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HYBRID TYPE GENERALIZED MULTIVALUED VECTOR COMPLEMENTARITY PROBLEMS

УЗАГАЛЬНЕНІ БАГАТОЗНАЧНІ ВЕКТОРНІ ЗАДАЧІ ДОПОВНЮВАНОСТІ ГІБРИДНОГО ТИПУ

We introduce a new type of generalized multivalued vector complementarity problems with moving pointed cone. We discuss the existence results for generalized multivalued vector complementarity problems under inclusive assumptions and obtain results on the equivalence between the generalized multivalued vector complementarity problems and the generalized multivalued vector variational inequality problems.

Введено новий тип узагальнених багатозначних векторних задач доповнюваності з рухомим загостреним конусом. Розглянуто питання про існування розв'язків узагальнених багатозначних векторних задач доповнюваності при умовах включення та отримано результати щодо еквівалентності між узагальненими багатозначними векторними задачами доповнюваності та узагальненими багатозначними векторними задачами для варіаційних нерівностей.

1. Introduction and preliminaries. The purpose of this paper is to introduce and discuss a new type of generalized multivalued vector complementarity problem with moving pointed cone which is a variable ordering relation. We derive existence of solutions for this class of generalized multivalued vector complementarity problems under inclusive type assumptions. This inclusive conditions require that any two of the family which is closed and convex satisfy an inclusion relation so long as their corresponding variable satisfy certain conditions. We have also obtained some equivalence results among a generalized multivalued vector complementarity problem, a generalized multivalued vector variational inequality problem, a generalized multivalued weak minimal element problem and a generalized multivalued vector unilateral optimization problem under some monotonicity conditions and some inclusive type assumptions in ordered Banach spaces. The theorems presented in this paper improved, extended and developed some earlier and very recent results in the literature including [2, 4 – 6, 8, 9, 11].

Motivated and inspired by these works [1, 3, 7, 8, 12 – 16], in this paper we initiate three types of generalized multivalued vector complementarity problems. Let X, Y be the two Banach spaces, $P\colon D \to 2^Y$ a multivalued mapping such that for each $x \in D, P(x)$ is a proper, closed, convex and pointed moving cone with apex at the origin and $\operatorname{int} P(x) \neq \varnothing$. Let $Q(\cdot,\cdot)\colon D \times L(X,Y) \to \to L(X,Y)$, $g\colon D \to Y$ be the single-valued mappings and $A\colon X \to 2^{L(X,Y)}$ be the multivalued mapping, where $2^{L(X,Y)}$ is a collection of all nonempty subsets of $L(X,Y)$. We consider the following three kinds of generalized multivalued vector complementarity problems.

Weakly generalized multivalued vector complementarity problem (**WGMVCP**): finding $x \in D$ and $u \in A(x)$ such that

$$\langle Q(x,u), g(x)\rangle \ngeq_{\operatorname{int} P(x)} 0, \qquad \langle Q(x,u), g(y)\rangle \nleq_{\operatorname{int} P(x)} 0 \quad \forall y \in D. \tag{1.1}$$

Positive generalized multivalued vector complementarity problem (**PGMVCP**): finding $x \in D$ and $u \in A(x)$ such that

$$\langle Q(x,u), g(x)\rangle \not>_{\operatorname{int} P(x)} 0, \qquad \langle Q(x,u), g(y)\rangle \geq_{P(x)} 0 \quad \forall y \in D. \tag{1.2}$$

Strong generalized multivalued vector complementarity problem (**SGMVCP**): finding $x \in D$ and $u \in A(x)$ such that

$$\langle Q(x,u), g(x)\rangle = 0, \qquad \langle Q(x,u), g(y)\rangle \geq_{P(x)} 0 \quad \forall y \in D. \tag{1.3}$$

***Remark* 1.1.** If we take g an identity mapping and $Q(x,u) = Q(u)$, then these generalized multivalued vector complementarity problems reduce to the following types of complementarity problems in Ceng and Lin [1].

Generalized weak vector complementarity problem (**GWVCP**): finding $x \in D$ and $u \in A(x)$ such that

$$\langle Q(u), x\rangle \not\geq_{\operatorname{int} P(x)} 0, \qquad \langle Q(u), y\rangle \not\leq_{\operatorname{int} P(x)} 0 \quad \forall y \in D. \tag{1.4}$$

Generalized positive vector complementarity problem (**GPVCP**): finding $x \in D$ and $u \in A(x)$ such that

$$\langle Q(u), x\rangle \not>_{\operatorname{int} P(x)} 0, \qquad \langle Q(u), y\rangle \geq_{P(x)} 0 \quad \forall y \in D. \tag{1.5}$$

Generalized strong vector complementarity problem (**GSVCP**): finding $x \in D$ and $u \in A(x)$ such that

$$\langle Q(u), x\rangle = 0, \qquad \langle Q(u), y\rangle \geq_{P(x)} 0 \quad \forall y \in D. \tag{1.6}$$

Again we remark that, if $A\colon X \to L(X,Y)$ and Q an identity mapping then aforesaid three kinds of problems reduce to the problem of Huang et al [8].

(Weak) vector complementarity problem (**VCP**): finding $x \in D$ such that

$$\langle A(x), x\rangle \not\geq_{\operatorname{int} P(x)} 0, \qquad \langle A(x), y\rangle \not\leq_{\operatorname{int} P(x)} 0 \quad \forall y \in D. \tag{1.7}$$

Positive vector complementarity problem (**PVCP**): finding $x \in D$ such that

$$\langle A(x), x\rangle \not>_{\operatorname{int} P(x)} 0, \qquad \langle A(x), y\rangle \geq_{P(x)} 0 \quad \forall y \in D. \tag{1.8}$$

Strong vector complementarity problem (**SVCP**): finding $x \in D$ such that

$$\langle A(x), x\rangle = 0, \qquad \langle A(x), y\rangle \geq_{P(x)} 0 \quad \forall y \in D. \tag{1.9}$$

We note that if $P(x) = P$ for all $x \in D$, where P is a closed, pointed and convex cone in Y with nonempty interior $\operatorname{int} P(x)$, then all these problems are equivalent to problems considered in Chen and Yang [3].

2. Existence of a solution for GMVCP. In this section, we extend their results to the cases involving the multivalued mappings.

Let X be an arbitrary real Hausdorff topological vector space and Y a Banach space. Let $L(X,Y)$ denotes the space of all continuous linear mappings from X to Y. Let D be a nonempty subset of X and $P\colon D \to 2^Y$ a multivalued mapping such that for each $x \in D$, $P(x)$ is a proper, closed, convex and pointed moving cone with apex at the origin and $\operatorname{int} P(x) \neq \varnothing$. Let K be a subset of Y. For each $x \in D$, a point $z \in K$ is called a minimal point of K with respect to the cone $P(x)$

if $K \cap (z - P(x)) = \{z\}$; $\mathrm{Min}_{P(x)} K$ is the set of all minimal points of K with respect to the cone $P(x)$; a point $z \in K$ is called a weakly minimal point of K with respect to the cone $P(x)$ if $K \cap (z - \operatorname{int} P(x)) = \varnothing$; $\mathrm{Min}^w_{P(x)} K$ is the set of all weakly minimal points of K with respect to the cone $P(x)$, for details see [11].

Let $Q\colon D \times L(X,Y) \to L(X,Y)$ and $g\colon D \to Y$ be the single-valued mappings and $A\colon X \to \to 2^{L(X,Y)}$ be the multivalued mapping. Consider the generalized multivalued vector complementarity problem (**GMVCP**): finding $x \in D$ and $u \in A(x)$ such that

$$\langle Q(x,u), g(x)\rangle \not\geq_{\operatorname{int} P(x)} 0, \qquad \langle Q(x,u), g(y)\rangle \not\leq_{\operatorname{int} P(x)} 0 \quad \forall y \in D. \tag{2.1}$$

A feasible set of (**GMVCP**) is

$$\Omega = \Big\{(x,u) \in D \times A(D)\colon\ u \in A(x),\ \langle Q(x,u), g(y)\rangle \not\leq_{\operatorname{int} P(x)} 0 \ \ \forall y \in D\Big\}. \tag{2.2}$$

We consider the following generalized multivalued vector optimization problem (**GMVOP**):

$$\mathrm{Min}_P \langle Q(x,u), g(x)\rangle \quad \text{subject to} \quad (x,u) \in \Omega. \tag{2.3}$$

A point $(x,u) \in \Omega$ is called a weakly minimal solutions of (**GMVOP**) with respect to the cone $P(x)$ if $\langle Q(x,u), g(x)\rangle$ is a weakly minimal point of (**GMVOP**) with respect to the cone $P(x)$, that is

$$\langle Q(x,u), g(x)\rangle \in \mathrm{Min}^w_{P(x)} \big\{\langle Q(x,u), g(x)\rangle\colon (x,u) \in \Omega\big\}.$$

We represent the set of all weakly minimal solutions of (**GMVOP**) with respect to the cone $P(x)$ by $\zeta^w_{P(x)}$ and the set of all weakly minimal solutions of (**GMVOP**) by ζ^w, that is

$$\zeta^w = \bigcup_{x \in D} \zeta^w_{P(x)}. \tag{2.4}$$

Theorem 2.1. *If $\zeta^w \neq \varnothing$ and, for some $x \in D$, there exists $(x,u) \in \zeta^w_{P(x)}$ such that*

$$\langle Q(x,u), g(x)\rangle \not\geq_{\operatorname{int} P(x)} 0,$$

then generalized multivalued vector complementarity problem **(GMVCP)** *is solvable.*

Proof. Let $(x,u) \in \zeta^w_{P(x)}$ and

$$\langle Q(x,u), g(x)\rangle \not\geq_{\operatorname{int} P(x)} 0, \tag{2.5}$$

then $x \in D$, $u \in A(x)$ and

$$\langle Q(x,u), g(x)\rangle \not\geq_{\operatorname{int} P(x)} 0, \qquad \langle Q(x,u), g(x)\rangle \not\leq_{\operatorname{int} P(x)} 0 \quad \forall y \in D.$$

Therefore, x is a solution of (**GMVCP**).

Theorem 2.1 is proved.

Remark **2.1.** If $Q(x,u) = Q(u)$ and g is an identity mapping, then Theorem 2.1 coincides with Theorem 2.1 in Ceng and Lin [1]. Again if $Q \equiv I$, the identity mapping of $L(X,Y)$ and A is a single-valued mapping from $D = X$ to $L(X,Y)$, then Theorem 2.1 coincides with Theorem 2.1 of Huang et al. [8].

Definition 2.1. *Let $A\colon D \to 2^{L(X,Y)}$, $P\colon D \to 2^Y$ be the two multivalued mappings with* $\operatorname{int} P(x) \neq \varnothing$ *for every $x \in D$, $Q\colon D \times L(X,Y) \to L(X,Y)$ and $g\colon D \to Y$ be the two single-valued mappings and Ω a subset of $D \times A(D)$. We say that P is inclusive with respect to Ω if for any $(x,u), (y,v) \in \Omega$,*

$$\langle Q(x,u), g(x)\rangle \leq_{\operatorname{int} P(y)} \langle Q(x,v), g(y)\rangle \quad \textit{implies that} \quad P(x) \subset P(y). \tag{2.6}$$

It is easy to see that, if $P(x) = P$ for all $x \in D$, where P is a closed, pointed and convex moving cone in Y, then P is inclusive with respect to Ω.

Theorem 2.2. *Suppose that P is inclusive with respect to Ω. If there exists at most finite number of solutions for* **(GMVCP)**, *then* **(GMVCP)** *is a solvable if and only if $\zeta^w \neq \varnothing$ and there exists $(x,u) \in \zeta^w_{P(x)}$ such that*

$$\langle Q(x,u), g(x)\rangle \not\geq_{\operatorname{int} P(x)} 0.$$

Proof. Let ρ_1 be a solution of (**GMVCP**), then there exists $u_1 \in A_{\rho_1}$ such that

$$\langle Q(\rho_1,u_1), g(\rho_1)\rangle \not\geq_{\operatorname{int} P(\rho_1)} 0, \quad \langle Q(\rho_1,u_1), g(y)\rangle \not\leq_{\operatorname{int} P(\rho_1)} 0 \quad \forall y \in D. \tag{2.7}$$

If $(\rho_1,u_1) \in \zeta^w_{P(\rho_1)}$, then

$$\langle Q(\rho_1,u_1), g(\rho_1)\rangle \not\geq_{\operatorname{int} P(\rho_1)} 0, \tag{2.8}$$

and hence the conclusion holds. If $(\rho_1,u_1) \notin \zeta^w_{P(\rho_1)}$, by the definition of a weakly minimal solution there exists $(\rho_2,u_2) \in \Omega$ such that

$$\langle Q(\rho_2,u_2), g(y)\rangle \not\leq_{\operatorname{int} P(\rho_2)} 0 \quad \forall y \in D, \tag{2.9}$$

$$\langle Q(\rho_2,u_2), g(\rho_2)\rangle \leq_{\operatorname{int} P(\rho_1)} \langle Q(\rho_1,u_1), g(\rho_1)\rangle \not\geq_{\operatorname{int} P(\rho_1)} 0.$$

This implies that

$$\langle Q(\rho_2,u_2), g(\rho_2)\rangle \not\geq_{\operatorname{int} P(\rho_1)} 0. \tag{2.10}$$

Since

$$\langle Q(\rho_2,u_2), g(\rho_2)\rangle \leq_{\operatorname{int} P(\rho_1)} \langle Q(\rho_1,u_1), g(\rho_1)\rangle,$$

and P is inclusive with respect to Ω, it follows that $P(\rho_2) \subset P(\rho_1)$, and this implies that

$$\langle Q(\rho_2,u_2), g(\rho_2)\rangle \not\geq_{\operatorname{int} P(\rho_2)} 0. \tag{2.11}$$

Thus ρ_2 is a solution of (**GMVCP**) and $\rho_2 \neq \rho_1$. Continuing this process there exists $(\rho_n,u_n) \in \Omega$ such that ρ_n is a solution of (**GMVCP**) and $(\rho_n,u_n) \in \zeta^w_{P(\rho_n)}$, since (**GMVCP**) has almost a finite number of solutions. Thus

$$\langle Q(\rho_n,u_n), g(\rho_n)\rangle \in \operatorname{Min}^w_{P(\rho_n)}\big\{\langle Q(x,u), g(\rho_n)\rangle \colon (\rho_n,u) \in \Omega\big\}$$

and

$$\langle Q(\rho_n,u_n), g(\rho_n)\rangle \not\geq_{\operatorname{int} P(\rho_n)} 0. \tag{2.12}$$

Combining this result and Theorem 2.1, we have the conclusion of the theorem.

***Remark* 2.2.** If $Q(x,u)=Q(u)$ and g is an identity mapping then Theorem 2.2 reduces to the Theorem 2.4 of Ceng and Lin [1]. If $Q\equiv I$ is an identity mapping, A is single-valued mapping from X to $L(X,Y)$ and $P(x)=P$ for all $x\in X$, where P is a closed pointed and convex moving cone in Y, then $P(x)$ satisfies the inclusive assumption with respect to Ω and Theorem 2.2 is equivalent to the Theorem 3.2 of Chen and Yang [3]. We note that if $Q\equiv I$, the identity mapping of $L(X,Y)$ and A is a single-valued mapping from $D=X$ to $L(X,Y)$, then Theorem 2.2 coincides with Theorem 2.2 of Huang et al. [8].

Next we consider the positive generalized multivalued vector complementarity problem (**PGMVCP**): finding $x\in D, u\in A(x)$ such that

$$\langle Q(x,u),g(x)\rangle \not\geq_{\operatorname{int}P(x)} 0, \qquad \langle Q(x,u),g(y)\rangle \geq_{P(x)} 0, \quad y\in D. \tag{2.13}$$

Let

$$\psi=\{(x,u)\in D\times A(D)\colon u\in A(x),\ \langle Q(x,u),g(y)\rangle\geq_{P(x)} 0\ \ \forall y\in D\}. \tag{2.14}$$

Consider the following generalized multivalued vector optimization problem (**GMVOP**) to be

$$\operatorname{Min}_P\langle Q(x,u),g(x)\rangle \quad \text{subject to} \quad (x,u)\in\psi. \tag{2.15}$$

We denote the set of all minimal point of (**GMVOP**) with respect to the moving cone $P(x)$ by $\Gamma_{P(x)}$, that is

$$\Gamma_{P(x)}=\operatorname{Min}_{P(x)}\{\langle Q(x,u),g(x)\rangle\colon (x,u)\in\psi\}$$

and denote the set of all minimal point of (**GMVOP**) by

$$\Gamma=\bigcup_{x\in D}\Gamma_{P(x)}. \tag{2.16}$$

Theorem 2.3. *If* $\Gamma\neq\varnothing$ *and there exists* $(x,u)\in\Gamma_{P(x)}$ *such that*

$$\langle Q(x,u),g(x)\rangle \not\geq_{\operatorname{int}P(x)} 0,$$

then **(PGMVCP)** *is solvable.*

Theorem 2.4. *Suppose that* P *is inclusive with respect to* ψ. *If there exists at most a finite number of solutions of* **(PGMVCP)**, *then* **(PGMVCP)** *is solvable if and only if* $\Gamma\neq\varnothing$ *and there exists* $(x,u)\in\Gamma_{P(x)}$ *such that*

$$\langle Q(x,u),g(x)\rangle \not\geq_{\operatorname{int}P(x)} 0. \tag{2.17}$$

We note that if $Q(x,u)=Q(u)$ and g is an identity mapping then Theorem 2.3 and 2.4 coincide with Theorem 2.6 and 2.7 of Ceng and Lin [1]. Also again if $Q\equiv I$ is an identity mapping of $L(X,Y)$ and A is a single-valued mapping from $D=X$ to $L(X,Y)$, then Theorem 2.3 and 2.4 is similar to Theorem 2.3 and 2.4 of Huang et al. [8].

3. Equivalence between the generalized multivalued vector complementarity problems and generalized multivalued weak minimal element problems. Let X, Y be the two Banach spaces and $P\colon D\to 2^Y$ a multivalued mapping such that for each $x\in D, P(x)$ is a proper, closed, convex and pointed moving cone with apex at the origin and $\operatorname{int}P(x)\neq\varnothing$. Let $Q\colon D\times L(X,Y)\to L(X,Y)$

and $g\colon D \to D$ the single-valued mappings, $A : X \to 2^{L(X,Y)}$ the multivalued mapping, where $2^{L(X,Y)}$ is a collection of all nonempty subsets of $L(X,Y)$, and $f\colon X \to Y$ a given operator.

Define the feasible set associated with A and K,

$$\Omega = \{x \in D\colon \text{there exists } u \in A(x) \text{ such that } \langle Q(x,u), g(y)\rangle \not\leq_{\operatorname{int} P(x)} 0 \ \ \forall y \in D\}. \tag{3.1}$$

We consider the following problems:

1. Generalized multivalued vector optimization problem **(GMVOP)**$_l$ for a given $l \in L(X,Y)$: finding $x \in \Omega$ such that

$$l(x) \in \operatorname{Min}^w_{P(x)} l(\Omega). \tag{3.2}$$

2. Generalized multivalued weak minimal element problem (**GMWMEP**): finding $x \in \Omega$ such that

$$x \in \operatorname{Min}^w_D \Omega.$$

3. Generalized multivalued vector complementarity problem (**GMVCP**): finding $x \in \Omega$, $u \in$ $\in A(x)$ such that

$$\langle Q(x,u), g(x)\rangle \not\geq_{\operatorname{int} P(x)} 0.$$

4. Generalized multivalued vector variational inequality problem (**GMVVIP**): finding $x \in D$ and $u \in A(x)$ such that

$$\langle Q(x,u), g(y) - g(x)\rangle \not\leq_{\operatorname{int} P(x)} 0 \quad \forall y \in D. \tag{3.3}$$

5. Generalized multivalued vector unilateral optimization problem (**GMVUOP**): finding $x \in D$ such that

$$f(x) \in \operatorname{Min}^w_{P(x)} f(D).$$

Definition 3.1 [3]. *A linear operator $l\colon X \to Y$ is called weakly positive if for any $x, y \in X$, $x \not\geq_{\operatorname{int} P} y$ implies that*

$$l(x) \not\geq_{\operatorname{int} P(x)} l(y).$$

Definition 3.2. *Let X and Y be the two Banach spaces, and l be a linear operator from X to Y. If the image of any bounded set in X is a self-sequentially compact set in Y then l is called completely continuous. A mapping $f\colon X \to Y$ is said to be convex if*

$$f(\lambda x + (1-\lambda)y) \leq_{P(x)} \lambda f(x) + (1-\lambda) f(y) \tag{3.4}$$

for all $x, y \in X$ and $0 \leq \lambda \leq 1$.

Definition 3.3. *Let $Q\colon L(X,Y) \to L(X,Y)$ and $f\colon X \to Y$ be two mappings. f is said to be Q-subdifferential at $x_0 \in X$ if there exists $u_0 \in L(X,Y)$ such that*

$$f(x) - f(x_0) \geq_{P(x_0)} \langle Q(u_0), x - x_0\rangle \quad \forall x \in X.$$

If f is Q-subdifferentiable at $x_0 \in X$, then we define the Q-subdifferential of f at x_0 as follows:

$$\partial_Q f(x_0) = \{u \in L(X,Y)\colon f(x) - f(x_0) \geq_{P(x_0)} \langle Q(u),\ x - x_0\rangle\} \quad \forall x \in X.$$

If f is Q-subdifferentiable at each $x \in X$, then f is Q-subdifferential on X.

***Remark* 3.1.** We note that if X and Y are two Banach spaces, a mapping $f\colon X \to Y$ is Frechet differentiable at $x_0 \in X$ if there exists a linear bounded operator $Df(x_0)$ such that

$$\lim_{x\to 0} \frac{\|f(x_0+x) - f(x_0) - \langle Df(x_0), x\rangle\|}{\|x\|} = 0,$$

where $Df(x_0)$ is said to be the Frechet derivative of f at x_0. The mapping f is said to be Frechet differentiable at each point of X if $f\colon X \to Y$ is convex and Frechet differentiable on X. Then

$$f(y) - f(x) \geq_{P(x)} \langle Df(x), y - x\rangle \quad \forall x, y \in X.$$

If f is a Frechet differentiable on X, then for each $x, y \in X$ we have

$$f(y) - f(x) \geq_{P(x)} \langle A(u), y - x\rangle \quad \forall y \in \partial_Q f(x).$$

Definition 3.4. *Let X be a Banach space, $D \subset X$ be a proper, closed, convex and moving pointed cone with apex at origin and* $\operatorname{int} D \neq \varnothing$*. The norm $\|\cdot\|$ in X is called strictly monotonically increasing in D if for each $y \in D$*

$$x \in (\{y\} - \operatorname{int} D) \cap D \Rightarrow \|x\| \leq \|y\|. \tag{3.5}$$

Theorem 3.1. *Let X, Y be the two Banach spaces, $D \subset X$ a proper, closed, convex moving pointed cone with apex at origin and* $\operatorname{int} D \neq \varnothing$*. Let $P\colon D \to 2^Y$ be a multivalued mapping with closed, convex moving pointed cone values such that* $\operatorname{int} P(x) \neq \varnothing$ *for all $x \in D$. Suppose that*

1) $T = \partial_Q f$ *is the subdifferential of a convex operator $f\colon X \to Y$*;

2) l *is a weakly positive linear operator*;

3) *there exists $x \in \Omega$ such that $Q(u)$ is one to one and completely continuous, where $u \in A(x)$ is associated with x in the definition of* Ω;

4) *X is a topological dual space of a real normed space and the norm $\|\cdot\|$ in X is strictly monotonically increasing on D.*

If **(GMVVIP)** *is solvable, then* **(GMVOP)**$_l$, **(GMWMEP)**, **(GMVCP)** *and* **(GMVUOP)** *are also solvable.*

We need the following proposition to prove Theorem 3.1.

Proposition 3.1. *Let $Q\colon D \times L(X,Y) \to L(X,Y)$ and $f\colon X \to Y$ be the two single-valued mappings, $A\colon D \to 2^{L(X,Y)}$ be the multivalued mapping and let $T = \partial_Q f$ be the Q-subdifferential of f. Then x solves* **(GMVUOP)** *which implies that x solves* **(GMVVIP)**. *If in addition, f is a convex mapping then conversely x solves* **(GMVVIP)** *which implies that x solves* **(GMVUOP)**.

Proof. Let x be a solution of (**GMVUOP**). Then $x \in D$ and

$$f(x) \in \operatorname{Min}^w_{P(x)} f(D)$$

that is,

$$f(x) \ngeq_{\operatorname{int} P(x)} f(y) \quad \forall y \in D.$$

Since D is convex cone, therefore

$$f(x) \ngeq_{\operatorname{int} P(x)} f(x + t(w - x)), \qquad 0 < t < 1, \quad w \in D. \tag{3.6}$$

Also, since f is Q-subdifferential on X, it follows that

$$f(x) \ngeq_{\operatorname{int} P(x)} f(x+t(w-x)) \geq_{P(x)} f(x)+\langle Q(x,u), \qquad t(w-x)\rangle, \quad 0<t<1, \quad w \in D. \tag{3.7}$$

This implies that

$$\langle Q(x,u), t(w-x)\rangle \nleq_{\operatorname{int} P(x)} 0, \qquad 0<t<1, \quad w \in D, \tag{3.8}$$

and hence

$$\langle Q(x,u), w-x\rangle \nleq_{\operatorname{int} P(x)} 0 \quad \forall w \in D. \tag{3.9}$$

Therefore, x is the solution of (**GMVVIP**).

Conversely, let x solve (**GMVVIP**). Then there exists $u_0 \in A(x) = \partial_Q f(x)$ such that

$$\langle Q(x,u_0), w-x\rangle \nleq_{\operatorname{int} P(x)} 0 \quad \forall w \in D. \tag{3.10}$$

Since f is Q-subdifferentiable on X, we have for all $u \in A(x) = \partial_Q f(x)$

$$f(w)-f(x) \geq_{P(x)} \langle Q(x,u), w-x\rangle \quad \forall w \in D. \tag{3.11}$$

Since f is convex, therefore we have

$$f(w)-f(x) \geq_{P(x)} \langle Q(x,u_0), w-x\rangle \nleq_{\operatorname{int} P(x)} 0 \quad \forall w \in D. \tag{3.12}$$

This implies that

$$f(w) \nleq_{\operatorname{int} P(x)} f(x) \quad \forall w \in D. \tag{3.13}$$

Therefore, x solves (**GMVUOP**).

Proposition 3.1 is proved.

Proposition 3.2. *If x solves* (**GMVVIP**), *then x also solves* (**GMVCP**). *Conversely, if*

$$\langle Q(x,u), g(x)\rangle \leq_{P(x)} 0 \qquad \forall w \in D, \quad u \in A(x),$$

then x solve (**GMVCP**) *which implies that x solve* (**GMVVIP**).

Proof. Let x be a solution of (**GMVVIP**), then there exists $u \in A(x)$ such that

$$\langle Q(x,u), g(y)-g(x)\rangle \nleq_{\operatorname{int} P(x)} 0 \quad \forall y \in D. \tag{3.14}$$

Letting $y=0$, we get

$$\langle Q(x,u), g(x)\rangle \ngeq_{\operatorname{int} P(x)} 0.$$

Since g is convex and letting $y = w + x$ such that

$$g(y) = g(w+x) \leq g(w)+g(x),$$

we have

$$\langle Q(x,u), g(w)\rangle \nleq_{\operatorname{int} P(x)} 0 \quad \forall w \in D. \tag{3.15}$$

Thus x is a solution of (**GMVCP**).

Conversely, let x solve (**GMVCP**). Then there exists $u \in Ax$ such that

$$\langle Q(x,u), g(x)\rangle \leq_{P(x)} 0 \not\geq_{\operatorname{int} P(x)} \langle Q(x,u), g(y)\rangle \quad \forall y \in D. \tag{3.16}$$

This implies that

$$\langle Q(x,u), g(x)\rangle \not\geq_{\operatorname{int} P(x)} \langle Q(x,u), g(y)\rangle \quad \forall y \in D \tag{3.17}$$

and so

$$\langle Q(x,u), g(y) - g(x)\rangle \not\geq_{\operatorname{int} P(x)} 0 \quad \forall y \in D. \tag{3.18}$$

Proposition 3.2 is proved.

Proposition 3.3. *Let l be a weakly positive linear operator. Then x solve* **(GMWMEP)** *which implies that x solve* **(GMVOP)**.

Proof. Let x be a solution of (**GMWMEP**). Then $x \in \Omega$ and $x \in \operatorname{Min}_D^w \Omega$. That is for any $z \in \Omega$,

$$x \not\geq_{\operatorname{int} D} z.$$

Since l is a weakly positive linear operator, it follows that

$$l(x) \not\geq_{P(x)} l(z)$$

and so

$$l(x) \in \operatorname{Min}^w_{P(x)} l(\Omega), \tag{3.19}$$

hence x solve (**GMVOP**).

Proposition 3.3 is proved.

Definition 3.5 [8]**.** *Let X be a Banach space, $D \subset X$ be a proper closed convex and moving pointed cone with apex at the origin and* $\operatorname{int} D \neq \varnothing, N$ *a nonempty subset of X.*

1. *If $x \in X, N_x = (\{x\} - D) \cap N \neq \varnothing$, then N_x is called a section of the set N.*
2. *N is called weakly closed if $\{x_n\} \subset N$, $x \in X$,*

$$\langle x^*, x_n\rangle \to \langle x^*, x\rangle \quad \forall x^* \in X^*,$$

then $x \in N$.

3. *N is called bounded below if there exists a point $b \in X$ such that*

$$N \subset b + D.$$

Lemma 3.1 [10]**.** *Let X be a Banach space, $D \subset X$ a proper closed convex moving pointed cone with apex at the origin and* $\operatorname{int} D \neq \varnothing, N$ *a nonempty subset of X and X be the topological dual space of a real normed space $(Z, \|\cdot\|_z)$. Suppose there exists $x \in X$ such that the section N_x is weakly closed and bounded below and the norm $\|\cdot\|$ in X is strictly monotonically increasing. Then the set N has at least one weakly minimal point.*

Lemma 3.2. *If* **(GMVVIP)** *is solvable, then the feasible set Ω is nonempty.*

Proof. Let x be a solution of (**GMVVIP**), then there exists $u \in A(x)$ such that

$$\langle Q(x,u), g(y) - g(x)\rangle \not\leq_{\operatorname{int} P(x)} 0 \quad \forall y \in D. \tag{3.20}$$

Since g is convex, taking $y = z + x$ with any $z \in \Omega$ we know that $y \in D$ and

$$\langle Q(x,u), g(z)\rangle \not\leq_{\operatorname{int} P(x)} 0 \quad \forall z \in D. \tag{3.21}$$

Thus $x \in \Omega$, this completes the proof.

Lemma 3.3 [3]. *If the norm $\|\cdot\|$ in an ordered Banach space X is strictly monotonically increasing, then the order intervals in X are bounded.*

Proposition 3.4. *Suppose that* **(GMVVIP)** *is solvable and*

1) *there exists x in Ω such that $Q(x,u)$ is one to one and completely continuous, where $u \in A(x)$ is associated with x in the definition of Ω;*

2) *X is the topological dual space of a real normed space $(Z, \|\cdot\|_Z)$ and the norm $\|\cdot\|$ in X is strictly monotonically increasing, then* **(GMWMEP)** *has at least one solution.*

Proof. By assumption and Lemma 3.2, $\Omega \neq \varnothing$. Let $x \in \Omega$ be a point such that $Q(x,u)$ is one-to-one and completely continuous where $u \in A(x)$ is associated with x in the definition of Ω and let $\{y_n\} \subset \Omega$ with $y_n \to y$ (weakly). Since

$$\Omega_x = (\{x\} - D) \cap \Omega \subset (\{x\} - D) \cap D = [0, x], \tag{3.22}$$

by Lemma 3.3. $[0, x]$ is bounded and so is Ω_x. Since $Q(x,u)$ is completely continuous, $\langle Q(x,u), \Omega_x\rangle$ is a self-sequentially compact set and

$$\big\{\langle Q(x,u), g(y_n)\rangle\big\} \subset \langle Q(x,u), \Omega_x\rangle$$

implies that there exists a subsequence $\big\{\langle Q(x,u), g(y_{n_k})\rangle\big\}$ which converges to $z \in \langle Q(x,u), \Omega_x\rangle$. We get a point $y_0 \in \Omega_x$ such that

$$\langle Q(x,u), g(y_{n_k})\rangle \to \langle Q(x,u), g(y_0)\rangle \quad \text{(strongly)}. \tag{3.23}$$

On the other hand, since $y_n \to y$, $g(y_n) \to g(y)$ (weakly) and $Q(x,u)$ is completely continuous. Then

$$\langle Q(x,u), g(y_n)\rangle \to \langle Q(x,u), g(y)\rangle \quad \text{(strongly)}. \tag{3.24}$$

By the uniqueness of limits, we get

$$\langle Q(x,u), g(y)\rangle = \langle Q(x,u), g(y_0)\rangle.$$

Since $Q(x,u)$ is one-to-one, $y = y_0$ and so $y \in \Omega_x$. Since Ω_x is weakly closed, it follows from Lemma 3.1 that Ω has a weakly minimal point b such that

$$b \not\geq_{\operatorname{int} P(b)} x \quad \forall x \in \Omega.$$

Hence (**GMWMEP**) has at least one solution.

Proposition 3.4 is proved.

Definition 3.6. *Let X, Y be the two Banach spaces, $D \subset X$ a proper closed convex moving pointed cone with apex at the origin and* $\operatorname{int} D \neq \varnothing$, *and $P\colon D \to 2^Y$ a multivalued mapping with closed convex moving pointed cone such that* $\operatorname{int} P(x) \neq \varnothing$ *for all $x \in D$. Let $Q : D \times L(X,Y) \to$ $\to L(X,Y)$, $g\colon D \to D$ be the single-valued mappings and $A\colon X \to 2^{L(X,Y)}$ the multivalued mapping. Q is called Q-positive if*

$$\langle Q(x,u), g(y)\rangle \geq_{P(x)} 0 \quad \forall x, y \in D, \quad u \in A(x). \tag{3.25}$$

Now consider the positive generalized multivalued vector complementarity problem (**PGMVCP**): Finding $x \in D$ and $u \in A(x)$ such that

$$\langle Q(x,u), g(y)\rangle \ngeq_{\operatorname{int} P(x)} 0, \qquad \langle Q(x,u), g(y)\rangle \geq_{P(x)} 0, \quad y \in D. \tag{3.26}$$

The feasible set related to (**PGMVCP**) is defined as $\Omega = \{x \in D$: there is $u \in A(x)$ such that

$$\langle Q(x,u), g(y)\rangle \geq_{P(x)} 0 \quad \forall y \in D. \tag{3.27}$$

Let us consider the following problems:

The generalized multivalued vector optimization problem (**GMVOP**)$_{l_0}$: finding $x \in \Omega_0$ such that

$$l(x) \in \operatorname{Min}_P^w l(\Omega_0).$$

The generalized multivalued weak minimal element problem (**GMWMEP**)$_0$: finding $x \in \Omega_0$ such that

$$x \in \operatorname{Min}_D^w \Omega_0.$$

The positive generalized multivalued vector complementarity problem (**PGMVCP**): finding $x \in$ $\in \Omega_0$ such that

$$\langle Q(x,u), g(x)\rangle \ngeq_{\operatorname{int} P(x)} 0, \tag{3.28}$$

where $u \in A(x)$ and $Q\colon D \times L(X,Y) \to L(X,Y)$, $g\colon D \to D$ associated with x in the definition of Ω_0.

The generalized multivalued vector variational inequality problem (**GMVVIP**): finding $x \in D$ and $u \in A(x)$ such that

$$\langle Q(x,u), g(y) - g(x)\rangle \nleq_{\operatorname{int} P(x)} 0 \quad \forall y \in D. \tag{3.29}$$

The generalized multivalued vector unilateral optimization problem (**GMVUOP**): for a given mapping $f\colon X \to Y$ finding $x \in D$ such that

$$f(x) \in \operatorname{Min}_D^w f(D).$$

Definition 3.7. *Let $Q\colon D \times L(X,Y) \to L(X,Y)$, $g\colon D \to D$ be the single-valued mappings, $A\colon D \to 2^{L(X,Y)}$ be the multivalued mapping. Q is called g-strongly monotone with first variable of Q, if*

$$\langle Q(\cdot,u) - Q(\cdot,v), g(x) - g(y)\rangle \geq_{P(x)} 0 \quad \forall x, y \in D, \quad x \neq y, \quad u \in A(x), \quad v \in A(y). \tag{3.30}$$

Definition 3.8 [8]. *We say that $P(x)$ satisfies an inclusive condition if for any $x, y \in X$, $x \leq_{\operatorname{int} P} y$ only implies that*

$$P(x) \subset P(y). \tag{3.31}$$

Proposition 3.5. *Let $Q\colon D \times L(X,Y) \to L(X,Y)$, $g\colon D \to D$ be the two single-valued mappings, $A\colon X \to 2^{L(X,Y)}$ be the multivalued mappings, Q is g-strongly monotone with respect to the first variable and x is a solution of* **(PGMVCP)**. *If P satisfies the inclusive condition then x is a weakly minimal point of Ω_0, i.e., x solve* **(GMWMEP)**$_0$.

Proof. It is easy to see that $x \in \Omega_0 \subset D$. If $x \in bd(D)$ (where $bd(D)$ denotes the boundary of D), then x solve **(GMWMEP)**$_0$, otherwise there exists $x_0 \in \Omega_0$ such that

$$g(x) \geq_{\operatorname{int} D} g(x_0),$$

$$g(x) = g(x - x_0 + x_0) \leq g(x - x_0) + g(x) \in \operatorname{int} D + D \subset \operatorname{int} D,$$

which is a contradiction. If $x \in \operatorname{int} D$, by the Q-strict monotonicity, we have

$$\langle Q(x,u), g(x) - g(y)\rangle \geq_{\operatorname{int} P(x)} \langle Q(x',v'), g(x) - g(y)\rangle \quad \forall y \in \Omega_0, \quad y \neq x, \quad v' \in A(y).$$

Suppose $g(x) \geq_{\operatorname{int} D} g(y)$. Since Q is g-positive

$$\langle Q(x,u), g(x) - g(y)\rangle \geq_{P(y)} 0$$

and

$$\langle Q(x,u), g(x) - g(y)\rangle \geq_{\operatorname{int} P(x)} \langle Q(x,v), g(x) - g(y)\rangle \geq_{P(x)} 0.$$

By the assumption, we get

$$P(y) \subset P(x)$$

and so

$$\langle Q(x,u), g(x) - g(y)\rangle \in \langle Q(x,v), g(x) - g(y)\rangle + \operatorname{int} P(x) \subset P(y) +$$

$$+ \operatorname{int} P(x) \subset P(x) + \operatorname{int} P(x) = \operatorname{int} P(x).$$

It follows that

$$\langle Q(x,u), g(x) - g(y)\rangle \geq_{\operatorname{int} P(x)} 0,$$

and thus

$$0 \not\leq_{\operatorname{int} P(x)} \langle Q(x,u), g(x)\rangle \geq_{P(x)} \langle Q(x,u), g(y)\rangle + k$$

for some $k \in \operatorname{int} P(x)$. This implies

$$\langle Q(x,u), g(y)\rangle + k \not\geq_{\operatorname{int} P(x)} 0.$$

Since $k \in \operatorname{int} P(x)$ and $x \in \Omega_0$

$$\langle Q(x,u), g(y)\rangle + k \in P(x) + \operatorname{int} P(x) \subset \operatorname{int} P(x)$$

and so

$$\langle Q(x,u), g(y)\rangle + k \geq_{\operatorname{int} P(x)} 0,$$

which leads to a contradiction. Therefore

$$g(x) \geq_{\operatorname{int} D} g(y)$$

does not hold, that is

$$g(x) \not\geq_{\operatorname{int} D} g(y) \quad \forall y \in \Omega_0.$$

It follows that x solve **(GMWMEP)**$_0$.

Proposition 3.5 is proved.

Proposition 3.6. *If x solve* **(PGMVCP)**, *then x also solve* **(GMVVIP)**.

Proof. Suppose x solve (**PGMVCP**). Then $x \in D$ and there exists $u \in A(x)$ such that

$$\langle Q(x,u), g(x)\rangle \not\geq_{\operatorname{int} P(x)} 0, \qquad \langle Q(x,u), g(y)\rangle \geq_{P(x)} 0 \quad \forall y \in D.$$

If $\langle Q(x,u), g(y) - g(x)\rangle \leq_{\operatorname{int} P(x)} 0$, then

$$\langle Q(x,u), g(x)\rangle = -\langle Q(x,u), g(y) - g(x)\rangle + \langle Q(x,u), g(y)\rangle \in \operatorname{int} P(x) + P(x) \subset \operatorname{int} P(x)$$

and so

$$\langle Q(x,u), g(x)\rangle \geq_{\operatorname{int} P(x)} 0,$$

which is a contradiction. It follows that

$$\langle Q(x,u), g(y) - g(x)\rangle \not\leq_{\operatorname{int} P(x)} 0,$$

and x solve (**GMVVIP**).

Proposition 3.6 is proved.

Similarly we can obtain other equivalence condition. We have the following theorem.

Theorem 3.2. *Let X, Y be two Banach spaces, $D \subset X$ a proper closed convex moving pointed cone with apex at the origin and* $\operatorname{int} D \neq \varnothing$ *and* $\{P(x) \colon x \in X\}$ *a family of closed moving pointed cone in Y such that* $\operatorname{int} P(x) \neq \varnothing$ *for all $x \in X$. Let $g \colon D \to D$, $Q \colon D \times L(X,Y) \to Y$ and $A \colon X \to 2^{L(X,Y)}$ be the mappings. Suppose that P satisfies the inclusive condition and*

1) *$T = \partial_Q f$ is the Q-subdifferential for the convex operator $f \colon X \to Y$;*

2) *l is a weakly positive operator;*

3) *A is Q-strictly monotone.*

If **(GMWMEP)**$_0$, **(PGMVCP)** *is solvable, then* **(GMVOP)**$_{l_0}$, **(GMVVIP)** *and* **(GMVUOP)** *have at least a common solution.*

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