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BOGOLYUBOV AVERAGING AND NORMALIZATION PROCEDURES IN NONLINEAR MECHANICS. II*

By using a proposed method, we study systems which are linear at the zero approximation and have perturbations in the form of polynomials. This class of systems is widely used in applications. The following fact is even more important: This method shows how the classical method of Poincaré–Birkhoff normal form can be generalized and new results can be obtained due to the group-theory technique. After a short exposition of the general theory of the asymptotic method decomposition, the new technique of normalization is demonstrated on the models based on the Lotka–Volterra equations.

За допомогою нового методу досліджуються системи, лінійні у нульовому наближенні, що мають збурення у формі поліномів. Цей клас систем має широке застосування. Однак більш значним є той факт, що новий метод показує, як можна узагальнити класичний метод нормальної форми Пуанкаре–Біркгофа і як можуть бути одержані нові результати завдяки застосуванню теоретико-групового апарату. Після стислого викладення загальної теорії ілюструється техніка нормалізації за методом асимптотичної декомпозиції на моделях, заснованих на рівняннях Лотке–Вольтерра.

1. Asymptotic decomposition in the space of homogeneous polynomials.

1.1. *Passing to the matrix of a simple structure.* Consider the system of almost linear ordinary differential equations

$$\dot{x}' = \mathcal{A}x' + \varepsilon\tilde{\omega}(x'), \qquad x'(t_0) = x_0, \tag{1}$$

where $\tilde{\omega}(x')$ is a column vector, the elements of which are polynomials in n variables $x'_1, \ldots, x'_n$ with the degree not greater than k.

Assume that the matrix $\mathcal{A}$ of coefficients of a system of zero approximation corresponding to system (1) is semi-simple (e.g., diagonable). Such a restriction on $\mathcal{A}$ is not essential since system (1) of dimension n with a nondiagonable matrix can be transformed into a system of dimension $n+1$ with a diagonable matrix. For the described transformation, the matrix $\mathcal{A}$ should be decomposed into diagonable and nilpotent components [2, p. 112].

$$\mathcal{A} = \mathcal{A}_d + \mathcal{A}_n, \quad \mathcal{A}_d\mathcal{A}_n \equiv \mathcal{A}_n\mathcal{A}_d, \quad \mathcal{A}_n^k \equiv 0, \ k \le n,$$

by one of the known techniques. Having substituted the obtained identities into the initial system and upon a change of variables

$$x' = \exp\mathcal{A}_n(t-t_0)\,y' = \sum_{i=0}^{k} \mathcal{A}_n^i \frac{(t-t_0)^i}{i!}\,y',$$

we obtain the system $\dot{y}' = \mathcal{A}_d y' + \varepsilon\tilde{\Omega}(t, y')$, where $\tilde{\Omega}(y') = \exp(-\mathcal{A}_n(t-t_0))\,\tilde{\omega}(\exp\mathcal{A}_n(t-t_0)\,y')$ is the homogeneous polynomials of finite power. By introducing the additional variable $y'_{n+1} \equiv t$, $\dot{y}'_{n+1} \equiv 1$, we obtain the system of $n+1$ equations with a diagonable matrix.

1.2. *General settings.* Along with the linear space V over P generated by the elements $x_1, \ldots, x_n$, we consider the linear space $V_{\otimes^\nu}$ over P, which is equal to the direct product of the space V taken ν times,

$$V_{\otimes^\nu} = \underbrace{V \otimes \ldots \otimes V}_{\nu}.$$

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The basis of the space $V_{\otimes \nu}$ is generated by all possible monomials of the form $x_1^{m_1} \dots x_n^{m_n}$, $m_1 + \dots + m_n = \nu$. Its dimension is denoted by m_ν. The row vector composed of the basis elements of $V_{\otimes \nu}$ is denoted by $\hat{\hat{x}}_{m_\nu}$. It is evident that $m_1 = n$ and $\hat{\hat{x}}_{m_1} = \| x_1, \dots, x_n \|$.

By $T(V)$, we denote the infinite-dimensional space equal to the direct sum of subspaces $V_{\otimes \nu}$.

$$T(V) = V \oplus (V \otimes V) \oplus \dots \oplus \left(\underbrace{V \otimes \dots \otimes V}_{r} \right) \oplus \dots .$$

The infinite-dimensional Lie algebra is closely connected with the space $T(V)$. Now we describe the construction of this algebra.

Let Q be a constant matrix of dimension $m_\nu \times n$ with elements $q_{ij} \in P$, where $i = \overline{1, m_\nu}$, $j = \overline{1, n}$, and $q_1, \dots, q_n$ are the elements of the row in the equality $q = \hat{\hat{x}}_{m_\nu} Q$, $q \overset{\text{def}}{=} \| q_1, \dots, q_n \|$.

For an arbitrary sequence of matrices Q, the totality of differential operators

$$X = q_1 \frac{\partial}{\partial x_1} + \dots + q_n \frac{\partial}{\partial x_n}, \quad q_i \in V_{\otimes \nu},$$

yields the linear space over P which is denoted by $\mathcal{B}(V_{\otimes \nu})$. Matrix Q will be called the matrix of operator X. It can be easily verified that the operator X maps all vectors from the subspace $V_{\otimes l}$ into the subspace $V_{\otimes l+\nu-1}$.

The action of the operator X upon the basis $V_{\otimes l}$ implies

$$X \hat{\hat{x}}_{m_l} = \hat{\hat{x}}_{m_{l+\nu-1}} \mathcal{F}_{l+\nu-1}, \tag{2}$$

where $\mathcal{F}_{l+\nu-1}$ is the rectangular matrix of the dimension $m_{l+\nu-1} \times m_\nu$. The matrix $\mathcal{F}_{l+\nu-1}$ of the operator X, which maps the space $\mathcal{B}(V_{\otimes l})$ into the space $\mathcal{B}(V_{\otimes \nu})$ and is defined by relation (2), will be called the representation matrix of the operator X in the space $V_{\otimes l}$. Let Y be the differential operator of $\mathcal{B}(V_{\otimes r})$.

$$Y = h_1 \frac{\partial}{\partial x_1} + \dots + h_n \frac{\partial}{\partial x_n}, \quad h_i \in V_{\otimes r},$$

$$h = \hat{\hat{x}}_{m_r} \mathcal{H}, \quad h = \| h_1, \dots, h_n \|, \quad \mathcal{H} \in R^{(m_r, n)}.$$

The following statement can be easily proved:

Lemma 1. *Let the operators X, Y belong to the subspaces $\mathcal{B}(V_{\otimes \nu})$ and $\mathcal{B}(V_{\otimes r})$. Let Q, $\mathcal{H}$ be the matrices of the representations of these operators in the subspace V; then the Poisson bracket*

$$[X, Y] = c_1 \frac{\partial}{\partial x_1} + \dots + c_n \frac{\partial}{\partial x_n}, \quad c = \hat{\hat{x}}_{m_{r+\nu-1}} C, \quad c = \| c_1, \dots, c_n \|,$$

belongs to the subspace $\mathcal{B}(V_{\otimes r+\nu-1})$ and its representation matrix is $C = \mathcal{R}_{\nu+r-1} \mathcal{H} - \mathcal{F}_{\nu+r-1} Q$, where

$$\mathcal{R}_{\nu+r-1} \in R^{(\nu+r-1,\, m-r)} \quad \text{and} \quad \mathcal{F}_{\nu+r-1} \in R^{(\nu+r-1,\, m_\nu)}$$

are the representation matrices of the operators X, Y, *respectively, in the subspaces* $V_{\otimes r}$ *and* $V_{\otimes v}$.

Theorem 1. *The space* $\mathcal{B}(T(V))$ *generates an infinite-dimensional Lie algebra.*

Proof. Lemma 1 implies the proof, since $[\mathcal{B}(V_{\otimes v}), \mathcal{B}(V_{\otimes r})] \in \mathcal{B}(V_{\otimes v+r-1})$ and the infinite dimensional space $\mathcal{B}(T(V))$ is closed under the usual operation of taking the Poisson bracket of two arbitrary operators of $\mathcal{B}(T(V))$.

Construct the generating Lie algebra $\tilde{\mathcal{B}}$. Denote the totality of the operators $\{U, \tilde{U}\}$ by σ. Introduce the recursive sequence of the sets $\sigma^1 = [\sigma, \sigma], \ldots, \sigma^k = [\sigma^{k-1}, \sigma^{k-1}]$, where $\sigma^i = \{Z: Z = [X, Y], X, Y \in \sigma^{k-1}\}$, $[\ ,\ ]$ is the Poisson bracket. It is clear that the obtained totality yields Lie algebra $\tilde{\mathcal{B}}$ over P and $\tilde{\mathcal{B}} \subseteq \mathcal{B}(T(V))$.

1.3. *The structure of the operators of the system of zero approximation.* Return to the initial system (1). Write down the differential operator associated with (1) $U_0 = U + \varepsilon\tilde{U}$, where

$$U = \omega_1^{(1)}(x)\frac{\partial}{\partial x_1} + \ldots + \omega_n^{(1)}(x)\frac{\partial}{\partial x_n},$$

$$\tilde{U} = \tilde{\omega}_1^{(1)}(x)\frac{\partial}{\partial x_1} + \ldots + \tilde{\omega}_n^{(1)}(x)\frac{\partial}{\partial x_n},$$

ε is a small parameter.

The vector of the coefficients $\omega^{(1)} = \| \omega_1^{(1)}, \ldots, \omega_n^{(1)} \|$ is defined by the matrix equality $\omega^{(1)}(x) = x^T \mathcal{F}$, $\mathcal{F} = \mathcal{A}^T$, i.e. $U \in \mathcal{B}(V_{\otimes 1})$. The vector $\tilde{\omega}^{(2)}(x) = \| \tilde{\omega}_1^{(1)}(x), \ldots, \tilde{\omega}_n^{(1)}(x) \|$ can be represented as $\tilde{\omega}^{(1)}(x) = \hat{\tilde{x}}_{m_1} Q_{m_1} + \ldots + \hat{\tilde{x}}_{m_k} Q_{m_k}$, where $Q_{m_1}, \ldots, Q_{m_k}$ are the rectangular constant matrices of dimensions $m_1 \times n, \ldots, m_k \times n$.

The vectors $\hat{\tilde{x}}_{m_i} Q_{m_i}$ belong to the space $V_{\otimes i}$, $i = \overline{1, k}$. Hence, the operator $\tilde{U}$ is represented by the sum, $\tilde{U} = U_{\otimes 1} + \ldots + U_{\otimes m}$, $U_{\otimes i} \in \mathcal{B}(V_{\otimes i})$, $i = \overline{1, k}$.

Further, we shall use the representation matrices of the linear operator U in the subspaces $V_{\otimes 1}, V_{\otimes 2}, \ldots$, determined by the relations $U\hat{\tilde{x}}_{m_1} = \hat{\tilde{x}}_{m_1}\mathcal{F}_1, \ldots, U\hat{\tilde{x}}_{m_j} = \hat{\tilde{x}}_{m_j}\mathcal{F}_j, \ldots$. It is easy to see that $\mathcal{F}_1 \equiv \mathcal{A}^T$. In further exposition, we shall use for matrix $\mathcal{A}^T$ double notation $\mathcal{F}_1 \equiv \mathcal{A}^T$ and $\mathcal{F} \equiv \mathcal{A}^T$. All the matrices $\mathcal{F}_j$ are quadratic as operator U maps space $V_{\otimes j}$, $j = 1, 2, \ldots$, into itself. We calculate the dimension of the quadratic matrices $\mathcal{F}_j$, $j = 1, 2, \ldots$, since this substantially influences the dimension of the corresponding systems of linear algebraic equations. The order of the matrix $\mathcal{F}_j$ coincides with the number of elements in the basis $\hat{\tilde{x}}_{m_j}$. The index j denotes the power of homogeneity of the polynomials in the subspace $V_{\otimes j}$. The following theorem is of very great importance in future exposition:

Theorem 2. *Let* n *denote the number of dependent variables in the system and* j *denote the power of the homogeneity of polynomials in the subspace* $V_{\otimes j}$, *then the dimension* m_j *of the matrix* $\mathcal{F}_j$ *has the order of growth* $m_j \sim j^{n-1}$.

Proof. The magnitude of the coefficient m_j is determined by the formula of combining recurrences of n elements in j.

$$m_j = C_{j+n-1}^{n-1} = \frac{(j+n-1)!}{(n-1)!\,j!} = \frac{j!(j+1)\dots(j+n-1)}{(n-1)!\,j!} \asymp j^{n-1}.$$

2. Reduction of a solution of operator equations to a solution of a system of algebraic equations.

2.1. *Basic equations.* Further development of the algorithm of asymptotic decomposition is related to the solution of the operator equations (see [1], subsection 2.4)

$$[U, S_\nu] = F_\nu, \quad \nu = 1, 2, \dots . \tag{3}$$

The right-hand sides F_ν of equations (3) are obtained by calculating Poisson bracket of the operators U, $\tilde{U}$, and S_ν finitely many times and, hence,

$$F_\nu \in \mathcal{B}(V_{\otimes 1}) + \dots + \mathcal{B}(V_{\otimes g_\nu}), \tag{4}$$

i.e., F_ν has polynomials of the maximum power g_ν in the coefficients. By $F_{\otimes i\nu}$, $i = \overline{1, g_\nu}$, we denote the components of the operator in the sum of the subspaces (4): $F_\nu = F_{\otimes 1\nu} + \dots + F_{\otimes g_\nu \nu}$. The rectangular matrix Q_{ν_i} of dimension $m_i \times n$ is the matrix of representation of the operator $F_{\otimes i\nu}$ in V. This operator can be represented in the vector form:

$$F_{\otimes i\nu} = \hat{\hat{x}}_{m_i} Q_{\nu_i} \partial, \quad \partial \stackrel{\text{def}}{=} \| \partial/\partial x_1, \dots, \partial/\partial x_n \|^T, \quad i = \overline{1, g_\nu}. \tag{5}$$

The operator S_ν should also be obtained in the form of the sum

$$S_\nu = S_{\otimes 1\nu} + \dots + S_{\otimes g_\nu \nu},$$

where $S_{\otimes i\nu} = \hat{\hat{x}}_{m_i} \Gamma_{\nu i} \partial$, $i = \overline{1, g_\nu}$.

Theorem 3. *Let the matrices $\mathcal{F}_1 \equiv \mathcal{A}^T$, $\mathcal{F}_2, \dots, \mathcal{F}_{g_\nu}$ of dimensions $m_1 \times m_1, \dots, m_{g_\nu} \times m_{g_\nu}$ be representation matrices of the operator U in the subspaces $V_{\otimes 1}, \dots, V_{\otimes g_\nu}$ and let matrices $\Gamma_{\nu 1}, \dots, \Gamma_{\nu g_\nu}$ of dimensions $m_1 \times n, \dots, m_\nu \times n$ be matrices of the operators $S_{\otimes 1\nu}, \dots, S_{\otimes g_\nu \nu}$. Then the solution of the operator equation (3) is reduced to a solution of g_ν independent matrix equations*

$$\begin{aligned} &\mathcal{F}_1 \Gamma_{\nu 1} - \Gamma_{\nu 1} \mathcal{F} = Q_{\nu 1}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &\mathcal{F}_{g_\nu} \Gamma_{\nu g_\nu} - \Gamma_{\nu g_\nu} \mathcal{F} = Q_{\nu g_\nu}. \end{aligned} \tag{6}$$

Proof. The Poisson bracket of operator U and operator from $\mathcal{B}(V_{\otimes i})$ is operator from $\mathcal{B}(V_{\otimes i})$, i.e., $[F, \mathcal{B}(V_{\otimes i})] = \mathcal{B}(V_{\otimes i})$. Hence, substituting the values F_ν, S_ν into the initial equation (3), we obtain g_ν independent equations

$$[U, S_{\otimes i\nu}] = F_{\otimes i\nu}, \quad i = \overline{1, g_\nu}. \tag{7}$$

By immediate calculations, we can easily see that

$$[U, S_{\otimes i\nu}] = (U \hat{\hat{x}}_{m_i} \Gamma_{\nu i} - S_{\otimes i\nu} \hat{\hat{x}}_{m_i} \mathcal{F}) \partial.$$

Here, $U = \hat{\hat{x}}_{m_i} \mathcal{F} \partial$, $U \hat{\hat{x}}_{m_i} \equiv \hat{\hat{x}}_{m_i} \mathcal{F}_i$, $S_{\otimes i\nu} \hat{\hat{x}}_{m_i} \equiv \hat{\hat{x}}_{m_i} \Gamma_{\nu i}$, hence, this expression takes the form

$$[U, S_{\otimes jv}] = \hat{\xi}_{m_j}(\mathcal{F}_j\Gamma_{vi} - \Gamma_{vi}\mathcal{F})\partial. \tag{8}$$

Substituting (5) and (8) into equation (7) and equating the matrices of the coefficients of bilinear form with respect to $\hat{\xi}_{m_j}$, ∂, we obtain the matrix system (6).

2.2. *Three approaches to a solution of the basic equations.* Describe three possible approaches to the solution of the basic equations (6).

The first approach. The notation $R^{(k,n)}$ stands for the set of rectangular matrices of dimension $k \times n$. Along with the linear space $R^{(k,n)}$ over the field P, introduce a linear space $\hat{R}^{(k,n)}$ isomorphic to it and consisting of column vectors formed by the rows of the matrices that belong to $R^{(k,n)}$. From the system of matrix equations (6) in the linear space $R^{(k,n)}$, we pass to the equivalent system in the isomorphic linear space $\hat{R}^{(k,n)}$

$$G_{ij}^{(1)}\hat{\Gamma}_{v1} = \hat{Q}_{v1}, \ldots, G_{ij}^{(g_v)}\hat{\Gamma}_{vg_v} = \hat{Q}_{vg_v},$$

where $G_{ij}^{(i)} = \mathcal{F}_i \otimes \mathcal{E}_n - \mathcal{E}_{m_i} \otimes \mathcal{F}^T$, $i = \overline{1, g_v}$, $\mathcal{E}_{m_i}$, $\mathcal{E}_n$ are the unit matrices of the dimensions m_i, n.

This matrix has the dimension $m_i n \times m_j n$ and acts in the space $\hat{R}^{(m_i, n)}$. The described approach effectively used theoretically leads to matrices of large dimensions. Although these matrices are discharged, to work with them is difficult.

The second approach. Write the system representative of system (6) as

$$\mathcal{F}_i\Gamma - \Gamma\mathcal{F} = Q. \tag{9}$$

Introduce the notation $\gamma_1, \ldots, \gamma_n$, $q_1, \ldots, q_n$ for the columns of matrices Γ and Q, respectively. Let the matrices S, S^{-1} reduce the matrix $\mathcal{F}$ to the diagonal form $S^{-1}\mathcal{F}S = \mathcal{F}_d = \operatorname{diag}\|\lambda_1, \ldots, \lambda_n\|$. Then multiplying equation (9) at the right by S and introducing the notation

$$\tilde{\Gamma} = \Gamma S, \quad \tilde{Q} = QS, \quad \tilde{\gamma}_j = \gamma_j S, \quad \tilde{h}_j = q_j S, \quad j = \overline{1, n},$$

we obtain n systems of independent equations

$$(\mathcal{F}_i - \lambda_1\mathcal{E}_m)\tilde{\gamma}_1 = \tilde{h}_n, \ldots, (\mathcal{F}_i - \lambda_n\mathcal{E}_m)\tilde{\gamma}_n = \tilde{h}_n,$$

where m is the dimension of the matrix $\mathcal{F}_i$. After necessary calculations, we return to the previous variables via the formula $\Gamma = \tilde{\Gamma}S^{-1}$, $Q = \tilde{Q}S^{-1}$.

The intermediate stage in the given approach is a reduction of $\mathcal{F}$ to a block form.

The third approach. Upon a change of variables $x = Sy$ in the initial system of equations (1), the system of zero approximation has been diagonalized. The representation matrices $\mathcal{F}_j$ of the operator U in $V_{\otimes j}$ have become diagonalized, too. Such an approach is used in the normal form method. The described approach lacks the possibility of physical interpretation of new coordinates which are usually complex. Moreover, since the matrices are reduced to a diagonal form for fixed values of the parameters (analytical calculation of roots of the characteristic equations is impossible in a general situation), we cannot follow the influence of the parameters of zero approximation system upon the system as a whole.

Thus, we have to find compromise between the three described approaches which depends upon the situation.

3. Construction of a centralized system and finding the reducing transformations.

3.1. *The basic algorithm.* Study the conditions of solvability of the basic equations

$$\begin{aligned} G_{\mathcal{J}}^{(1)}\hat{\Gamma}_{\nu 1} &= \hat{Q}_{\nu 1}, \\ &\cdots\cdots\cdots\cdots \\ G_{\mathcal{J}}^{(g_\nu)}\hat{\Gamma}_{\nu g_\nu} &= \hat{Q}_{\nu g_\nu}, \end{aligned} \tag{10}$$

where $G_{\mathcal{J}}^{(i)} = \mathcal{F}_i \otimes \mathcal{E}_n - \mathcal{E}_{m_i} \otimes \mathcal{F}^T$, $\mathcal{F} \equiv \mathcal{F}_1 \equiv \mathcal{A}^T$, $i = \overline{1, g_\nu}$, $\nu = 1, 2, \ldots$. The equations of system (10) are solved independently and it is enough to consider only one of them,

$$G_{\mathcal{J}}^{(r)}\hat{\Gamma}_{\nu r} = \hat{Q}_{\nu r}. \tag{11}$$

The homogeneous equation $G_{\mathcal{J}}^{(r)}\hat{\Gamma}_{\nu r} = 0$ and the conjugate equation $G_{\mathcal{J}}^{(r)^*}\hat{\Gamma}_{\nu r} = 0$, where $G_{\mathcal{J}}^{(r)^*} = \mathcal{F}_r^* \otimes \mathcal{E}_n - \mathcal{E}_{m_r} \otimes (\mathcal{F}^*)^T$ is the matrix complex conjugate to $G_{\mathcal{J}}^{(r)}$, correspond to system (11).

Denote the kernels of the operators $G_{\mathcal{J}}^{(r)}$, $G_{\mathcal{J}}^{(r)^*}$ by $\hat{N}_{\mathcal{J}}^{(r)}$, $\hat{N}_{\mathcal{J}}^{(r)^*}$ and the images of these operators by $\hat{T}_{\mathcal{J}}^{(r)}$, $\hat{T}_{\mathcal{J}}^{(r)^*}$. The space $\hat{R}^{(m_r, n)}$ can be decomposed into the direct sum of subspaces $\hat{N}_{\mathcal{J}}^{(r)}$, $\hat{T}_{\mathcal{J}}^{(r)}$ in a unique way $\hat{R}^{(m_r, n)} = \hat{N}_{\mathcal{J}}^{(r)} \oplus \hat{T}_{\mathcal{J}}^{(r)}$. The component $\hat{Q}_{\nu rN}$ is found from the condition of orthogonality of the difference $\hat{Q}_{\nu rT} = \hat{Q}_{\nu rN} - \hat{Q}_{\nu r}$ to the subspace $\hat{N}_{\mathcal{J}}^{(r)^*}$ (a kernel of $G_{\mathcal{J}}^{(r)^*}$). For details, see [3, p. 186]. Represent the right-hand side of equation (11) in the form of the sum $\hat{Q}_{\nu r} = \hat{Q}_{\nu rN} + \hat{Q}_{\nu rT}$, $\hat{Q}_{\nu rN} \in \hat{N}_{\mathcal{J}}^{(r)}$, $\hat{Q}_{\nu rT} \in \hat{T}_{\mathcal{J}}^{(r)}$. In the space $R^{(m_r, n)}$, the vector $\hat{Q}_{\nu rN}$ of the space $\hat{R}^{(m_r, n)}$ is associated with the rectangular matrix $Q_{\nu rN}$. This matrix, in its turn, determines the differential operator

$$N_{\nu r} = \hat{\hat{x}}_{m_r} Q_{\nu rN} \partial, \quad \partial = \operatorname{col} \| \partial/\partial x_1; \ldots, \partial/\partial x_n \|.$$

The constructed operators $N_{\nu r}$, $r = \overline{1, g_\nu}$, $\nu = 1, 2, \ldots$, commute with the operator U: $[U, N_{\nu r}] \equiv 0$. According to the general theory of the asymptotic decomposition algorithm [5, subsection 2.2], the expression $\operatorname{pr} F_\nu \overset{\text{def}}{=} N_\nu \overset{\text{def}}{=} N_{\nu 1} + \ldots + N_{\nu g_\nu}$ should be taken as the projection of the operator F_ν.

Thus, after these transformations, the operator U_0 turns into the operator

$$U_0 = U + \varepsilon N_1 + \varepsilon^2 N_2 + \ldots, \tag{12}$$

where

$$N_\nu = \sum_{j=1}^{g_\nu} N_{\nu j}.$$

The system associated with the operator (12) be

$$\frac{dx_j}{dt} = (U + \varepsilon N_1 + \varepsilon^2 N_2 + \ldots) x_j, \quad j = \overline{1, n},$$

or

$$\frac{dx_j}{dt} = \omega_j(x) + \varepsilon q_{1j}(x) + \varepsilon^2 q_{2j}(x) + \ldots + \varepsilon^{\nu} q_{\nu j}(x) + \ldots,$$

where $q_{\nu 1}, \ldots, q_{\nu n}$ are components of the vector q_{ν} determined by the equality

$$q_{\nu} = \hat{\hat{x}}_{m_{\nu}} \sum_{r=1}^{g_{\nu}} Q_{\nu r N}$$

is a centralized system for system (1).

The operator S_{ν} is determined only by the matrix $\Gamma_{\nu r}$ which can be found as the solution of the equation $G_{\mathcal{J}}^{(r)} \hat{\Gamma}_{\nu r} = \hat{Q}_{\nu r T}$. Upon writing the operators $S_{\nu r} = \hat{\hat{x}}_{m_r} \Gamma_{\nu r} \partial$, we can represent the operator S of equations (3) by the sum $S_{\nu} = S_{\nu 1} + \ldots + S_{\nu g_{\nu}}$.

3.2. *A sufficient criterion of decomposability.* In practical calculations, we restrict ourselves to finitely many approximations m $(m \leq 2)$ in the centralized system. The operator associated with the centralized system can be written in the form

$$U_0 = U + \varepsilon N_1 + \ldots + \varepsilon^m N_m. \tag{13}$$

Restore the centralized system

$$\frac{dx_j}{dt} = \sum_i a_{ji} x_i + \varepsilon f_{1xj}(x) + \ldots + \varepsilon^m f_{mxj}(x), \quad j = \overline{1, n}, \tag{14}$$

where

$$f_{1xj}(x) = N_1 x_j, \ldots, f_{mxj}(x) = N_m x_j, \tag{15}$$

by the operator (13).

In [1, Ch. 3], a few theorems on separation of variables are given. Here, we give a new sufficient criterion of decomposability and an algorithm to separate fast and slow variables in system (14).

Obtain the solution of the equation

$$U f = 0 \tag{16}$$

in the subspaces $V_{\otimes 1}, \ldots, V_{\otimes N}$ in the form of the sums

$$\rho_1 = \hat{\hat{x}}_{m_1} \alpha^{(1)}, \ldots, \rho_N = \hat{\hat{x}}_{m_N} \alpha^{(N)}, \tag{17}$$

where

$$\alpha^{(1)} = \operatorname{col} \| \alpha_{11}, \ldots \alpha_{1m_1} \|, \ldots, \alpha^{(N)} = \operatorname{col} \| \alpha_{N1}, \ldots \alpha_{Nm_N} \|.$$

N is the maximum degree of polynomials in the right-hand sides of equations (14).

Substituting relations (17) into (16), we obtain the system of algebraic equations

$$\mathcal{F}_1 \alpha^{(1)} = 0, \ldots, \mathcal{F}_N \alpha^{(N)} = 0, \tag{18}$$

where $\mathcal{F}_1, \ldots, \mathcal{F}_N$ are representation matrices of the operator U in the subspaces $V_{\otimes 1}, \ldots, V_{\otimes N}$.

Two integrals $\rho_1(x)$ and $\rho_2(x)$ obtained in this way are functionally dependent if there is a polynomial $F(y)$ such that $\rho_1(x) \equiv F(\rho_2(x))$. Denote the totality of all functionally independent integrals obtained while solving system (18) by σ_0,

$$\sigma_0 = \{ \rho_1(x), \ldots, \rho_k(x) \}. \tag{19}$$

Fix one of the element of set (19) $\rho_j(x)$ and form the functions

$$N_1\rho_j(x), \ldots, N_m\rho_j(x). \tag{20}$$

By virtue of the commutativity of the operators U and $N_1, \ldots, N_m$, all functions of series (20) are also solutions of equation (16). Let σ_j be a subset of set (20), composed of functionally independent integrals. Assume that intersection of the sets

$$\sigma = \sigma_0 \cup \left(\bigcap_{j=1}^{k} \sigma_j \right)$$

is nonempty and contains k elements $\rho_1, \ldots, \rho_k$. If $k = n$, then, after a change of variables

$$y_1 = \rho_1(x), \ldots, y_n = \rho_n(x), \tag{21}$$

the centralized system (14) is transformed into

$$\frac{dy_j}{dt} = \varepsilon \tilde{f}_{1j}(y) + \ldots + \varepsilon^m \tilde{f}_{mj}(y), \quad j = \overline{1,n}.$$

If $k < n$, then the problem of obtaining additional functions generating fast variables appears. This problem, as with that previously described, can be solved constructively if we consider the equation

$$Uf = \lambda f, \quad \lambda = \text{const}. \tag{22}$$

Let f_0 be a solution of equation (22). Then any function $\rho(x)f_0$, where $\rho(x)$ is a polynomial solution of equation (16), satisfies equation (22). Two solutions of equation (22) f_1 and f_2 will be called dependent if $f_1 = \rho(x)f_2$, where $\rho(x)$ is a polynomial solution of the system (16) which functionally depends on set (19). Assume that in the space $V_{\otimes 1}$, equation (22) has h solutions $q_i = \tilde{\hat{x}}_{m_1} b^{(i)}$, $b^{(i)} = \operatorname{col} \| b_{i1}, \ldots, b_{im_1} \|$, $i = \overline{1,h}$, where the vector $b^{(i)}$ is a solution of the matrix equation $(\mathcal{F}_1 - \lambda_i \mathcal{E}) b^{(i)} = 0$, $i = \overline{1,h}$, λ_i is the eigenvalue of the characteristic equation of the matrix $\mathcal{F}_1$.

By virtue of the commutativity of the operators U and $N_1, \ldots, N_m$, the functions

$$N_1 q_i, \ldots, N_m q_i \tag{23}$$

are also eigenfunctions of the operator U. Denote all independent functions of the totality (23) by Ω_i. Let the number of them be p_i, $p_i \geq 1$.

Supplement basis (21) of the number of functions Ω_i to n variables and find the change of variables decomposing the centralized system (14) into r slow and $n - r$ fast variables. It is clear that, for fast variables, for example, $p_1, \ldots, p_l$, $l < q$, the condition $p_1 + \ldots + p_l = n - r$ would hold. In the most simple case where $p_1 = \ldots = p_l = 1$, functions (21) are supplemented with the functions

$$z_1 = q_1(x), \ldots, z_l = q_l(x), \quad l = n - r. \tag{24}$$

After the change (21), (24), the centralized system can be written in the following way:

$$\dot{y} = \varepsilon\varphi_1(y) + \ldots + \varepsilon^m \varphi_m(y),$$

$$\dot{z}_j = \lambda_j z_j + \varepsilon(\psi_{j1}(y) + \ldots + \varepsilon^{m-1}\psi_{jm}(y)) z_j, \quad j = \overline{1,l}.$$

Let some operator W commuting with U be known. Then we can choose solutions

of the equation as a fast variable. In this case, we must be sure that for these solutions, for example, $f_1(x), \ldots, f_l(x)$, the relations $N_j f_1 = F_{j1}(f_1, \ldots, f_l), \ldots, N_j f_l = F_{jl}(f_1, \ldots, f_l)$, where $F_{j1}, \ldots, F_{jl}$ are polynomials of their arguments, would be valid.

***Example* 1.** Choose $U_0 = U + \varepsilon N_1 + \varepsilon^2 N_2$, where

$$U = x_1 \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2} - 3x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4} + 4x_5 \frac{\partial}{\partial x_5},$$

$$N_1 = (\alpha_1 x_2 x_4 + \alpha_2 x_3 x_5) \frac{\partial}{\partial x_1} + \alpha_3 x_1^2 \frac{\partial}{\partial x_2} + \alpha_4 x_2 x_3 \frac{\partial}{\partial x_4} + \alpha_5 x_2^2 \frac{\partial}{\partial x_5},$$

$$N_2 = (\beta_1 x_1^2 x_4 + \beta_2 x_2^2 x_3) \frac{\partial}{\partial x_1} + \beta_3 x_4^2 x_5 \frac{\partial}{\partial x_2} + \beta_4 x_1 x_4^2 \frac{\partial}{\partial x_4} + \beta_5 x_1^2 x_2 \frac{\partial}{\partial x_5}.$$

Write the corresponding centralized system

$$\dot{x}_1 = x_1 + \varepsilon(\alpha_1 x_1 x_4 + \alpha_2 x_3 x_5) + \varepsilon^2(\beta_1 x_1^2 x_4 + \beta_2 x_2^2 x_3),$$

$$\dot{x}_2 = 2x_2 + \varepsilon\alpha_3 x_1^2 + \varepsilon^2 \beta_3 x_4^2 x_5,$$

$$\dot{x}_3 = -3x_3,$$

$$\dot{x}_4 = -x_4 + \varepsilon\alpha_4 x_2 x_3 + \varepsilon\beta_4 x_1 x_4^2,$$

$$\dot{x}_5 = 4x_5 + \varepsilon\alpha_5 x_2^2 + \varepsilon^2 \beta_5 x_1^2 x_2.$$

The coefficients of the operators belong to subspaces $V_{\otimes 1}$, $V_{\otimes 2}$, $V_{\otimes 3}$. Let us begin the realization of the algorithm from two solutions of equation (16)

$$\rho_1 = x_1 x_4, \quad \rho_2 = x_2 x_4^2. \tag{25}$$

By immediate calculation, we obtain

$$\begin{aligned} N_1\rho_1 &= \alpha_1 x_2 x_4^2 + \alpha_2 x_3 x_4 x_5 + \alpha_4 x_1 x_2 x_3, \\ N_1\rho_2 &= \alpha_3 (x_1 x_4)^2 + 2\alpha_4 (x_2^2 x_3 x_4), \\ N_2\rho_1 &= \beta_1 (x_1 x_4)^2 + \beta_2 x_2^2 x_3 x_4 + \beta_4 (x_1 x_4)^2, \\ N_2\rho_2 &= \beta_3 x_4^4 x_5 + 2\beta_4 x_1 x_2 x_4^3. \end{aligned} \tag{26}$$

The new integrals $\rho_3 = x_3 x_4 x_5$, $\rho_4 = x_1 x_2 x_3$, $\rho_5 = x_2^2 x_3 x_4$, $\rho_6 = x_4^4 x_5$ appear on the right-hand sides of relations (26).

The obtained system of integrals $\rho_1, \ldots, \rho_6$ is not closed with respect to the action of the operators N_1, N_2. Really, $N_1\rho_3 = \alpha_4 x_2 x_3^2 x_5 + \alpha_5 x_2^2 x_3 x_4$. Here, $\rho_7 = x_2 x_3^2 x_5$ is a new integral.

For the system of integrals (25) to be closed with respect to the action of the operators N_1, N_2, we should put $\alpha_2 = \alpha_5 = 0$, $\beta_2 = \beta_3 = 0$. Choose eigenfunctions $z_1 = x_1$, $z_2 = x_2 x_4$, $z_5 = x_3 x_5$ of the operator U as fast variables. Under suggestion $\alpha_5 = 0$, $\beta_5 = 0$ these set of functions is translated into itself by operators N_1, N_2:

$$N_1 x_1 = \alpha_1 x_2 x_4, \quad N_2 x_1 = \beta_1 x_1 (x_1 x_4).$$

$$N_1 x_2 x_4 = \alpha_3 x_1 (x_1 x_4), \quad N_1 (x_3 x_5) = 0,$$

$$N_2 x_2 x_4 = \beta_4 x_1 (x_2 x_4^2), \quad N_2 (x_3 x_5) = 0.$$

Hence, upon a change of variables in the centralized system

$$y_1 = x_1 x_4, \quad y_2 = x_2 x_4^2, \quad z_1 = x_1, \quad z_2 = x_2 x_4, \quad z_3 = x_3 x_5,$$

these variables are separated into two fast and three slow ones. After simple calculations, we obtain

$$\dot{y}_1 = \varepsilon \alpha_1 y_2 + \varepsilon^2 (\beta_1 + \beta_2) y_1^2,$$

$$\dot{y}_2 = \varepsilon \alpha_3 y_1^2 + \varepsilon^2 \beta_4 y_1 y_2,$$

$$\dot{z}_1 = z_1 + \varepsilon \alpha_1 z_2 + \varepsilon^2 \beta_1 z_1 y_1,$$

$$\dot{z}_2 = z_2 + \varepsilon \alpha_3 z_1 y_1 + \varepsilon^2 \beta_4 y_1 z_2,$$

$$\dot{z}_3 = z_3.$$

4. Models based on Lotka–Volterra system. Consider the model "prey-predator" described by the system of second-order ordinary differential equations* (Svirizhev Yu., Logofet D. O. [4, p. 94], Freedman H. I. [5, p. 33])

$$\frac{dx}{dt} = \alpha x - V(x) y; \quad \frac{dy}{dt} = -my + kyV(x). \tag{27}$$

where $x(t)$ and $y(t)$ are the number of preys and predators, respectively. Let the number of preys, in the absence of predators, grow exponentially with the relative velocity α. The predators die out in the absence of preys with relative velocity m. The function $V = V(x)$ is the number (or biomass) of preys consumed by one predator for a unit of time. The function $V(x)$ is usually called a trophic function of the predator or the functional response of the predator to the density of population of the prey.

Let us find a solution of system (27) in the neighborhood of zero, i.e., for small x and y. $V(x)$ is assumed to be a linear function, i.e., $V(x) = b_1 x$. For $k = \text{const}$, the system takes the form

$$\frac{dx}{dt} = \alpha x - b_1 x y; \quad \frac{dy}{dt} = k b_1 x y - my.$$

This system is called a classical Lotka–Volterra "prey–predator" model.

In the described study, we take a trophic function in the form of a polynomial of the second degree

$$V(x) = b_1 x + b_2 x^2. \tag{28}$$

Let us introduce the new variables $x_1' = x$, $x_2' = y$. Assume that dimensionless variables are introduced and the system contains a small parameter ε $(0 < \varepsilon < 1)$.

$$\frac{dx_1'}{dt} = \alpha x_1' - \varepsilon (b_1 x_1' x_2' + b_2 x_1'^2 x_2');$$

$$\frac{dx_2'}{dt} = -m x_2' + \varepsilon k (b_1 x_1' x_2' + b_2 x_1'^2 x_2'). \tag{29}$$

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To show some peculiarities of the proposed method, we consider a more general form of the trophic function (28) in comparison with the classical model.

Apply the method described above to system (29). Make some preliminary remarks. The exponential function $x_1 = \exp(\alpha t)c_1$, $x_2 = \exp(-mt)c_2$, where c_1, c_2 are the arbitrary constants in the general solution of the system of zero approximation

$$\frac{dx_1}{dt} = \alpha x_1; \quad \frac{dx_2}{dt} = -mx_2.$$

The asymptotic method (Bogolyubov N. N., Mitropolsky Yu. A. [6]) cannot be applied to the perturbed system (29) by virtue of the exponential character of the solution of the system of zero approximation. Using the mentioned general solution, we reduce the perturbed system (29) to the standard form. Upon a change of variables

$$x_1' = \exp(\alpha t)c_1, \quad x_2' = \exp(-mt)c_2,$$

system (29) turns into

$$\frac{dc_1}{dt} = -\varepsilon F_1(t, c_1, c_2), \quad \frac{dc_2}{dt} = -\varepsilon k F_2(t, c_1, c_2),$$

where

$$F_1(t, c_1, c_2) = b_1 e^{-mt} c_1 c_2 + b_2 e^{(\alpha - m)t} c_1^2 c_2,$$

$$F_2(t, c_1, c_2) = b_1 c_1 c_2 e^{\alpha t} + b_2 e^{2\alpha t} c_1^2 c_2.$$

By immediate calculation, we obtain

$$\int_0^T F_2(t, c_1, c_2)\,dt = \frac{1}{\alpha} b_1 c_1 c_2 (e^{\alpha T} - 1) + \frac{1}{2\alpha} b_2 c_1^2 c_2 (e^{2\alpha T} - 1)$$

and, correspondingly,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T F_2(t, c_1, c_2)\,dt = \infty.$$

Thus, there is no average value of the right-hand sides of the system in the standard form and the averaging method cannot be applied.

According to the general technique of this chapter, we write the differential operator $U_0' = U' + \varepsilon \tilde{U}'$, where

$$U' = \alpha x_1' \frac{\partial}{\partial x_1'} - m x_2' \frac{\partial}{\partial x_2'}; \quad \tilde{U} = (b_1 x_1' x_2' + b_2 x_1'^2 x_2') \left(-\frac{\partial}{\partial x_1'} + k \frac{\partial}{\partial x_2'} \right),$$

associated with system (29).

We represent the coefficients of the nonlinear part of the initial system in the form

$$\| -b_1 x_1' x_2' - b_2 x_1'^2 x_2', \; k b_1 x_1' x_2' + k b_2 x_1'^2 x_2' \| = \hat{\tilde{x}}'_{m_2} Q_{m_2, 1} + \hat{\tilde{x}}'_{m_3} Q_{m_3, 1}.$$

The vectors

$$\hat{\tilde{x}}'_{m_2} = \| x_1'^2, x_1 x_2', x_2'^2 \|, \quad \hat{\tilde{x}}'_{m_3} = \| x_1'^3, x_1'^2 x_2', x_1' x_2'^2, x_2'^3 \|$$

are composed of the basis of spaces $V_{\otimes 2}$, $V_{\otimes 3}$; the matrices $Q_{m_2, 1}$ and $Q_{m_3, 1}$ are determined as

$$Q_{m_2,1} = \begin{Vmatrix} 0 & 0 \\ -b_1 & kb_1 \\ 0 & 0 \end{Vmatrix}; \quad Q_{m_3,1} = \begin{Vmatrix} 0 & 0 \\ -b_2 & kb_2 \\ 0 & 0 \\ 0 & 0 \end{Vmatrix}.$$

The differential operators $\tilde{U}_2'$, $\tilde{U}_3'$ are determined by these matrices

$$\tilde{U}_2' = \tilde{\hat{x}}'_{m_2} Q_{m_2,1} \partial' \equiv -b_1 x_1' x_2' \frac{\partial}{\partial x_1'} + kb_1 x_1' x_2' \frac{\partial}{\partial x_2'};$$

$$\tilde{U}_3' = \tilde{\hat{x}}'_{m_3} Q_{m_3,1} \partial' \equiv -b_2 x_1'^2 x_2' \frac{\partial}{\partial x_1'} - kb_2 x_1'^2 x_2' \frac{\partial}{\partial x_2'},$$

where $\partial' = \operatorname{col} \| \partial/\partial x_1', \partial/\partial x_2' \|$, and belong to the spaces $\mathcal{B}(V_{\otimes 2})$, $\mathcal{B}(V_{\otimes 3})$.

Thus, $\tilde{U} = \tilde{U}_2' + \tilde{U}_3'$, $\tilde{U}_i' \in \mathcal{B}(V_{\otimes i})$, $i = 2, 3$. Consider the two approximations in the transformed operator (12), i.e., the sum involves terms with ε in a power not higher than two,

$$\tilde{U}_0 = U + \varepsilon N_1 + \varepsilon^2 N_2. \tag{30}$$

Calculate the operators S_1 and S_2 in the change of variables (20) from [1]. Find them from the equations

$$[U, S_1] = \tilde{U} - \operatorname{pr} \tilde{U}; \quad [U, S_2] = \left\{ -\left[\tilde{U}, S_1\right] - \frac{1}{2}[S_1[U, S_1]] \right\} - \operatorname{pr}\{\ldots\}. \tag{31}$$

According to the general theory, the equations are solved recursively and are reduced to the solution of systems of linear algebraic equations. The solution of the first equation is obtained in the form $S_1 \equiv S_{\otimes 21} + S_{\otimes 31}$, where $S_{\otimes i1} \equiv \tilde{\hat{x}}_{m_i} \Gamma_{1i} \partial$, $i = 2, 3$, Γ_{1i} are rectangular matrices of dimensions $m_i \times n$, which are solutions of the system of independent algebraic equations

$$\mathcal{J}_i \Gamma_{1i} - \Gamma_{1i} \mathcal{J} = Q_{m_i,1} - \operatorname{pr} Q_{m_i,1}, \quad i = 2, 3. \tag{32}$$

Now consider the second equation. We find the solution in the form of the sum $S_2 = S_{\otimes 22} + S_{\otimes 32} + S_{\otimes 42} + S_{\otimes 52}$, where $S_{\otimes i2} \equiv \tilde{\hat{x}}_{m_i} \Gamma_{2i} \partial$, $i = \overline{2, 5}$; Γ_{2i} is the solution of the system of independent algebraic equations

$$\mathcal{J}_i \Gamma_{2i} - \Gamma_{2i} \mathcal{J} = Q_{m_i,2} - \operatorname{pr} Q_{m_i,2}, \quad i = \overline{2, 5}. \tag{33}$$

From the systems of equations (32), (33), we pass to

$$G_{ij}^{(i)} \hat{\Gamma}_{1i} = \hat{Q}_{m_i,1} - \operatorname{pr} \hat{Q}_{m_i,1}, \quad i = 2, 3, \tag{34}$$

and

$$G_{ij}^{(i)} \hat{\Gamma}_{2i} = \hat{Q}_{m_i,2} - \operatorname{pr} \hat{Q}_{m_i,2}, \quad i = \overline{2, 5}, \tag{35}$$

where $G_{ij}^{(i)} = \mathcal{J}_i \otimes \mathcal{E}_2 - \mathcal{E}_{m_i} \otimes \mathcal{J}^T$, $i = \overline{2, 5}$.

Before solving equations (32), (33) or (34), (35), we should define projections of their right-sides. This deals with the solution of the homogeneous system $G_{ij}^{(i)} \hat{\Gamma} = 0$.

Represent the elements involved in the described relations in an explicit form. $\lambda_1 = \alpha$, $\lambda_2 = -m$ are characteristic numbers of the matrix

$$\mathcal{J} = \begin{Vmatrix} \alpha & 0 \\ 0 & -m \end{Vmatrix}.$$

The matrix $\mathcal{A} = \operatorname{diag} \| \lambda_1, \lambda_2 \|$ is a matrix of the operator U.

$\mathcal{F}_2, \ldots, \mathcal{F}_5$ are the representation matrices of the operator U in the subspaces $V_{\otimes 2}, \ldots V_{\otimes 5}$ and are determined by the identities $U \hat{\hat{x}}_{m_i} = \hat{\hat{x}}_{m_i} \mathcal{F}_i$, $i = \overline{2, 5}$. Represent these matrices in an explicit form,

$$\mathcal{F}_2 = \operatorname{diag} \| 2\lambda_1, \lambda_1 + \lambda_2, 2\lambda_2 \|;$$

$$\mathcal{F}_3 = \operatorname{diag} \| 3\lambda_1, 2\lambda_1 + \lambda_2, \lambda_1 + 2\lambda_2, 3\lambda_2 \|;$$

$$\mathcal{F}_4 = \operatorname{diag} \| 4\lambda_1, 3\lambda_1 + \lambda_2, 2\lambda_1 + 2\lambda_2, \lambda_1 + 3\lambda_2, 4\lambda_2 \|;$$

$$\mathcal{F}_5 = \operatorname{diag} \| 5\lambda_1, 4\lambda_1 + \lambda_2, 3\lambda_1 + 2\lambda_2, 2\lambda_1 + 3\lambda_2, \lambda_1 + 4\lambda_2, 5\lambda_2 \|.$$

The explicit form of vectors $\hat{\hat{x}}_{m_2}$, $\hat{\hat{x}}_{m_3}$ was given above. The vectors $\hat{\hat{x}}_{m_4}$, $\hat{\hat{x}}_{m_5}$ are composed of the basis elements of the subspaces $V_{\otimes 4}$, $V_{\otimes 5}$. Represent these vectors in an explicit form,

$$\hat{\hat{x}}_{m_4} = \| x_1^4, x_1^3 x_2, x_1^2 x_2^2, x_1 x_2^3, x_2^4 \|;$$

$$\hat{\hat{x}}_{m_5} = \| x_1^5, x_1^4 x_2, x_1^3 x_2^2, x_1^2 x_2^3, x_1 x_2^4, x_2^5 \|.$$

Represent the matrices $G_{\mathcal{F}}^{(2)}, \ldots, G_{\mathcal{F}}^{(5)}$ in an explicit form, too. By definition,

$$\mathcal{F}_2 \otimes \mathcal{E}_2 = \operatorname{diag} \| 2\lambda_1, 2\lambda_1, \lambda_1 + \lambda_2, \lambda_1 + \lambda_2, 2\lambda_2, 2\lambda_2 \|;$$

$$\mathcal{E}_3 \otimes \mathcal{F}^T = \operatorname{diag} \| \lambda_1, \lambda_2, \lambda_1, \lambda_2, \lambda_1, \lambda_2 \|.$$

Hence,

$$G_{\mathrm{A}}^{(2)} = \mathcal{F}_2 \otimes \mathcal{E}_2 - \mathcal{E}_3 \otimes \mathcal{F}^T = \operatorname{diag} \| \lambda_1, 2\lambda_1 - \lambda_2, \lambda_2, \lambda_1, 2\lambda_2 - \lambda_1, \lambda_2 \|.$$

In an analogous way, we obtain

$$G_{\mathcal{F}}^{(3)} = \operatorname{diag} \| 2\lambda_1, 3\lambda_1 - \lambda_2,$$
$$(2\lambda_1 + \lambda_2) - \lambda_1, (2\lambda_2 + \lambda_2) - \lambda_2,$$
$$(\lambda_1 + 2\lambda_2) - \lambda_1, (\lambda_1 + 2\lambda_2) - \lambda_2, 3\lambda_2 - \lambda_1, 2\lambda_2 \|;$$

$$G_{\mathcal{F}}^{(4)} = \operatorname{diag} \| 3\lambda_1, 4\lambda_1 - \lambda_2, (3\lambda_1 + \lambda_2) - \lambda_1, (3\lambda_1 + \lambda_2) - \lambda_2,$$
$$(2\lambda_1 + 2\lambda_2) - \lambda_1, (2\lambda_1 + 2\lambda_2) - \lambda_2, (\lambda_1 + 3\lambda_2) - \lambda_1,$$
$$(\lambda_1 + 3\lambda_2) - \lambda_2, 4\lambda_2 - \lambda_1, 3\lambda_2 \|;$$

$$G_{\mathcal{F}}^{(5)} = \operatorname{diag} \| 4\lambda_1, 5\lambda_1 - \lambda_2, (4\lambda_1 + \lambda_2) - \lambda_1, (4\lambda_1 + \lambda_2) - \lambda_2,$$
$$(3\lambda_1 + 2\lambda_2) - \lambda_1, (3\lambda_1 + 2\lambda_2) - \lambda_2, (2\lambda_2 + 3\lambda_2) - \lambda_1,$$
$$(2\lambda_1 + 3\lambda_2) - \lambda_2, (\lambda_1 + 4\lambda_2) - \lambda_1, (\lambda_1 + 4\lambda_2) - \lambda_2, 5\lambda_2 - \lambda_1, 4\lambda_2 \|.$$

Calculate the components N_1 and N_2 in operator (30), which yields the centralized system and make a change of variables upon which the separation occurs. To determine the matrices of the coefficients Γ_{12}, Γ_{13} of the transformation S_1, we consider equation (34). The matrices $G_{\mathcal{F}}^{(2)}$ and $G_{\mathcal{F}}^{(3)}$ have the form

$$G_{\mathcal{F}}^{(2)} = \operatorname{diag} \| \alpha, 3\alpha, -\alpha, \alpha, -3\alpha, -\alpha \|;$$

$$G_{\mathcal{J}}^{(3)} = \operatorname{diag} \| 2\alpha,\ 4\alpha,\ 0,\ 2\alpha,\ -2\alpha,\ 0,\ -4\alpha,\ -2\alpha \|.$$

The space $\hat{N}_{\mathcal{J}}^{(2)}$ is defined as solutions of the homogeneous equations $G_{\mathcal{J}}^{(2)} \hat{\Gamma} = 0$. The matrix $G_{\mathcal{J}}^{(2)}$ is nonsingular and $\hat{N}_{\mathcal{J}}^{(2)} \equiv \varnothing$. Hence, $\operatorname{pr} \hat{Q}_{m_2,1} \equiv 0$ and the equation $G_{\mathcal{J}}^{(2)} \hat{\Gamma}_{12} = \hat{Q}_{m_2,1}$, where $\hat{Q}_{m_2,1} = \operatorname{col} \| 0,\ 0,\ -b_1,\ kb_1,\ 0,\ 0 \|$ has the unique solution $\hat{\Gamma}_{12} = \operatorname{col} \| 0,\ 0,\ b_1/\alpha,\ kb_1/\alpha,\ 0,\ 0 \|$. The space $\hat{N}_{\mathcal{J}}^{(3)}$ is defined as the solutions of the homogeneous equation $\hat{G}_{\mathcal{J}}^{(3)} \hat{\Gamma}_{12} = 0$. Take

$$\hat{Z}_{13} = \operatorname{col} \| 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0 \|; \quad \hat{Z}_{23} = \operatorname{col} \| 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0 \|$$

as the basis $\hat{N}_{\mathcal{J}}^{(2)}$. It is easy to see that the projection of the vector $\hat{Q}_{m_3,1} = \operatorname{col} \| 0,\ 0,\ -b_2,\ kb_2,\ 0,\ 0,\ 0,\ 0 \|$ is equal to $\operatorname{pr} \hat{Q}_{m_3,1} = \operatorname{col} \| 0,\ 0,\ -b_2,\ 0,\ 0,\ 0,\ 0,\ 0 \|$. The equation determining the matrix $\hat{\Gamma}_{13}$: $G_{\mathcal{J}}^{(3)} \hat{\Gamma}_{13} = \hat{Q}_{m_3,1} - \operatorname{pr} \hat{Q}_{m_3,1}$ has the solution $\hat{\Gamma}_{13} = \operatorname{col} \| 0,\ 0,\ 0,\ kb_2/2\alpha,\ 0,\ 0,\ 0,\ 0 \|$. In operator notation, we obtain

$$N_1 = \hat{\hat{x}}_{m_3} \operatorname{pr} Q_{m_3,1} \partial = -b_2 x_1^2 x_2 \partial/\partial x_1;$$
$$S_1 = S_{\otimes 21} + S_{\otimes 31}, \tag{36}$$

where

$$S_{\otimes 21} = \hat{\hat{x}}_{m_2} \Gamma_{12} \partial = \frac{b_1}{\alpha} x_1 x_2 \left(\frac{\partial}{\partial x_1} + k \frac{\partial}{\partial x_2} \right);$$

$$S_{\otimes 31} = \hat{\hat{x}}_{m_3} \Gamma_{12} \partial = \frac{kb_2}{2\alpha} x_1^2 x_2 \frac{\partial}{\partial x_2}.$$

Thus, the operator U_0 (30) in the first approximation is defined as

$$U_0 = \alpha \left(x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} \right) - \varepsilon b_2 x_1^2 x_2 \frac{\partial}{\partial x_1} \tag{37}$$

and the centralized system of the first approximation takes the form

$$\frac{dx_1}{dt} = \alpha x_1 - \varepsilon b_2 x_1^2 x_2; \quad \frac{dx_2}{dt} = -\alpha x_2. \tag{38}$$

We may use the new variables

$$y_1 = x_1 x_2, \quad y_2 = x_1, \tag{39}$$

which separate variables in the new system. We pass from the operator U_0 (37) to

$$U_0(y) = -\varepsilon b_2 y_1 \frac{\partial}{\partial y_1} + (\alpha y_2 - \varepsilon b_2 y_1 y_2) \frac{\partial}{\partial y_2}$$

and, respectively, we pass from system (38) to

$$\frac{dy_1}{dt} = -\varepsilon b_2 y_1^2; \quad \frac{dy_2}{dt} = \alpha y_1 - \varepsilon b_2 y_1 y_2. \tag{40}$$

Apply Theorem 2.2 [1] to the centralized system (40) in the first approximation. The solution of this system may be presented in the form

$$y = \exp \mathcal{A}(t - t_0)\, \bar{y}, \quad \bar{y} = \operatorname{col} \| \bar{y}_1,\ \bar{y}_2 \|. \tag{41}$$

Here, the vector $\bar{y}$ is obtained from the system of equations

$$\frac{d\bar{y}_1}{d\tau} = -b_2\bar{y}_1^2; \quad \frac{d\bar{y}_2}{d\tau} = -b_2\bar{y}_1\bar{y}_2, \quad \tau = \varepsilon t. \tag{42}$$

System (42) can be easily integrated $\bar{y}_1 = 1/(c_1 + b_2\tau)$; $\bar{y}_2 = c_2/(c_1 + b_2\tau)$, where c_1 and c_2 are arbitrary constants.

Finally, we can write solution (41) of system (40) in the form

$$y_1 = \frac{e^{\alpha(t-t_0)}}{c_1 + b_2\varepsilon t}; \quad y_2 = \frac{c_2 e^{-\alpha(t-t_0)}}{c_1 + b_2 t}.$$

Now find the operator N_2 and the second approximation of the centralized system. For this purpose, we should calculate the right-hand side of equation (31)

$$-[\tilde{U}, S_1] + 1/2[[U, S_1]S_1], \tag{43}$$

where the operator S_1 has been obtained previously by formula (36),

$$S_1 = \frac{b_1}{\alpha}x_1x_2\left(\frac{\partial}{\partial x_1} + k\frac{\partial}{\partial x_2}\right) + \frac{kb_2}{2\alpha}x_1^2x_2\frac{\partial}{\partial x_2}.$$

Represent the final form of the operators $[S_1, \tilde{U}]$, $1/2[U, [S_1, S_1]]$ omitting the intermediate calculation

$$[\tilde{U}, S_1] = \left(\frac{2kb_1^2}{\alpha}x_1^2x_2 + \frac{b_1b_2}{\alpha}x_1^2x_2^2 + \frac{5}{2}\frac{kb_1b_2}{\alpha}x_1^3x_2 + \frac{kb_2^2}{2\alpha}x_1^4x_2\right)\frac{\partial}{\partial x_1} +$$

$$+ \left(\frac{2kb_1^2}{\alpha}x_1x_2^2 + 4\frac{kb_1b_2}{\alpha}x_1^2x_2^2 + \frac{kb_2^2}{2\alpha}x_1^3x_2^2\right)\frac{\partial}{\partial x_2};$$

$$1/2[[U, S_1]S_1] = \left(2\frac{kb_1^2}{\alpha}x_1^2x_2 + \frac{3}{2}\frac{kb_1b_2}{\alpha}x_1^3x_2\right)\frac{\partial}{\partial x_1} -$$

$$- \left(2\frac{kb_1^2}{\alpha}x_1^2x_2^2 + 3\frac{kb_1b_2}{\alpha}x_1^2x_2^2\right)\frac{\partial}{\partial x_2}.$$

Using the previously introduced notation, we can represent operator (43) in the form of a sum,

$$-[\tilde{U}, S_1] + 1/2[[U, S_1]S_1] = F_{\otimes 23} + F_{\otimes 24} + F_{\otimes 25},$$

where

$$F_{\otimes 23} = -\frac{kb_1^2}{\alpha}x_1^2x_2\frac{\partial}{\partial x_1} + \frac{kb_1^2}{\alpha}x_1x_2^2\frac{\partial}{\partial x_2};$$

$$F_{\otimes 24} = -\left(\frac{b_1b_2}{\alpha}x_1^2x_2^2 + \frac{7}{4}\frac{kb_1b_2}{\alpha}x_1^3x_2\right)\frac{\partial}{\partial x_1} + \frac{5}{2}\frac{kb_1b_2}{\alpha}x_1^2x_2^2\frac{\partial}{\partial x_2};$$

$$F_{\otimes 25} = -\frac{kb_2^2}{2\alpha}x_1^4x_2\frac{\partial}{\partial x_1} + \frac{kb_2^2}{\alpha}x_1^3x_2^2\frac{\partial}{\partial x_2}.$$

Repeating the calculation carried out above, we obtain the following result:

$$N_2 = \operatorname{pr} F_{\otimes 23},$$

where

$$\operatorname{pr} F_{\otimes 23} = -\frac{kb_1^2}{\alpha} x_1 x_2 \left(x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} \right).$$

Further, find the expression of the operator $S_2 = S_{\otimes 24} + S_{\otimes 25}$, where

$$S_{\otimes 24} = -\left(-\frac{7}{4} \frac{kb_1 b_2}{\alpha} x_1^3 x_2 + \frac{b_1 b_2}{\alpha} x_1^2 x_2 \right) \frac{\partial}{\partial x_1} + \frac{5}{2} \frac{kb_1 b_2}{\alpha^2} x_1^2 x_2^2 \frac{\partial}{\partial x_2};$$

$$S_{\otimes 25} = -\frac{kb_2^2}{4\alpha^2} x_1^4 x_2 \frac{\partial}{\partial x_1} + \frac{kb_2^2}{\alpha} x_1^3 x_2 \frac{\partial}{\partial x_2}.$$

The operator U_0 (30) in the second approximation is defined as

$$U_0 = \alpha \left(x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} \right) - \varepsilon b_2 x_1^2 x_2 \frac{\partial}{\partial x_1} -$$

$$- \varepsilon^2 \frac{kb_1^2}{\alpha} x_1 x_2 \left(x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} \right) \tag{44}$$

and the centralized system takes the form

$$\frac{dx_1}{dt} = \alpha x_1 - \varepsilon b_2 x_1^2 x_2 - \frac{\varepsilon^2 kb_1^2}{\alpha} x_1^2 x_2;$$

$$\frac{dx_2}{dt} = -\alpha x_2 - \frac{\varepsilon^2 kb_1^2}{\alpha} x_1 x_2^2. \tag{45}$$

By change (39), we separate the slow and fast variables in the centralized system of the second approximation. The operator U_0, (44) is turned into

$$U_0(y) = -\varepsilon b_2 y_1^2 \frac{\partial}{\partial y_1} + \left(\alpha y_2 - \varepsilon b_2 y_1 y_2 - \varepsilon^2 \frac{kb_1^2}{\alpha} y_1 y_2 \right) \frac{\partial}{\partial y_2}.$$

Respectively, the centralized system takes the form

$$\frac{dy_1}{dt} = -\varepsilon b_2 y_1^2; \quad \frac{dy_2}{dt} = \alpha y_2 - \varepsilon b_2 y_1 y_2 - \frac{\varepsilon^2 kb_1^2}{\alpha} y_1 y_2^2.$$

Apply Theorem 2.2 [1] to the centralized system of the second approximation (45)

$$y = \exp \mathcal{A}(t - t_0)\, \bar{y}, \quad \bar{y} = \operatorname{col} \| \bar{y}_1, \bar{y}_2 \|. \tag{46}$$

The vector $\bar{y}$ is obtained from the system of equations

$$\frac{d\bar{y}_1}{d\tau} = -b_2 \bar{y}_1^2; \quad \frac{d\bar{y}_2}{d\tau} = (-b_2 + \varepsilon k b_1^2 / \alpha)\, \bar{y}_1 \bar{y}_2, \quad \tau = \varepsilon t.$$

The written system can be easily integrated. Hence, the final solution (46) can be represented in the form

$$y_1 = \frac{e^{\alpha(t - t_0)}}{c_1 + b_2 \varepsilon t}; \quad y_2 = \frac{c_2}{(c_1 + b_2 \varepsilon t)^{\varepsilon k b_1^2 / \alpha b_2 - 1}}.$$

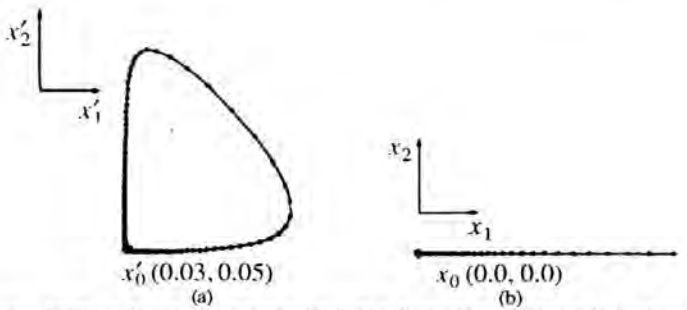

Fig. 1 (a). Solution in the phase plane for Lotka–Volterra system $\varepsilon = 0.1$. (b). Solution in the phase plane for a centralized system in the first approximation $\varepsilon = 0.1$ (dot curve).

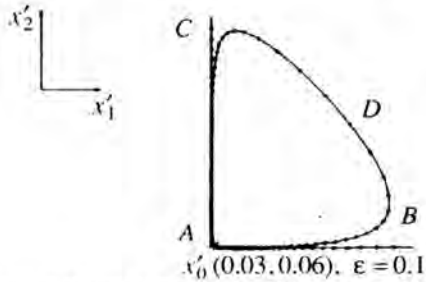

Fig. 2. Approximation of the solution of Lotka–Volterra system in the phase plane by the solution of centralized system in the first approximation, which is enumerated to initial coordinates, $\varepsilon = 0.1$

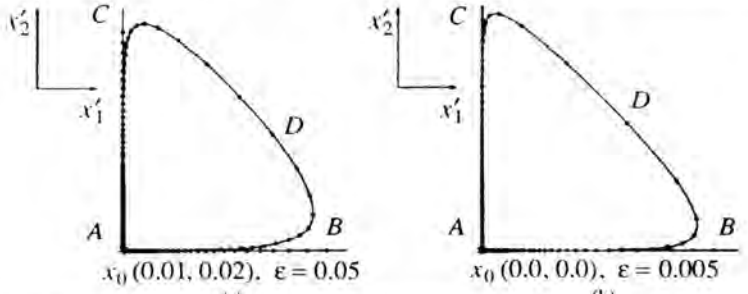

Fig. 3. Approximation of the solution of Lotka-Volterra system in the phase plane by the solution of centralized system in the first approximation, which is enumerated to initial coordinates (dot curves

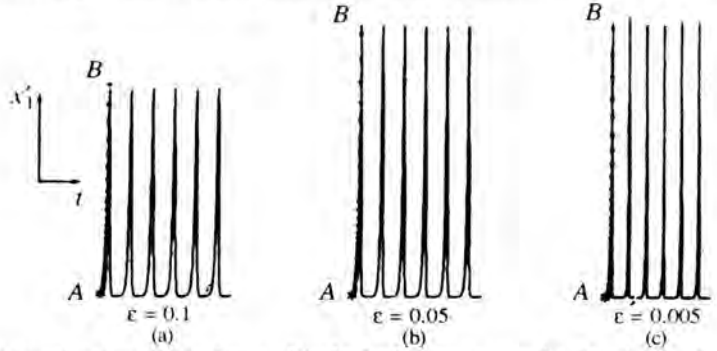

Fig. 4. Approximation of the solution of the Lotka–Volterra system by the solution of a centralized system in the first approximation (time dependence).

The second case. Neither one of the resonance relations is fulfilled. This case holds when the inequalities $\alpha \neq m$, $2\alpha \neq m$, $\alpha \neq 2m$, $3\alpha \neq m$, $4\alpha \neq m$, $3\alpha \neq 2m$, $2\alpha \neq 3m$, $\alpha \neq 4m$ are valid. The centralized system in the second approximation is linear,

$$\frac{dx_1}{dt} = \alpha x_1; \quad \frac{dx_2}{dt} = -mx_2.$$

The preceding analysis is illustrated in Figures 1 – 4. The solution of the Lotka–Volterra system in the phase plane obtained from system (29) for $m = 1$, $\alpha = 2$, $b_1 = k = 1$, $b_2 = 0$, $\varepsilon = 0, 1$

$$\dot{x}_1' = 2x_1' - \varepsilon x_1' x_2', \quad \dot{x}_2' = -x_2' + \varepsilon x_1' x_2' \tag{47}$$

is given in Fig. 1a. The solution in the phase plane corresponding to the centralized system (38) in the first approximation

$$\dot{x}_1 = 2x_1, \quad \dot{x}_2 = -x_2. \tag{48}$$

which is linear, is given in Fig. 1b. The initial system is transformed into the centralized system under the transformation (see (36))

$$x_1' = x_1 + \varepsilon x_1 x_2, \quad x_2' = x_2 + (\varepsilon/2) x_1 x_2 \tag{49}$$

and under the corresponding inverse transformation

$$x_1 = x_1' - \varepsilon x_1' x_2', \quad x_2 = x_2' - (\varepsilon/2) x_1' x_2 x_2'. \tag{50}$$

Both transformations are taken up to ε. The initial conditions of system (47), while transforming to system (48), were calculated by (50).

Approximation of solutions of system (47) in the phase plane, which correspond to solutions of system (48), is given in Fig. 2. Curve AB corresponding to the time interval $[0, T]$, is obtained from the points of the phase plane in Fig. 1b of system (48) under the transformation (49). Curve AC corresponds to the time interval $[0, -T]$.

The structure of Fig. 3 is identical to that of Fig. 2, but the values of the parameter ε are different. The degree of exactness of the described approximation can be evaluated in the following way: If we take T as the time of one cycle passing in the phase plane for the Lotka–Volterra system (see Fig. 1a), then the ratio of the length of curves CDB and T in Figs. 2, 3a, and 3b is the following:

$$CDB/T = 0.2, \quad CDB/T = 0.1, \quad CDB/T = 0.04.$$

Thus, for $\varepsilon \to 0$, the weakly approximated part of curve CDB also tends to zero.

The time dependence of the coordinate x_1' for the Lotka–Volterra system as well as its approximation by the solutions of the centralized system (48) (curve AB) in the first approximation are given in Figs. 4a, 4b, and 4c. It is more illuminating to present different scales in Fig. 4.

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