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ON THE RELATION BETWEEN CURVATURE, DIAMETER AND VOLUME OF A COMPLETE RIEMANNIAN MANIFOLD

ПРО СПІВВІДНОШЕННЯ МІЖ КРИВИЗНОЮ, ДІАМЕТРОМ ТА ОБ’ЄМОМ ПОВНОГО РІМАНОВОГО МНОГОВИДУ

In this note, we prove that if $N$ is a compact totally geodesic submanifold of a complete Riemannian manifold $(M,g)$, whose sectional curvature $K$ satisfies the relation $K \geq k > 0$, then $d(m,N) \leq \frac{\pi}{2\sqrt{k}}$ for any point $m \in M$. In the case where $\dim M = 2$, a Gaussian curvature $K$ satisfies the relation $K \geq k \geq 2$, and $\gamma$ has the length $l$, we get $\text{Vol}(M,g) \leq \frac{2l}{\sqrt{k}}$ if $k \neq 0$ and $\text{Vol}(M,g) \leq 2l\text{diam}(M)$ if $k = 0$.

Доведено, що якщо $N$ — компактний підкомпункт геодезичний підмножник повного ріманового многовіду $(M,g)$ з секційною кривизною $K$, що задовольняє умову $K \geq k > 0$, то для будь-якої точки $m \in M$ виконується нерівність $d(m,N) \leq \frac{\pi}{2\sqrt{k}}$. У випадку, коли $\dim M = 2$, гаусова кривизна $K$ задовольняє умову $K \geq k \geq 2$, і крива $\gamma$ має довжину $l$, отримано співвідношення $\text{Vol}(M,g) \leq \frac{2l}{\sqrt{k}}$ для $k \neq 0$ і $\text{Vol}(M,g) \leq 2l\text{diam}(M)$ для $k = 0$.

1. Introduction. As well known, one of the most interesting problems in Riemannian geometry is to study a relation between geometrical notions as curvature, diameter and volume. This problem have been studied by many authors for the concrete manifolds. In [1], Y. C. Wong considered this problem for the Grassmann manifolds. W. Klingenberg proved that in a compact simply connected even-dimensional Riemannian manifold with sectional curvature $K$ belonging to $[0,k)$, $k > 0$, the length of any closed geodesic is greater than $\frac{2\pi}{\sqrt{k}}$ (see [2]). The relation between curvature and topology of Riemannian manifolds is exposed in [3, 4].

Generalizing the known results for the sphere $S^2$ in Euclidian three-space, we obtained the following results.

Theorem 1. Let $(M, \langle \cdot, \cdot \rangle)$ be a complete Riemannian manifold, whose sectional curvature $K$ satisfies $K \geq k > 0$. Let $N$ be a compact totally geodesic submanifold of $M$, then for any $m \in M$ we have $d(m,N) \leq \frac{\pi}{2\sqrt{k}}$.

Theorem 2. Let $(M, g)$ be a complete Riemannian manifold of dimension 2, whose sectional curvature $K$ satisfies $K \geq k \geq 0$, $k$ is constant. Let $\gamma$ be a closed geodesic in $M$ of length of $l$. Then we have

\[
\text{Vol}(M) \leq \begin{cases} 
\frac{2l}{\sqrt{k}} & \text{if } k > 0, \\
2l\text{diam}(M) & \text{if } k = 0.
\end{cases}
\]

The basic notions used in this article are from [5, 6].

2. Proof of Theorem 1. Since $N$ is the compact totally geodesic submanifold of $M$, then every geodesic of $N$ is also a geodesic of $M$.

Let $m$ be a point of $M$, we have to prove that $d(m,N) \leq \frac{\pi}{2\sqrt{k}}$. We assume that $m \notin N$, then $d(m,N) = l > 0$, and $\exists p \in N$ such that $d(m,N) = d(m,p) = l$. By
the Hopf – Rinow theorem, there exists the minimal geodesic parametrized by arc length:

\[ c : [0, L] \to M, \quad c(0) = m, \quad c(L) = p, \quad \|c'(s)\| = 1. \]

Let \( \gamma \) be a geodesic in \( N \) passing through \( p \), then \( \gamma \) is also a geodesic in \( M \). Suppose that \( \gamma \) is parametrized by arc length:

\[ \gamma : (-\rho, \rho) \to M, \quad \rho > 0, \quad \gamma(0) = p, \quad \|\gamma'(t)\| = 1. \]

Let \( Y_L = \gamma'(0) \in T_p M \), take a parallel vector field \( Y(s) \) along \( c \) such that \( Y(L) = Y_L \). We have \( \|Y(s)\| = \|Y_L\| = 1 \) for all \( s \in [0, L] \).

Set \( X(s) = \sin \frac{\pi s}{2L} Y(s) \), \( X(s) \) is a vector field along \( c \), \( X(0) = 0 \), \( X(L) = Y(L) = \gamma'(0) \).

We now consider the variation \( H \) of \( c \) as follows:

\[ H : [0, L] \times (-\rho, \rho) \to M, \quad (s, t) \mapsto H(s, t) = \exp_{c(s)} tX(s), \]

\( H \) is well defined by the completeness of manifold \( M \). We have \( H(0, t) = m \).

Set \( c_t(s) = H(s, t) \), \( c_0(s) = H(s, 0) = c(s) \), then

\[ H(L, t) = \exp_{c(L)} tX(L) = \exp_p t\gamma'(0) = \gamma(t), \]

\[ H \circ \frac{\partial}{\partial t} (s, 0) = X(s). \]

By construction of the variation \( H \), the length function \( L(c_t) \) attain a minimum at \( t = 0 \). Hence,

\[ \frac{d}{dt} L(c_t) \big|_{t=0} = 0 \quad (1) \]

and

\[ \frac{d^2}{dt^2} L(c_t) \big|_{t=0} \geq 0. \quad (2) \]

Using the first variation formula together with remark that \( c \) is a geodesic, i.e., \( \Delta c c' = 0 \), we have

\[ (1) \Leftrightarrow \left\langle X(s), c'(s) \right\rangle \big|_0^L - \int_0^L \left\langle X(s), \nabla c c'(s) \right\rangle ds = 0 \Leftrightarrow \left\langle X(L), c'(L) \right\rangle = 0. \]

On the other hand,

\[ \frac{d}{ds} \left\langle Y(s), c'(s) \right\rangle = \left\langle Y(s), \frac{D}{ds} c'(s) \right\rangle + \left\langle \frac{D}{ds} Y(s), c'(s) \right\rangle, \]

where \( \frac{D}{ds} Y(s) = \nabla_{\frac{D}{ds}} Y(s) \).

Since \( \frac{D}{ds} Y(s) = 0 \), \( \frac{D}{ds} c'(s) = 0 \), we have \( \frac{d}{ds} \left\langle Y(s), c'(s) \right\rangle = 0. \) Thus,

\[ \left\langle Y(s), c'(s) \right\rangle = \left\langle Y(L), c'(L) \right\rangle = 0 \quad \forall s \in [0, L] \]

and \( \{Y(s), c'(s)\} \) is an orthonormal system.

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Set \( \overline{X}(s, t) = H \ast \frac{\partial}{\partial t}(s, t) \Rightarrow \overline{X}(s, 0) = X(s) \).

Using the second variation formula, we have

\[
(2) \Leftrightarrow \langle \nabla_{\frac{\partial}{\partial x}} \overline{X}(s, 0), c'(s) \rangle \bigg|_0^L + \int_0^L \left( \left| \overline{X}' \right|^2 - \langle R(\overline{X}, c', c'), \overline{X} \rangle - \langle c', \overline{X}' \rangle^2 \right) ds \geq 0,
\]

where \( \overline{X}' = \nabla_{\frac{\partial}{\partial x}} \overline{X}(s, 0) \).

We have

\[
\overline{X}(s, 0) = X(s) = \sin \left( \frac{\pi s}{2L} \right) Y(s),
\]

\[
\overline{X}'(s, 0) = X'(s) = \frac{\pi}{2L} \cos \left( \frac{\pi s}{2L} \right) Y(s) + \sin \left( \frac{\pi s}{2L} \right) \frac{D}{2L} Y(s) = \frac{\pi}{2L} \cos \left( \frac{\pi s}{2L} \right) Y(s),
\]

\[
\langle c'(s), \overline{X}'(s, 0) \rangle = \frac{\pi}{2L} \cos \left( \frac{\pi s}{2L} \right) \langle c'(s), Y(s) \rangle = 0,
\]

\[
\overline{X}(0, t) = H \ast \frac{\partial}{\partial t}(0, t) = 0 \quad \text{(since} \quad H(0, t) = m \quad \forall t) \Rightarrow \nabla_{\frac{\partial}{\partial t}} \overline{X}(0, 0) = 0,
\]

\[
\overline{X}(L, t) = H \ast \frac{\partial}{\partial t}(L, t) = \gamma'(t) \Rightarrow \nabla_{\frac{\partial}{\partial t}} \overline{X}(L, 0) = \frac{D}{dt} \gamma'(t) \bigg|_{t=0} = 0.
\]

Thus,

\[
(3) \Leftrightarrow \int_0^L \left[ \left( \frac{\pi}{2L} \right)^2 \cos^2 \left( \frac{\pi s}{2L} \right) |Y(s)|^2 - \sin^2 \left( \frac{\pi s}{2L} \right) \langle R(Y(s), c'(s), c(s)), Y(s) \rangle \right] ds \geq 0 \Leftrightarrow
\]

\[
\Leftrightarrow \int_0^L \left[ \left( \frac{\pi}{2L} \right)^2 \sin^2 \left( \frac{\pi s}{2L} \right) - \sin^2 \left( \frac{\pi s}{2L} \right) R(Y(s), c'(s)) \right] ds \geq 0 \Rightarrow
\]

\[
\Rightarrow \int_0^L \left[ \left( \frac{\pi}{2L} \right)^2 - k \right] \sin^2 \frac{\pi s}{2L} ds \geq 0 \Leftrightarrow \left( \frac{\pi}{2L} \right)^2 - k \geq 0 \Rightarrow L \leq \frac{\pi}{2 \sqrt{k}}.
\]

Theorem 1 is proved.

3. Proof of Theorem 2. In order to prove Theorem 2, the following lemma is necessary:

**Lemma.** Suppose that \( x : (\alpha, \beta) \rightarrow \mathbb{R} \) is a differentiable function defined on \( (\alpha, \beta) \), where \( -\frac{\pi}{2} \leq \alpha < 0 < \beta \leq \frac{\pi}{2} \), and satisfies

\[
x(0) = 0, \quad x' + x^2 \leq -1.
\]

Then

\[
x(t) \leq -\tan(t), \quad t \in [0, \beta),
\]

\[
x(t) \geq -\tan(t), \quad t \in (\alpha, 0].
\]

**Proof.** Set \( x(t) = -\tan \varphi(t) \), then \( \varphi(t) = -\arctan x(t) \), \( \varphi(t) \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \).

\[
(4) \Leftrightarrow \varphi(0) = 0 \quad \text{and} \quad -\frac{\varphi'}{\cos^2 \varphi} + \frac{\sin^2 \varphi}{\cos^2 \varphi} \leq -1 \Leftrightarrow \frac{\varphi'}{\cos^2 \varphi} \geq \frac{1}{\cos^2 \varphi} \Leftrightarrow \varphi' \geq 1.
\]

Thus \( (\varphi(t) - t) \) is the increasing function.
\[
\Rightarrow \begin{cases} 
\varphi(t) \geq t & \forall t \in [0, \beta), \\
\varphi(t) \leq t & \forall t \in (\alpha, 0].
\end{cases}
\]

Since the function \( \tan t \) is increasing on \((\alpha, \beta)\), we get
\[
x(t) = -\tan \varphi(t) \leq -\tan (t) & \forall t \in [0, \beta), \\
x(t) = -\tan \varphi(t) \geq -\tan (t) & \forall t \in (\alpha, 0].
\]

The lemma is proved.

Now we prove Theorem 2.

Suppose that \( \gamma \) is a closed geodesic parametrized by arc length:
\[
\gamma : I \to M, \quad \text{where } [0, l] \subset I, \quad \gamma(0) = \gamma(l), \quad \gamma'(0) = \gamma'(l), \quad \| \gamma'(t) \| = 1
\]
\( \forall t \in I, l \) is interval on \( R \).

Since \( \gamma \) is a closed geodesic, there exists a unit parallel vector field \( Y \) along \( \gamma \) such that \( Y(0) = Y(l) \) and \( \langle Y(0), \gamma'(0) \rangle = 0 \).

For \( s \in [0, l] \), put
\[
\rho(s) = \sup \{ t \geq 0 | t = d(\exp_{\gamma(s)} t Y(s), \gamma) \},
\]
\[
\varphi(s) = \inf \{ t \leq 0 | -t = d(\exp_{\gamma(s)} t Y(s), \gamma) \}.
\]
(5)

We will prove that \( \rho(s) \) is upper semicontinuous and \( \varphi(s) \) is lower semicontinuous.

We consider \( s_n \to s \), set \( t_0 = \lim_{n \to \infty} \rho(s_n) \), and claim that \( t_0 \leq \rho(s) \). In fact, since
\[
t_0 = \lim_{n \to \infty} \rho(s_n), \quad \text{we have } \exists \{ s_{n_k} \} \subset \{ s_n \} | \rho(s_{n_k}) \to t_0.
\]

Put \( t_k = \max \left\{ 0, \rho(s_{n_k}) - \frac{1}{k} \right\} \), \( t_k \to t_0 \to \exp_{\gamma(s_{n_k})} t_k Y(s_{n_k}) \to \exp_{\gamma(s)} t_0 Y(s) \) and
\[
d(\exp_{\gamma(s)} t_0 Y(s), \gamma) = \lim_{k \to \infty} (\exp_{\gamma(s_{n_k})} t_k Y(s_{n_k}), \gamma) = \lim_{k \to \infty} t_k = t_0 \Rightarrow \rho(s) \geq t_0.
\]

Thus, \( \rho(s) \) is upper semicontinuous. Similarly, \( \varphi(s) \) is lower semicontinuous. This implies that \( \rho(s) \) and \( \varphi(s) \) are the measurable functions.

We have
\[
A = \{ (s, t) | 0 \leq s < l, \quad \varphi(s) < t < \rho(s) \} \quad \text{is measurable set and } \quad \operatorname{mes} B = 0,
\]
where \( B = \{ (s, t) | 0 \leq s < l, \varphi(s) = 0 \text{ or } \rho(s) = t \} \).

Consider the sets
\[
C = \{ \exp_{\gamma(s)} t Y(s) | (s, t) \in A \},
\]
\[
D = \{ \exp_{\gamma(s)} t Y(s) | (s, t) \in B \}.
\]

It is clear that \( C, D \) are the images of \( A \) and \( B \), respectively, under a continuous mapping. Thus, \( C \) and \( D \) are measurable and \( \operatorname{mes} (D) = 0 \).

We will prove that \( M = C \cup D \) and \( C \cap D = \emptyset \).

In fact, since \( \gamma \) is closed, \( \gamma \) is compact.

This implies that, for arbitrary \( p \in M \), \( \exists q \in \gamma | r = d(p, q) = d(p, \gamma), \quad q = \gamma(s) \).

Suppose that \( c \) is a minimal geodesic length-parametrized joining \( p \) and \( q \):
\[
c : [c, r] \to M, \quad c(0) = q, \quad c(r) = p.
\]

\( X \) is the parallel vector field along \( c \) such that \( X(0) = \gamma'(s) \). We consider the variation

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\[
H : [0, r] \times (-\varepsilon, \varepsilon) \to M, \quad H(u, v) = \exp_{c(u)} \left( v \cos \frac{\pi u}{2r} \right) X(u).
\]

Put \( c_v(u) = H(u, v) \). It is clear that \( c_0(u) = c(u) \), \( H(0, v) = \exp_p(v \gamma'(s)) \in \gamma \), so the function \( L(v) \) attain a minimum at \( v = 0 \).

Using the first variation formula, we have
\[
\frac{d}{dv} \left( L(c_v) \right) \bigg|_{v=0} = 0 \iff \left\langle \cos \frac{\pi u}{2r} \ X(u), c'(u) \right\rangle \bigg|_0^r - \int_0^r \left( \cos \frac{\pi u}{2r} \ X(u), \nabla_c c'(u) \right) du =
\]
\[
= 0 \iff \left\langle X(0), c'(0) \right\rangle = 0 \iff \left\langle \gamma'(0), c'(0) \right\rangle = 0 \iff c'(0) = \pm Y(s).
\]
Without loss of generality, we can suppose that \( c'(0) = Y(s) \). Hence,
\[
p = c(r) = \exp_{c(0)} r c'(0) = \exp_{\gamma(s)} r Y(s) \quad \text{and} \quad d(p, \gamma) = r.
\]
By the definition of \( \varphi(s) \), \( \rho(s) \), we have \( \varphi(s) \leq r \leq \rho(s) \), whence \( (s; r) \in A \cup B \) and \( p \in C \cup D \). Thus, \( M = C \cup D \).

In order to prove \( C \cap D = \emptyset \), we suppose that there exists \( p_1 \in C \cap D \). Then \( \exists s_1 \neq s_2 \) such that
\[
p_1 = \exp_{\gamma(s_1)} t_1 Y(s_1) = \exp_{\gamma(s_2)} t_2 Y(s_2),
\]
where \( \varphi(s_1) < t_1 < \rho(s_1) \), \( t_2 = \varphi(s_2) \) or \( t_2 = \rho(s_2) \).

Choose a number \( t_3 \) such that
\[
t_1 < t_3 < \rho(s_1) \quad \text{if} \quad t_1 \geq 0 \quad \text{and} \quad \varphi(s_1) < t_3 < t_1 \quad \text{if} \quad t_1 < 0.
\]
Put \( q_1 = \exp_{\gamma(s_1)} t_3 Y(s_1) \). We get
\[
d(q_1, \gamma) = d(q_1, p_1) + d(p_1, \gamma(s_1)) = d(p_1, q_1) + d(p_1, \gamma) =
\]
\[
= d(p_1, q_1) + d(p_1, \gamma(s_2)) > d(q_1, \gamma(s_2)).
\]
This contradiction proves our assertion.

Consider the variation \( \overline{H} : I \times R \to M \) defined by \( \overline{H}(s, t) = \exp_{\gamma(s)} t Y(s) \). Put \( H = \overline{H} \big|_A \). It is clear that \( H(A) = C \) by definition of the set \( C \). We now prove that \( H \) is injective. In fact, suppose (inversely) that \( H \) isn't injective, then \( \exists (s_1, t_1) \neq (s_2, t_2), (s_i, t_i) \in A, i = 1, 2, \) such that \( H(s_1, t_1) = H(s_2, t_2) = q \), hence \( d(q, \gamma) = |t_1| = |t_2| = 0 \) by (4). There are two cases:

**Case 1.** If \( s_1 = s_2, \ t_2 = -t_1 > 0 \), then there exists \( t_3 : 0 < t_2 < t_3 < \rho(s_1) \). Consider the geodesic \( c(t) = \exp_{\gamma(s_1)} t Y(s_1) \), we have \( c(t_1) = c(t_2) \). By a consequence of the Hopf – Rinow theorem (see [4, p. 100]), the geodesic \( c \) is no more minimal on the interval \( [0, t_3] \). This is a contradiction.

**Case 2.** If \( s_1 \neq s_2 \), we can suppose \( t_2 = t_0 > 0 \), for all \( 0 < t_2 < t_3 < \rho(s_2) \) \( p = \exp_{\gamma(s_2)} t_3 Y(s_2) \) we have
\[
d(p, \gamma) \leq d(p, \gamma(s_1)) < d(p, q) + d(q, \gamma(s_1)) =
\]
\[
= d(p, q) + d(q, \gamma(s_2)) = d(p, \gamma).
\]
This is a contradiction.
Thus, \( H \) is bijective from \( A \) on \( C \). Moreover, \( H \) is the homeomorphism. Let \( \mu \) be the canonical measure on \( M \). We have

\[
\text{Vol}(M) = \int_M \mu = \int_C \mu + \int_D \mu = \int_C \mu + \int_D \mu = \int_{H^{-1}(C)} G r\left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right)^{1/2} ds \ dt = \int_0^1 \int_{\varphi(s)} G r\left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right)^{1/2} dt \ ds. \quad (6)
\]

Since \( H(s, \cdot) \) is a geodesic, then

\[
\nabla_{\partial/\partial s} \frac{\partial}{\partial t} = 0 \Rightarrow \left| \frac{\partial}{\partial t}(s, t) \right| = |Y(s)| = 1.
\]

Because

\[
\frac{d}{dt} \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) = \left( \nabla_{\partial/\partial s} \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = \left( \nabla_{\partial/\partial s} \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = 1, 2 \frac{d}{ds} \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) = 0.
\]

we have

\[
\left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right)(s, t) = \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right)(s, 0) = \left( \gamma'(s), Y(s) \right) = 0.
\]

Thus,

\[
G r\left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) = \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right) \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) - \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right)^2 = \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right).
\]

Put \( f = \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right) \). We will estimate the function \( f(s, t) \).

From \( K \geq k \) we have

\[
\left( \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right) \geq k \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right) = kf,
\]

hence

\[
-k \left( \nabla_{\partial/\partial s} \nabla_{\partial/\partial s} \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right) \geq kf. \quad (7)
\]

Furthermore, since

\[
\left( \nabla_{\partial/\partial s} \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right) = \left( \nabla_{\partial/\partial s} \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right) = 1, 2 \frac{d}{ds} \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right) = 0,
\]

we have

\[
\nabla_{\partial/\partial s} \frac{\partial}{\partial s} = \nabla(s, t) \frac{\partial}{\partial s}. \quad (8)
\]

Thus,

\[
\nabla_{\partial/\partial s} \nabla_{\partial/\partial s} \frac{\partial}{\partial s} = \nabla_{\partial/\partial s} \nabla_{\partial/\partial s} \frac{\partial}{\partial s} = \nabla(y, t) \frac{\partial}{\partial s} + \nabla^2(s, t) \frac{\partial}{\partial s} = (\nabla' + \nabla^2) \frac{\partial}{\partial s}. \quad (9)
\]

On the other hand, we get from (8):
\[ \tilde{x}(s, 0) = \left( \nabla_{\partial_t} \tilde{\frac{\tilde{\partial}}{\partial s}}, \tilde{\frac{\tilde{\partial}}{\partial s}} \right)(s, 0) = \frac{d}{dt} \left( \frac{\tilde{\partial}}{\partial s}, \frac{\tilde{\partial}}{\partial s} \right)(s, 0) = 0, \] 

\[ f' = \frac{d}{dt} \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right) = 2 \left( \nabla_{\partial_t} \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right) = 2\tilde{x}f. \]

From (7) and (9), we obtain

\[ -(\tilde{x}' + \tilde{x}^2)f \geq kf \Leftrightarrow \tilde{x}' + \tilde{x}^2 \leq -k. \]

We now consider two cases.

a) In the case \( k > 0 \), set

\[ x(s, t) = \frac{1}{\sqrt{k}} \tilde{x} \left( s, \frac{t}{\sqrt{k}} \right), \]

\[ x'(s, t) = \frac{1}{\sqrt{k}} \tilde{x}' \left( s, \frac{t}{\sqrt{k}} \right). \]

So \( kx' + kx^2 \leq -k \Leftrightarrow x' + x^2 \leq -1. \)

Theorem 1 states that

\[ -\frac{\pi}{2\sqrt{k}} \leq \varphi(s) \leq 0 \leq \rho(s) \leq \frac{\pi}{2\sqrt{k}} \Rightarrow \frac{-\pi}{2} \leq -\sqrt{k}\varphi(s) \leq 0 \leq \sqrt{k}\rho(s) \leq \frac{\pi}{2}. \]

Using Lemma 1, we get

\[ \begin{cases} 
  x(s, t) \leq -\tan t, & s \in [0, l], \quad 0 \leq t < -\sqrt{k}\rho(s) \\
  x(s, t) \geq -\tan t, & s \in [0, l], \quad -\sqrt{k}\varphi(s) < t \leq 0 \\
 \end{cases} \]

\[ \Rightarrow \begin{cases} 
  \tilde{x}(s, t) \leq -\sqrt{k}\tan(\sqrt{k}t), & s \in [0, l], \quad 0 \leq t < \rho(s), \\
  \tilde{x}(s, t) \geq -\sqrt{k}\tan(\sqrt{k}t), & s \in [0, l], \quad \varphi(s) < t \leq 0. 
\end{cases} \]

From (10), we have \( \frac{f'}{f} = 2\tilde{x}. \)

For \( t \geq 0 \) \( \Rightarrow \frac{f'}{f} \leq -\sqrt{k}2\tan(\sqrt{k}t) \Leftrightarrow (\ln |f|)' \leq (\ln(\cos^2(\sqrt{k}t)))' \Rightarrow \)

\[ \Rightarrow \left[ \ln |f| - \ln(\cos^2(\sqrt{k}t)) \right]' \leq 0 \Rightarrow \left( \frac{|f(s, t)|}{\cos^2(\sqrt{k}t)} \right)' \leq 0 \Rightarrow \]

\[ \Rightarrow \ln \frac{|f(s, t)|}{\cos^2(\sqrt{k}t)} \leq \ln |f(s, 0)| \Rightarrow \]

\[ \Rightarrow \ln \frac{|f(s, t)|}{\cos^2(\sqrt{k}t)} \leq |f(s, 0)| = \langle \gamma'(s), \gamma'(s) \rangle = 1 \Rightarrow \]

\[ \Rightarrow |f| \leq \cos^2(\sqrt{k}t). \]

For \( t \leq 0 \) \( \Rightarrow \frac{f'}{f} \geq -\sqrt{k}2\tan(\sqrt{k}t) \Leftrightarrow (\ln |f|)' \geq (\ln(\cos^2(\sqrt{k}t)))' \Rightarrow \)

\[ \Rightarrow \frac{|f|}{\cos^2(\sqrt{k}t)} \leq |f(s, 0)| = 1 \Rightarrow |f| \leq \cos^2(\sqrt{k}t). \]
So, for all $t$ such that $\varphi(s) < t < \rho(s)$ and for $s \in [0, l)$, we get $|f(s, t)| \leq \cos^2(\sqrt{k}t)$.

Thus

$$\Vol(M) = \int \left( \int \frac{\partial}{\partial s} \sqrt{\frac{\partial}{\partial t}} \right)^{1/2} ds \right) dt = \int \left( \int \sqrt{f} \, dt \right) ds \leq \int \left( \int \cos(\sqrt{k}t) dt \right) ds \leq \int \left( \int \cos(\sqrt{k}t) dt \right) ds = \frac{2l}{\sqrt{k}}.$$

b) Consider the case $k = 0$.

From (11)

$$\Rightarrow a'_s + b_s^2 \leq 0 \Rightarrow a'_s \leq 0 \Rightarrow$$

$$\Rightarrow \begin{cases} a(s, t) \leq a(s, 0) = 0, & 0 \leq t < \rho(s) \\ a(s, t) \geq a(s, 0) = 0, & \varphi(s) < t \leq 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} f'_s \leq 0, & 0 \leq t < \rho(s) \\ f'_s \geq 0, & \varphi(s) < t \leq 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} f(s, t) \leq f(s, 0) = 1, & 0 \leq t < \rho(s) \\ f(s, t) \leq f(s, 0) = 1, & \varphi(s) < t \leq 0 \end{cases} \Rightarrow$$

$$\Rightarrow \Vol(M) = \int \left( \int \frac{\partial}{\partial s} \right) ds \leq \int \left( 2l \diam(M) \right) ds = 2l \diam(M).$$

Thus,

$$\Vol(M) = \begin{cases} \frac{2l}{\sqrt{k}} & \text{if } k > 0, \\ 2l \diam(M) & \text{if } k = 0. \end{cases}$$

Theorem 2 is proved.


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