Fréchet derivative of Π.

The plate force potential Π is bounded on bounded sets from $H_0^2(\Omega)$ and there exist $\eta < 1/2$ and $C \geq 0$ such that

$$\eta\|\Delta u\|_{\Omega}^2 + \Pi(u) + C \geq 0 \quad \forall u \in H_0^2(\Omega). \tag{44}$$

Examples of nonlinear feedback (elastic) forces $\mathcal{F}(u)$ satisfying Assumption 4.1 are described in [9] and [14], see also [13]. They represent different plate models and include Kirchhoff, von Karman, and Berger models.

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[2]We recall [35] that $H^{-1/2}(\Omega) = \left[H_*^{1/2}(\Omega)\right]' \not\supseteq \left[H_0^{1/2}(\Omega)\right]'$.

4.1. Well-possedness.

Definition 4.1. *A pair of functions $(v(t); u(t))$ is said to be a weak solution to (2)–(8) on a time interval $[0, T]$ if*

$v \in L_\infty(0, T; X) \bigcap L_2(0, T; V)$;

$u \in L_\infty(0, T; H_0^2(\Omega))$, $u_t \in L_\infty(0, T; \widehat{L}_2(\Omega))$, $u(0) = u_0$;

the equality in (23) *holds with* $G_{pl}(t) := -\mathcal{F}(u(t)) + G_{pl}(t)$;

the compatibility condition $v(t)|_\Omega = (0; 0; u_t(t))$ *holds for almost all* t.

Theorem 4.1. *Let Assumptions 2.1 and 4.1 be in force. Assume that $U_0 = (v_0; u_0; u_1) \in \mathcal{H}$, $G_f(t) \in L_2(0, T; V')$ and $G_{pl}(t) \in L_2(0, T; H^{-1/2}(\Omega))$. Then for any interval $[0, T]$ there exists a unique weak solution $(v(t); u(t))$ to (2)–(8) with the initial data U_0. This solution possesses the property*

$$U(t) \equiv (v(t); u(t); u_t(t)) \in C(0, T; \mathcal{H}), \tag{45}$$

where $\mathcal{H}$ is given by (25), *and satisfies the energy balance equality*

$$\mathcal{E}(v(t), u(t), u_t(t)) + \int\limits_0^t \left[\nu \|\nabla v\|_{\mathcal{O}}^2 + (Av, v)_{\mathcal{O}} \right] d\tau =$$

$$= \mathcal{E}(v_0, u_0, u_1) + \int\limits_0^t (G_f, v)_{\mathcal{O}} d\tau + \int\limits_0^t (G_{pl}, u_\tau)_{\Omega} d\tau \tag{46}$$

for every $t > 0$, where the energy functional $\mathcal{E}$ is defined by the relation

$$\mathcal{E}(v, u, u_t) = \frac{1}{2} \left(\|v\|_{\mathcal{O}}^2 + \|u_t\|_{\Omega}^2 + \|\Delta u\|_{\Omega}^2 \right) + \Pi(u).$$

Moreover, there exists a constant $a_{R,T} > 0$ such that for any couple of weak solutions $U(t) = \big(v(t); u(t); u_t(t) \big)$ and $\hat{U}(t) = \big(\hat{v}(t); \hat{u}(t); \hat{u}_t(t) \big)$ with the initial data possessing the property $\|U_0\|_{\mathcal{H}}, \|\hat{U}_0\|_{\mathcal{H}} \leq R$ we have

$$\|U(t) - \hat{U}(t)\|_{\mathcal{H}}^2 + \int\limits_0^t \|\nabla(v - \hat{v})\|_{\mathcal{O}}^2 d\tau \leq a_{R,T} \|U_0 - \hat{U}_0\|_{\mathcal{H}}^2, \quad \in [0, T]. \tag{47}$$

The spatial average of $u(t)$ is preserved. In particular, if $U_0 \in \widehat{\mathcal{H}}$, then $U(t) \in \widehat{\mathcal{H}}$ for every $t > 0$. We recall that $\widehat{\mathcal{H}}$ is defined by (26).

Proof. The proof of the local existence of an approximate solution is almost the same, as in the linear case (see Theorem 3.1). We use approximate solutions of the same structure which satisfy (31) with $-\mathcal{F}(u_n(t)) + G_{pl}(t)$ instead of $G_{pl}(t)$. Then using the standard argument we establish the energy relation in (46) for these approximate solutions. Now the positivity type estimate in (44) allow us to obtain the same a priori estimates as in (35) and (36). Therefore we can prove the global existence of approximate solutions and establish the existence of a weak solution $U(t) = (v(t); u(t); u_t(t))$ by the same argument as in the linear case. To make limit transition in the nonlinear term we use (43).

Next we consider the pair $(v(t); u(t))$ as a solution to the linear problem with $G_{pl}(t) :=$ $:= -\mathcal{F}(u(t)) + G_{pl}(t)$. This allow us to obtain (45) and also derive energy balance relation (46) from (27) using the potential structure of the force $\mathcal{F}$: $\mathcal{F}(u) = \Pi'(u)$.

Since the difference of two weak solution can be treated as a solution to the linear problem with $G_f \equiv 0$ and $G_{pl}(t) := \mathcal{F}(\hat{u}(t)) - \mathcal{F}(u(t))$, we can obtain (47) from the energy equality (27). The uniqueness follows from (47).

Preservation of the spatial average of $u(t)$ follows from the same property for approximate solutions.

Theorem 4.1 is proved.

We can derive from Theorem 4.1 the following assertion.

Corollary 4.1.   *In addition to the hypotheses of Theorem 4.1 we assume that $G_f(t) \equiv G_0 \in V'$ is independent of t and $G_{pl}(t) \equiv 0$. Then problem (2)–(8) generates dynamical systems $(S_t, \mathcal{H})$ and $(S_t, \widehat{\mathcal{H}})$ with the evolution operator defined by the formula $S_t U_0 = (v(t); u(t); u_t(t))$, where $(v; u)$ is a weak solution to (2)–(8) with the initial data $U_0 = (v_0; u_0; u_1)$.*

If we assume in addition that $G_0 = 0$ and Assumption 3.1 holds, then these systems are gradient with the full energy $\mathcal{E}(v_0, u_0, u_1)$ as a Lyapunov function. This means that (a) $U \mapsto \mathcal{E}(U)$ is continuous on $\mathcal{H}$, (b) $\mathcal{E}(S_t U_0)$ is not increasing in t, and (c) if $\mathcal{E}(S_t U_0) = \mathcal{E}(U_0)$ for some $t > 0$, then U_0 is a stationary point of S_t (i.e., $S_t U_0 = U_0$ for all $t \geq 0$). Moreover, the set $\mathcal{E}_R = \{U_0 : \mathcal{E}(U_0) \leq R\}$ is a bounded closed forward invariant set for every $R > 0$.

Proof.   The argument is the same as in [14]. We only note that under Assumption 3.1 from (46) (with $G_f = 0$ and $G_{pl} = 0$) we have that every stationary point U_* for S_t has the form $U_* = (0; u; 0)$, where $u \in H_0^2(\Omega)$.

4.2. Stationary solutions.   As above we assume that $G_{pl} \equiv 0$ and $G_f \equiv 0$ and Assumptions 2.1 and 3.1 holds. It follows from Definition 4.1 that a stationary (time-independent) solution is a pair $(v; u)$ from $\widetilde{V} \times H_0^2(\Omega)$ satisfying the relation

$$\nu(\nabla v, \nabla \psi)_\mathcal{O} + (L_0 v, \psi)_\mathcal{O} + (\Delta u, \Delta \beta)_\Omega + (\mathcal{F}(u), \beta)_\Omega = 0 \qquad (48)$$

for any $\psi \in W$ with $\psi^3\big|_\Omega = \beta$, where W is given by (24). Taking $\psi = v$ we conclude that $\nu\|\nabla v\|_\mathcal{O}^2 + (Av, v)_\mathcal{O} = 0$ and hence from Assumption 3.1 we have $v = 0$. Therefore we obtain the following variational problem for $u \in H_0^2(\Omega)$:

$$(\Delta u, \Delta \beta)_\Omega + (\mathcal{F}(u), \beta)_\Omega = 0 \quad \forall \beta \in \widehat{H}_0^2(\Omega). \qquad (49)$$

As in [14] we can show the existence of a family of solutions to (49) parameterized by a real parameter. In the case of the zero average of u we can fix this parameter and obtain the following assertion (see [14] for details).

Proposition 4.1 [14]. *In addition to Assumption 4.1 we assume that there exist $\eta < 1/2$ and $c \geq 0$ such that*

$$\eta\|\Delta u\|_\Omega^2 + (u, \mathcal{F}(u))_\Omega \geq -c \quad \forall u \in H_0^2(\Omega). \qquad (50)$$

Then the set $\mathcal{N}_0$ of solutions u to problem (49) with the property $\int_\Omega u\,dx = 0$ is nonempty compact set in $\widehat{H}_0^2(\Omega)$. This implies that the set $\mathcal{N}$ of all stationary points of S_t in the space $\widehat{\mathcal{H}}$ is nonempty compact set and has the form

$$\mathcal{N} = \left\{ (0; u; 0) \colon u \in \widehat{H}_0^2(\Omega) \ \ solves\ (49) \right\}. \tag{51}$$

4.3. Asymptotical behavior. In this section we are interested in global asymptotic behavior of the dynamical system $(S_t, \widehat{\mathcal{H}})$. Our main result states the existence of a compact global attractor of finite fractal dimension.

We recall (see, e.g., [3, 7, 33]) that a *global attractor* of the dynamical system $(S_t, \widehat{\mathcal{H}})$ is defined as a bounded closed set $\mathfrak{A} \subset \widehat{\mathcal{H}}$ which is invariant ($S(t)\mathfrak{A} = \mathfrak{A}$ for all $t > 0$) and uniformly attracts all other bounded sets:

$$\lim_{t \to \infty} \sup\{\mathrm{dist}_{\mathcal{H}}(S(t)y, \mathfrak{A}) \colon y \in B\} = 0 \qquad \text{for any bounded set} \quad B \quad \text{in} \quad \widehat{\mathcal{H}}.$$

Theorem 4.2. *Let Assumptions 2.1, 3.1, and 4.1 be in force. Assume that $G_{pl} \equiv 0$, $G_f \equiv 0$ and (50) hold. Then the dynamical system $(S_t, \widehat{\mathcal{H}})$ possesses a compact global attractor $\mathfrak{A}$ of finite fractal dimension*[3].

Moreover,

(1) *Any trajectory $\gamma = \{(v(t); u(t); u_t(t)) \colon t \in \mathbb{R}\}$ from the attractor $\mathfrak{A}$ possesses the properties*

$$(v_t; u_t; u_{tt}) \in L_\infty(\mathbb{R}; X \times \widehat{H}_0^2(\Omega) \times \widehat{L}_2(\Omega)) \tag{52}$$

and there is $R > 0$ such that

$$\sup_{\gamma \subset \mathfrak{A}} \sup_{t \in \mathbb{R}} \left(\|v_t\|_{\mathcal{O}}^2 + \|u_t\|_{2,\Omega}^2 + \|u_{tt}\|_{\Omega}^2 \right) \le R^2. \tag{53}$$

(2) *The global attractor $\mathfrak{A}$ consists of full trajectories $\{(v(t); u(t); u_t(t)) \colon t \in \mathbb{R}\}$ which are homoclinic to the set $\mathcal{N}$, i.e.,*

$$\lim_{t \to \pm\infty} \inf_{u_* \in \mathcal{N}_0} \left(\|v(t)\|_{\mathcal{O}}^2 + \|u - u_*\|_{2,\Omega}^2 + \|u_t\|_{\Omega}^2 \right) = 0,$$

where $\mathcal{N}_0 = \left\{ u \in \widehat{H}_0^2(\Omega) \ \ solves\ (49) \right\}$. In addition we have

$$\lim_{t \to +\infty} \mathrm{dist}_{\widehat{\mathcal{H}}}(S_t y, \mathcal{N}) = 0 \qquad \text{for any initial data} \quad y \in \widehat{\mathcal{H}}. \tag{54}$$

We emphasize that Theorem 4.2 deals with dynamics in the space $\widehat{\mathcal{H}}$ (the case of the zero spatial average of the deflection). For a possible approach to description of the system long-time behavior in the space $\mathcal{H}$ we refer to [14] (Remark 4.9).

To obtain the result stated in Theorem 4.2 it is sufficient to show that the system is quasi-stable in the sense of Definition 7.9.2 [11] (see also Section 4.4 in [12]). For this we can use the stability properties of linear problem (18)–(22) established in Theorem 3.2 and the same argument as in [14] which yields the following assertion.

Lemma 4.1 (Quasi-stability). *Let the hypotheses of Theorem 4.2 be in force and $U^i(t) = = (v^i(t); u^i(t); u_t^i(t))$, $i = 1, 2$, be two weak solutions with initial data $U_0^i = (v_0^i; u_0^i; u_1^i)$ from $\widehat{\mathcal{H}}$ such that $\|U_0^i\|_{\mathcal{H}} \le R$, $i = 1, 2$. Then the difference*

$$Z(t) = U^1(t) - U^2(t) \equiv (v(t); u(t); u_t(t))$$

satisfies the relation

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[3]For the definition and some properties of the *fractal dimension*, see, e.g., [7] or [33].

$$\|Z(t)\|_{\mathcal{H}}^2 \leq M_R e^{-\gamma_* t}\|Z_0\|_{\mathcal{H}}^2 + M_R \int_0^t e^{-\gamma_*(t-\tau)}\|u(\tau)\|_{\Omega}^2 d\tau \tag{55}$$

for some positive constant M_R and γ_*.

Proof. See [14] for some details.

To complete the proof of Theorem 4.2 we note that by Proposition 7.9.4 [11] $(S_t, \widehat{\mathcal{H}})$ is asymptotically smooth, i.e., for any bounded set B in $\widehat{\mathcal{H}}$ such that $S_t B \subset B$ for $t > 0$ there exists a compact set K in the closure $\overline{B}$ of B, such that $S_t B$ converges uniformly to K. By Corollary 4.1 the system is gradient. Proposition 4.1 yields that the set $\mathcal{N}$ of the stationary points (see (51)) is bounded in $\widehat{\mathcal{H}}$. Therefore to prove the existence of a global attractor we can use well-known criteria for gradient systems (see, e.g., [31] (Theorem 4.6) or Corollary 2.29 in [10]).

The standard results on gradient systems with compact attractors (see, e.g., [3, 7, 33]) imply (54). Since $(S_t, \widehat{\mathcal{H}})$ is quasi-stable, the finiteness of fractal dimension $\dim_f \mathfrak{A}$ follows from Theorem 7.9.6 [11]. To obtain the result on regularity stated in (52) and (53) we apply Theorem 7.9.8 [11].

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