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ON SCALAR-TYPE SPECTRAL OPERATORS AND CARLEMAN ULTRADIFFERENTIABLE C_0-SEMIGROUPS

ПРО СПЕКТРАЛЬНІ ОПЕРАТОРИ СКАЛЯРНОГО ТИПУ ТА УЛЬТРАДИФЕРЕНЦІЙОВНІ C_0-НАПІВГРУПИ КАРЛЕМАНА

Necessary and sufficient conditions for a scalar-type spectral operator in a Banach space to be a generator of a Carleman ultradifferentiable C_0-semigroup are found. The conditions are formulated exclusively in terms of the operator's spectrum.

Знайдено необхідні та достатні умови для того, щоб спектральний оператор скалярного типу в банаховому просторі породжував ультрадиференційовну C_0-напівгрупу Карлемана. Ці умови сформульовано виключно у термінах спектра оператора.

1. Introduction. This paper is a natural sequal to [1, 2], where criteria of a scalar-type spectral operator in a complex Banach space being a generator of a C_0-semigroup, an analytic C_0-semigroup, an *infinite differentiable*, or a *Gevrey ultradifferentiable* C_0-semigroup were found.

Here, we are to generalize the results of [2] concerning the *Gevrey ultradifferentiability* by obtaining necessary and sufficient conditions for a scalar-type spectral operator in a complex Banach space to be a generator of a *Carleman ultradifferentiable* C_0-semigroup.

It is to be noted that such conditions, as well as those of [1, 2], will be formulated exclusively in terms of the operator's spectrum, no restrictions on its *resolvent* behavior necessary. This fact appears to be distinctive for *scalar-type spectral operators* making the results significantly more transparant than in general case [3 – 7] (see also [8, 9]) and purely qualitative.

Similar results for a *normal operator* in a complex Hilbert space are discussed in a more general context in [10 – 13].

2. Preliminaries. ***2.1. Scalar-type spectral operators.*** Henceforth, unless specified otherwise, A is a *scalar-type spectral operator* in a complex Banach space X with a norm $\|\cdot\|$ and $E_A(\cdot)$ is its *spectral measure* (the *resolution of the identity*), the operator's spectrum $\sigma(A)$ being the *support* for the latter [14, 15].

Note that, in a Hilbert space, the *scalar-type spectral operators* are those similar to the *normal* ones [16].

For such operators, there has been developed an *operational calculus* for Borel measurable functions on $\mathbb{C}$ $(\sigma(A))$ [14, 15], $F(\cdot)$ being such a function, a new *scalar-type spectral operator*

$$F(A) = \int\limits_{\mathbb{C}} F(\lambda)\, dE_A(\lambda) = \int\limits_{\sigma(A)} F(\lambda)\, dE_A(\lambda) \tag{2.1}$$

is defined as follows:

$$F(A)f := \lim_{n\to\infty} F_n(A)f, \quad f \in D(F(A)),$$

$$D(F(A)) := \left\{ f \in X \;\middle|\; \lim_{n\to\infty} F_n(A)f \text{ exists} \right\}$$

($D(\cdot)$ is the *domain* of an operator), where

$$F_n(\cdot) := F(\cdot)\chi_{\{\lambda\in\sigma(A)\,|\,|F(\lambda)|\le n\}}(\cdot), \quad n = 1, 2, \ldots$$

($\chi_\alpha(\cdot)$ is the *characteristic function* of a set α), and

$$F_n(A) := \int\limits_{\sigma(A)} F_n(\lambda)\, dE_A(\lambda), \quad n = 1, 2, \ldots,$$

being the integrals of *bounded* Borel measurable functions on $\sigma(A)$, are *bounded scalar-type spectral operators* on X defined in the same manner as for *normal operators* (see, e.g., [17, 18]).

The properties of the *spectral measure*, $E_A(\cdot)$, and the *operational calculus* underlying the entire subsequent discourse are exhaustively delineated in [14, 15]. Let's just outline here a few facts that will be especially important for us.

Observe first that, due to its *strong countable additivity*, the spectral measure $E_A(\cdot)$ is *bounded*, i.e., there is an $M > 0$ such that, for any Borel set δ in $\mathbb{C}$ [19],

$$\|E_A(\delta)\| \le M. \tag{2.2}$$

Note that, in (2.2), the notation $\|\cdot\|$ was used to designate the norm in the space of bounded linear operators on X. We shall adhere to this rather common economy of symbols in what follows adopting the same notation for the norm in the dual space X^* as well.

As we saw [2, 20], for any $f \in X$ and $g^* \in X^*$ (X^* is the *dual space*), the total variation $v(f, g^*, \cdot)$ of the complex-valued measure $\langle E_A(\cdot)f, g^*\rangle$ ($\langle\cdot,\cdot\rangle$ is the is the *pairing* between the space X and its dual, X^*) is *bounded*. Indeed,

$$v(f, g^*, \sigma(A)) \le 4M\|f\|\|g^*\|. \tag{2.3}$$

Also [2, 20], $F(\cdot)$ being an arbitrary Borel measurable function on $\mathbb{C}$ ($\sigma(A)$), for any $f \in D(F(A))$, $g^* \in X^*$ and arbitrary Borel sets $\delta \subseteq \sigma$,

$$\int\limits_{\sigma} |F(\lambda)|\, dv(f, g^*, \lambda) \le 4M\|E_A(\sigma)\|\|F(A)f\|\|g^*\|. \tag{2.4}$$

In particular,

$$\int\limits_{\sigma(A)} |F(\lambda)|\, dv(f, g^*, \lambda) \le 4M\|F(A)f\|\|g^*\|. \tag{2.5}$$

Observe also that, as follows from [1, 21, 22], if a scalar-type spectral operator A generates a C_0-semigroup, it's of the form $\left\{e^{tA} \;\middle|\; t \ge 0\right\}$, where the operator exponentials are defined in accordance with the *operational calculus* (2.1).

On account of compactness, the terms *spectral measure* and *operational calculus* for scalar-type spectral operators, frequently referred to, will be abbreviated to *s.m.* and *o.c.*, respectively.

2.2. Carleman ultradifferentiability. Let X be a Banach space with a norm $\|\cdot\|$, I be an interval of the real axis, $C^\infty(I,X)$ be the set of all X-valued functions strongly infinite differentiable on I, and $\{m_n\}_{n=0}^\infty$ be a sequence of positive numbers.

The sets

$$C_{\{m_n\}}(I,X) \stackrel{\text{df}}{=} \Big\{ g(\cdot) \in C^\infty(I,X) \,\big|\, \forall [a,b] \subseteq I \quad \exists \alpha > 0 \quad \exists c > 0 \colon$$

$$\max_{a \le t \le b} \|g^{(n)}(t)\| \le c\alpha^n m_n, \quad n = 0,1,2,\ldots \Big\}$$

and

$$C_{(m_n)}(I,X) \stackrel{\text{df}}{=} \Big\{ g(\cdot) \in C^\infty(I,X) \,\big|\, \forall [a,b] \subseteq I \quad \forall \alpha > 0 \quad \exists c > 0 \colon$$

$$\max_{a \le t \le b} \|g^{(n)}(t)\| \le c\alpha^n m_n, \quad n = 0,1,2,\ldots \Big\}$$

are called the *Carleman classes* of *strongly ultradifferentiable functions* corresponding to the sequence $\{m_n\}_{n=0}^\infty$ of *Roumieu's* and *Beurling's types*, respectively (for numeric functions, see [23–25]).

Obviously,

$$C_{(m_n)}(I,X) \subseteq C_{\{m_n\}}(I,X).$$

Observe that, for $m_n := [n!]^\beta$ or, due to *Stirling's formula*, $m_n := n^{\beta n}$, $n = 0,1,2,\ldots$, $0 \le \beta < \infty$, we obtain the well-known *Gevrey classes*, $\mathcal{E}^{\{\beta\}}(I,X)$ and $\mathcal{E}^{(\beta)}(I,X)$ (for numeric functions, see [26]). In particular, $\mathcal{E}^{\{1\}}(I,X)$ and $\mathcal{E}^{(1)}(I,X)$ are the classes of *real analytic* and *entire* vector functions, respectively.

2.3. Carleman classes of vectors. Let

$$C^\infty(A) \stackrel{\text{df}}{=} \bigcap_{n=0}^\infty D(A^n).$$

The vector sets

$$C_{\{m_n\}}(A) \stackrel{\text{df}}{=} \Big\{ f \in C^\infty(A) \,\big|\, \exists \alpha > 0 \quad \exists c > 0 \colon$$

$$\|A^n f\| \le c\alpha^n m_n, \quad n = 0,1,2,\ldots \Big\}$$

and

$$C_{(m_n)}(A) \stackrel{\text{df}}{=} \Big\{ f \in C^\infty(A) \,\big|\, \forall \alpha > 0 \quad \exists c > 0 \colon$$

$$\|A^n f\| \le c\alpha^n m_n, \quad n = 0,1,2,\ldots \Big\}$$

are called the *Carleman classes* of the operator A corresponding to the sequence $\{m_n\}_{n=0}^\infty$ of *Roumie's* and *Beurling's types*, respectively. Again

$$C_{(m_n)}(A) \subseteq C_{\{m_n\}}(A). \tag{2.6}$$

For $m_n := [n!]^\beta$ or $m_n := n^{\beta n}$, $n = 0, 1, 2, \ldots$, $0 \le \beta < \infty$, the above are the *Gevrey classes* of the operator A, $\mathcal{E}^{\{\beta\}}(A)$ and $\mathcal{E}^{(\beta)}(A)$ (see, e.g., [27–29]). In particular, $\mathcal{E}^{\{1\}}(A)$ and $\mathcal{E}^{(1)}(A)$ are the celebrated classes of *analytic* and *entire* vectors, respectively [30, 31].

3. The sequence $\{m_n\}_{n=0}^\infty$. The sequence $\{m_n\}_{n=0}^\infty$ being subject to the condition

(WGR) for any $\alpha > 0$, there exist such a $c = c(\alpha) > 0$ that

$$c\alpha^n \le m_n, \quad n = 0, 1, 2, \ldots ,$$

the scalar function

$$T(\lambda) := m_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{m_n}, \quad 0 \le \lambda < \infty, \quad 0^0 := 1, \tag{3.1}$$

first introduced by S. Mandelbrojt [25] is well-defined (see also [29]).

The function $T(\cdot)$ is, evidently, *continuous*, *strictly increasing* and $T(0) = 1$. Whenever the function $T(\cdot)$ is well defined, so is

$$M(\lambda) := \ln T(\lambda), \quad 0 \le \lambda < \infty. \tag{3.2}$$

The latter is also *continuous*, *strictly increasing* and $M(0) = 0$. Thus, it has an *inverse* $M^{-1}(\cdot)$ defined on $[0, \infty)$ and inheriting all the aforementioned properties of $M(\cdot)$.

According to [20], for a *scalar-type spectral operator* A in a complex Banach space X and $0 < \beta < \infty$, we have

$$\begin{aligned} C_{\{m_n\}}(A) &\supseteq \bigcup_{t>0} D(T(t|A|)), \\ C_{(m_n)}(A) &\supseteq \bigcap_{t>0} D(T(t|A|)), \end{aligned} \tag{3.3}$$

the function $T(\cdot)$ being replaceable by any *nonnegative*, *continuous*, and *increasing* function $L(\cdot)$ defined on $[0, \infty)$ such that

$$c_1 L(\gamma_1 \lambda) \le T(\lambda) \le c_2 L(\gamma_2 \lambda), \quad \lambda > R,$$

with some positive γ_1, γ_2, c_1, c_2, and a nonnegative R.

In particular, $T(\cdot)$ in (3.3) is replaceable by [29]

$$S(\lambda) := m_0 \sup_{n \ge 0} \frac{\lambda^n}{m_n}, \quad 0 \le \lambda < \infty,$$

or

$$P(\lambda) := m_0 \left[\sum_{n=0}^{\infty} \frac{\lambda^{2n}}{m_n^2} \right]^{1/2}, \quad 0 \le \lambda < \infty.$$

Observe that inclusions (3.3) turn into equalities provided the space X is *reflexive* [20].

The positive sequence $\{m_n\}_{n=0}^\infty$ will be subject to the following conditions:

(GR) for some $\alpha > 0$ and $c > 0$,

$$c\alpha^n n! \le m_n, \quad n = 0, 1, 2, \ldots;$$

(SBC) for some $l > 0$, $L > 0$ and $h > 1$, $H > 1$,

$$lh^n \le \sum_{k=0}^{n} \frac{m_n}{m_k m_{n-k}} \le LH^n, \quad n = 0, 1, 2, \ldots .$$

Obviously, condition (GR) is stronger than (WGR) and condition (SBC) resembles the fundemental property of the *binomial coefficients*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n, \quad n = 0, 1, 2, \ldots .$$

Actually, when $m_n = n!$, $n = 0, 1, 2, \ldots$, we positively arrive at the latter.

Observe also that there are sequences of positive numbers satisfying both (GR) and (SBC), e.g., $m_n = [n!]^\beta$, $n = 0, 1, 2, \ldots$, $1 \le \beta < \infty$.

As is easily seen, the sequence $m_n := \sqrt{n!}$, $n = 0, 1, 2, \ldots$, satisfies condition (SBC), but doesn't meet condition (GR).

We leave it to the reader to make sure that the sequence

$$m_n := \begin{cases} n^{2n} & \text{for } n = n(k), \\ e^{n^4} & \text{otherwise}, \end{cases}$$

where $n(0) := 1$, $n(1) := 2$, $n(k) := n(k-2) + n(k-1) + 1$, $k = 2, 3, \ldots$, satisfies condition (GR) but not (SBC).

Thus, conditions (GR) and (SBC) are independent.

Now, let's see what conditions (GR) and (SBC) imply for the function $M(\cdot)$ (3.2).

By condition (GR), for a certain $\alpha > 0$ and a certain $c > 0$,

$$T(\lambda) = m_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{m_n} \le m_0 c^{-1} \sum_{n=0}^{\infty} \frac{(\alpha^{-1}\lambda)^n}{n!} = m_0 c^{-1} e^{\alpha^{-1}\lambda}, \quad 0 \le \lambda < \infty.$$

Whence

$$M(\lambda) \le \ln(m_0 c^{-1}) + \alpha^{-1}\lambda, \quad 0 \le \lambda < \infty.$$

Therefore, there is such an $R = R(\alpha, c) > 0$ that

$$M(\lambda) \le 2\alpha^{-1}\lambda, \quad R \le \lambda < \infty.$$

Substituting $M^{-1}(\lambda)$ for λ, we arrive at the following estimate:

$$2\alpha^{-1} M^{-1}(\lambda) \ge \lambda, \quad M(R) \le \lambda < \infty, \tag{3.4}$$

with some $\alpha > 0$ and $R > 0$.

Condition (SBC) implies that with some $h > 1$ and $l > 0$

$$T^2(\lambda) =$$

Cauchy's product of series

$$= m_0^2 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{m_k m_{n-k}} \lambda^n \geq m_0^2 l \sum_{n=0}^{\infty} \frac{(h\lambda)^n}{m_n} = m_0 l T(h\lambda), \quad 0 \leq \lambda < \infty.$$

Whence

$$M(\lambda) \geq 2^{-1} M(h\lambda) + 2^{-1} \ln(m_0 l), \quad 0 \leq r < \infty.$$

Iductively, we infer that, for certain $h > 1$ and $l > 0$ and any natural n,

$$M(\lambda) \geq 2^{-n} M(h^n \lambda) + \left[\sum_{k=1}^{n} 2^{-k}\right] \ln(m_0 l) =$$

$$= 2^{-n} M(h^n \lambda) + [1 - 2^{-n}] \ln(m_0 l), \quad 0 \leq \lambda < \infty. \tag{3.5}$$

Analogously, condition (SBC) implies that, along with (3.5) the function $M(\cdot)$ satisfies the following estimate:

$$M(\lambda) \leq 2^{-n} M(H^n \lambda) + [1 - 2^{-n}] \ln(m_0 L), \quad 0 \leq \lambda < \infty. \tag{3.6}$$

4. Ultradifferentiability of an orbit. Let A be a scalar-type spectral operator generating a C_0-semigroup $\left\{e^{tA} \mid t \geq 0\right\}$.

Proposition 4.1. *Let I be a subinterval of $[0, \infty)$ and $\left\{m_n\right\}_{n=0}^{\infty}$ be a sequence of positive numbers. Then the restriction of an orbit $e^{tA} f$, $0 \leq t < \infty$, $f \in X$, to I belongs to $C_{\{m_n\}}(I, X)$ ($C_{(m_n)}(I, X)$) if and only if*

$$e^{tA} f \in C_{\{m_n\}}(A) \quad (C_{(m_n)}(A), \textit{respectively}) \quad \textit{for any} \quad t \in I.$$

Proof. *"Only if" part.* Assume that the restriction of an orbit $e^{tA} f$, $0 \leq t < \infty$, $f \in X$, to a subinterval I of $[0, \infty)$ belongs to $C^\infty(I, X)$.

Then by [2] the restriction of $e^{tA} f$, $0 \leq t < \infty$, $f \in X$, to I is strongly infinite differentiable on I, i.e., $e^{tA} \in C^\infty(I, X)$ and, for any natural n,

$$\frac{d^n}{dt^n} e^{tA} f = A^n e^{tA} f, \quad t \in I.$$

Furthermore, the fact that the restriction of $e^{tA} f$, $0 \leq t < \infty$, $f \in X$, to I belongs to the class $C_{\{m_n\}}(I, X)$ ($C_{(m_n)}(I, X)$) implies that, for an arbitrary $t \in I$, a certain (any) $\alpha > 0$, and a certain $c > 0$:

$$\|A^n e^{tA} f\| = \|\frac{d^n}{dt^n} e^{tA} f\| \leq c\alpha^n m_n, \quad n = 0, 1, \ldots.$$

Therefore,

$$e^{tA} f \in C_{\{m_n\}}(A) \ (C_{(m_n)}(A)), \quad t \in I.$$

"If" part. Let an orbit $e^{tA} f$, $0 \leq t < \infty$, $f \in X$, be such that

$$e^{tA} f \in C_{\{m_n\}}(A) \quad (C_{(m_n)}(A)), \quad t \in I,$$

where I is a subinterval of $[0, \infty)$.

Hence, for arbitrary $t \in I$ and some (any) $\alpha(t) > 0$, there is such a $c(t,\alpha) > 0$ that

$$\|A^n e^{tA} f\| \le c(t,\alpha)\alpha(t)^n m_n, \quad n = 0,1,2,\ldots. \tag{4.1}$$

The inclusions

$$C_{(m_n)}(A) \subseteq C_{\{m_n\}}(A) \subseteq C^\infty(A)$$

imply, by [2], that

$$e^{tA} f \in C^\infty(A) \quad \text{and} \quad \frac{d^n}{dt^n} e^{tA} f = A^n e^{tA} f, \quad n = 1,2,\ldots, \quad t \in I. \tag{4.2}$$

Let us fix an arbitrary subsegment $[a,b] \subseteq I$. For $n = 0,1,\ldots$, we have

$$\max_{a \le t \le b} \left\| \frac{d^n}{dt^n} e^{tA} f \right\| =$$

by (4.2);

$$= \max_{a \le t \le b} \|A^n e^{tA} f\| =$$

by the properties of the *o.c.* and the *Hahn – Banach Theorem*;

$$= \max_{a \le t \le b} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \left| \left\langle \int\limits_{\sigma(A)} \lambda^n e^{t\lambda}\, dE_A(\lambda) f, g^* \right\rangle \right| \le$$

by the properties of the *o.c.*;

$$\le \max_{a \le t \le b} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \left| \int\limits_{\sigma(A)} \lambda^n e^{t\lambda}\, d\langle E_A(\lambda) f, g^* \rangle \right| \le$$

$$\le \max_{a \le t \le b} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int\limits_{\sigma(A)} |\lambda|^n e^{t \operatorname{Re} \lambda}\, dv(f, g^*, \lambda) =$$

$$= \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \max_{a \le t \le b} \Bigg[\int\limits_{\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \le 0\}} |\lambda|^n e^{t \operatorname{Re} \lambda}\, dv(f, g^*, \lambda) +$$

$$+ \int\limits_{\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda > 0\}} |\lambda|^n e^{t \operatorname{Re} \lambda}\, dv(f, g^*, \lambda) \Bigg] \le$$

$$\le \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int\limits_{\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \le 0\}} |\lambda|^n e^{a \operatorname{Re} \lambda}\, dv(f, g^*, \lambda) +$$

$$+ \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int\limits_{\{\lambda\in\sigma(A)\mid \operatorname{Re}\lambda>0\}} |\lambda|^n e^{b\operatorname{Re}\lambda}\, dv(f,g^*,\lambda) \le$$

$$\le \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int\limits_{\sigma(A)} |\lambda|^n e^{a\operatorname{Re}\lambda}\, dv(f,g^*,\lambda)+$$

$$+ \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int\limits_{\sigma(A)} |\lambda|^n e^{b\operatorname{Re}\lambda}\, dv(f,g^*,\lambda) \le$$

by (2.5);

$$\le \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} 4M\|A^n e^{aA} f\|\|g^*\| +$$

$$+ \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} 4M\|A^n e^{bA} f\|\|g^*\| =$$

$$= 4M[\|A^n e^{aA} f\| + \|A^n e^{bA} f\|] \le$$

by (4.1);

$$\le 4M[c(a,\alpha)+c(b,\alpha)]\max[\alpha(a),\alpha(b)]^n m_n, \quad n=0,1,2,\ldots.$$

This implies that the restriction of $e^{tA}f$, $0 \le t < \infty$, $f \in X$, to the subinterval $I \subseteq [0,T)$ belongs to the Carleman class $C_{\{m_n\}}(I,X)$ ($C^{(m_n)}(I,X)$).

The proposition is proved.

5. Carleman ultradifferentiable C_0-semigroups. Let $\{m_n\}_{n=0}^\infty$ be a sequence of positive numbers. We shall call a C_0-semigroup $\{S(t) \mid t \ge 0\}$ in a Banach space X a $C_{\{m_n\}}$-semigroup (a $C_{(m_n)}$-semigroup) if, for any $f \in X$, the orbit $S(\cdot)f$ belongs to the Carleman class $C_{\{m_n\}}((0,\infty),X)$ ($C_{(m_n)}((0,\infty),X)$, respectively). We shall call a C_0-semigroup a *Carleman ultradifferentiable* semigroup if, for some positive sequence $\{m_n\}_{n=1}^\infty$, it is a $C_{\{m_n\}}$-semigroup or, which, due to inclusions (2.6), is the same, a $C_{(m_n)}$-semigroup.

Theorem 5.1. *Let a scalar-type spectral operator A generate a C_0-semigroup and a sequence of positive numbers $\{m_n\}_{n=0}^\infty$ satisfy conditions* (GR) *and* (SBC). *Then the C_0-semigroup is a $C_{\{m_n\}}$-semigroup if and only if there are a real a and a positive b such that*

$$\operatorname{Re}\lambda \le a - bM(|\operatorname{Im}\lambda|), \quad \lambda\in\sigma(A).$$

Proof. *"If" part.* Consider an arbitrary orbit $e^{tA}f$, $0 \le 0 < \infty$, $f \in X$.

According to Proposition 4.1, we need to show that

$$e^{tA}f \in C_{\{m_n\}}(A), \quad 0<t<\infty.$$

In view of inclusions (3.3), it suffices to show that

$$e^{tA}f \in \bigcup_{s>0} D(T(s|A|)), \quad 0<t<\infty.$$

Let's fix an arbitrary $t > 0$. Let's also fix a sufficiently large natural N so that

$$2^{-N}\gamma \le t,$$

where $\gamma := \max(1, 2b^{-1})$, and set

$$s := H^{-N}[2\alpha^{-1} + 1]^{-1} > 0,$$

where α and H are some positive constants from estimates (3.4) and (3.6), respectively.

For any $g^* \in X^*$,

$$\int_{\sigma(A)} T(s|\lambda|)e^{t \operatorname{Re} \lambda}\, dv(f, g^*, \lambda) =$$

$$= \int_{\{\lambda\in\sigma(A) \mid \operatorname{Re} \lambda\le\min(-M(R),a)\}} T(s|\lambda|)e^{t \operatorname{Re} \lambda}\, dv(f, g^*, \lambda) +$$

$$+ \int_{\{\lambda\in\sigma(A) \mid \min(-M(R),a)<\operatorname{Re} \lambda\le a\}} T(s|\lambda|)e^{t \operatorname{Re} \lambda}\, dv(f, g^*, \lambda) < \infty,$$

where R is a positive constant from (3.4).

Indeed, the latter integral is finite due to the boundedness of the set $\{\lambda \in \sigma(A) \mid \min(-M(R), a) < \operatorname{Re} \lambda \le a\}$ (note that, for $a \le -M(R)$, the set is, obviously, empty), the continuity of the integrated function, and the finiteness of the positive measure $v(f, g^*, \cdot)$ (see (2.3)).

For the former of the two above integrals, we have

$$\int_{\{\lambda\in\sigma(A) \mid \operatorname{Re} \lambda\le\min(-M(R),a)\}} T(s|\lambda|)e^{t \operatorname{Re} \lambda}\, dv(f, g^*, \lambda) \le$$

$$\text{for } \lambda \in \sigma(A),\ \operatorname{Re} \lambda \le \min(-M(R), a)$$

$$\operatorname{Re} \lambda \le -M(R) \text{ and } |\operatorname{Im} \lambda| \le M^{-1}[b^{-1}(a - \operatorname{Re} \lambda)];$$

$$\le \int_{\{\lambda\in\sigma(A) \mid \operatorname{Re} \lambda\le\min(-M(R),a)\}} e^{M(s[-\operatorname{Re} \lambda+M^{-1}(b^{-1}(a-\operatorname{Re} \lambda))])} e^{t \operatorname{Re} \lambda}\, dv(f, g^*, \lambda).$$

Let's consider separately the two posible cases: $a \le 0$ and $a > 0$.

If $a \le 0$, then $a - \operatorname{Re} \lambda \le -2 \operatorname{Re} \lambda$ for all λ's such that $\operatorname{Re} \lambda \le \min(-M(R), a)$, and we have

$$\int_{\{\lambda\in\sigma(A) \mid \operatorname{Re} \lambda\le\min(-R,a)\}} e^{M(s[-\operatorname{Re} \lambda+M^{-1}(b^{-1}(a-\operatorname{Re} \lambda))])} e^{t \operatorname{Re} \lambda}\, dv(f, g^*, \lambda) \le$$

$$\le \int_{\{\lambda\in\sigma(A) \mid \operatorname{Re} \lambda\le\min(-M(R),a)\}} e^{M(s[-\operatorname{Re} \lambda+M^{-1}(2b^{-1}[-\operatorname{Re} \lambda])])} e^{t \operatorname{Re} \lambda}\, dv(f, g^*, \lambda) \le$$

$$\text{by (3.4);}$$

$$\leq \int\limits_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda\leq\min(-M(R),a)\}} e^{M(s[2\alpha^{-1}M^{-1}(-\,\mathrm{Re}\,\lambda)+M^{-1}(2b^{-1}[-\,\mathrm{Re}\,\lambda])])}\times$$

$$\times\, e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)=$$

by the choice: $\gamma=\max(1,2b^{-1})$;

$$=\int\limits_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda\leq\min(-M(R),a)\}} e^{M(s[2\alpha^{-1}+1]M^{-1}(\gamma[-\,\mathrm{Re}\,\lambda]))}e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)=$$

by (3.6);

$$M(s[2\alpha^{-1}+1]M^{-1}(\gamma[-\,\mathrm{Re}\,\lambda]))\leq$$

$$\leq 2^{-N}M(H^N s[2\alpha^{-1}+1]M^{-1}(\gamma[-\,\mathrm{Re}\,\lambda]))+[1-2^{-N}]\ln(m_0L)=$$

with some $H>1$ and $L>0$;

$$=(m_0L)^{[1-2^{-N}]}\int\limits_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda\leq\min(-M(R),a)\}} e^{2^{-N}M(H^N s[2\alpha^{-1}+1]M^{-1}(\gamma[-\,\mathrm{Re}\,\lambda])}\times$$

$$\times\, dv(f,g^*,\lambda)e^{t\,\mathrm{Re}\,\lambda}=$$

by the choice: $s=H^{-N}[2\alpha^{-1}+1]^{-1}>0$;

$$=(m_0L)^{[1-2^{-N}]}\int\limits_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda\leq\min(-M(R),a)\}} e^{[t-2^{-N}\gamma]\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)\leq$$

by the choice: $2^{-N}\gamma\leq t$;

$$\leq(m_0L)^{[1-2^{-N}]}\int\limits_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda\leq\min(-M(R),a)\}} 1\,dv(f,g^*,\lambda)\leq$$

$$\leq(m_0L)^{[1-2^{-N}]}v(f,g^*,\sigma(A))\leq$$

by (2.3);

$$\leq(m_0L)^{[1-2^{-N}]}4M\|f\|\|g^*\|<\infty. \quad (5.1)$$

If $a>0$,

$$\int\limits_{\{\lambda\in\sigma(A)\,|\,\mathrm{Re}\,\lambda\leq\min(-M(R),a)\}} e^{s|\lambda|^{1/\beta}}e^{t\,\mathrm{Re}\,\lambda}\,dv(f,g^*,\lambda)=$$

$$= \int\limits_{\{\lambda\in\sigma(A) \mid \operatorname{Re}\lambda\le\min(-M(R),-a)\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}\, dv(f,g^*,\lambda) +$$

$$+ \int\limits_{\{\lambda\in\sigma(A) \mid \min(-M(R),-a)<\operatorname{Re}\lambda\le -M(R)\}} e^{s|\lambda|^{1/\beta}} e^{t\operatorname{Re}\lambda}\, dv(f,g^*,\lambda) < \infty.$$

Indeed, the latter integral is finite due to the boundedness of the set $\big\{\lambda \in \sigma(A) \ \big|\ \min(-a,-M(R)) < \operatorname{Re}\lambda \le -M(R)\big\}$ (note that, for $a \le M(R)$, the set is, obviously, empty), the continuity of the integrated function, and the finiteness of the positive measure $v(f,g^*,\cdot)$ (see (2.3)).

The former of the two above integrals is finite as well:

$$\int\limits_{\{\lambda\in\sigma(A) \mid \operatorname{Re}\lambda\le\min(-M(R),-a)\}} e^{M(s|\lambda|)} e^{t\operatorname{Re}\lambda}\, dv(f,g^*,\lambda)\times$$

$$\times \int\limits_{\{\lambda\in\sigma(A) \mid \operatorname{Re}\lambda\le\min(-M(R),-a)\}} e^{M(s[-\operatorname{Re}\lambda+M^{-1}(b^{-1}(a-\operatorname{Re}\lambda))])} e^{t\operatorname{Re}\lambda}\, dv(f,g^*,\lambda) \le$$

since, for $\operatorname{Re}\lambda \le -a$, $a - \operatorname{Re}\lambda \le -2\operatorname{Re}\lambda$;

$$\le \int\limits_{\{\lambda\in\sigma(A) \mid \operatorname{Re}\lambda\le\min(-M(R),-a)\}} e^{M(s[-\operatorname{Re}\lambda+M^{-1}(2b^{-1}[-\operatorname{Re}\lambda])])}\times$$

analogously to (5.1);

$$\times\, e^{t\operatorname{Re}\lambda}\, dv(f,g^*,\lambda) < \infty.$$

Thus, we have proved that, for an arbitrary Borel subset $\sigma(A) \subseteq \sigma(A)$, any $f \in X$ and $g^* \in X^*$,

$$\int\limits_{\sigma} (A) T(s|\lambda|) e^{t\operatorname{Re}\lambda}\, dv(f,g^*,\lambda) < \infty, \quad t>0, \tag{5.2}$$

with $s = s(t) = H^{-N}[2\alpha^{-1}+1]^{-1} > 0$.

Furthermore, for any $f \in X$, $g^* \in X^*$, $t>0$ and $s = s(t) = H^{-N}[2\alpha^{-1} + +1]^{-1} > 0$:

$$\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int\limits_{\{\lambda\in\sigma(A) | T(s|\lambda|)e^{t\operatorname{Re}\lambda}>n\}} T(s|\lambda|) e^{t\operatorname{Re}\lambda}\, dv(f,g^*,\lambda) \to$$

$$\to 0 \quad \text{as} \quad n\to\infty. \tag{5.3}$$

Indeed, as follows from the preceding arguments, the specific choice of

$$s = H^{-N}[2\alpha^{-1}+1]^{-1} > 0$$

allows to partition the set $\{\lambda \in \sigma(A) \mid T(s|\lambda|)e^{t\operatorname{Re}\lambda} > n\}$ into two Borel subsets σ_1 and σ_2 in such a way that σ_1 is bounded and

$$T(s|\lambda|)e^{t\operatorname{Re}\lambda} \le 1, \quad \lambda \in \sigma_2.$$

Therefore,

$$\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int\limits_{\{\lambda\in\sigma(A)|T(s|\lambda|)e^{t\operatorname{Re}\lambda}>n\}} T(s|\lambda|)e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda) \le$$

$$\le \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int\limits_{\{\lambda\in\sigma_1|T(s|\lambda|)e^{t\operatorname{Re}\lambda}>n\}} T(s|\lambda|)e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda) +$$

$$+ \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int\limits_{\{\lambda\in\sigma_2|T(s|\lambda|)e^{t\operatorname{Re}\lambda}>n\}} T(s|\lambda|)e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda) \le$$

since σ_1 is bounded, there is such a $C > 0$ that

$$T(s|\lambda|)e^{t\operatorname{Re}\lambda} \le C, \quad \lambda \in \sigma_1;$$

by (2.4);

$$\le \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} C4M\|E_A(\{\lambda\in\sigma_1|T(s|\lambda|)e^{t\operatorname{Re}\lambda}>n\})f\|\|g^*\| +$$

$$+ \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M\|E_A(\{\lambda\in\sigma_2|T(s|\lambda|)e^{t\operatorname{Re}\lambda}>n\})f\|\|g^*\| =$$

$$= 4CM\|E_A(\{\lambda\in\sigma_1|T(s|\lambda|)e^{t\operatorname{Re}\lambda}>n\})f\| +$$

$$+ 4M\|E_A(\{\lambda\in\sigma_2|T(s|\lambda|)e^{t\operatorname{Re}\lambda}>n\})f\| \to$$

by the strong continuity of the *s.m.*;

$$\to 0 \quad \text{as} \quad n \to \infty.$$

According to [22], Proposition 3.1, (5.2) and (5.3) imply that, for any $f \in X$ and $t > 0$,

$$e^{tA}f \in D(T(s|A|)),$$

where $s = s(t) = H^{-N}[2\alpha^{-1}+1]^{-1} > 0$.

Hence, for any $f \in X$,

$$e^{tA}f \in \bigcup_{s>0} D(T(s|A|)) \subseteq C_{\{m_n\}}(A), \quad 0 < t < \infty.$$

"Only if" part. Let's prove this part *by contrapositive*, i.e., we assume that for any real a and positive b,

$$\sigma(A) \setminus \big\{\lambda \in \mathbb{C} \;\big|\; \mathrm{Re}\,\lambda \le a - bM(|\mathrm{Im}\,\lambda|)\big\} \ne \varnothing.$$

Therefore, for any natural n,

$$\sigma(A) \setminus \left\{\lambda \in \mathbb{C} \;\middle|\; \mathrm{Re}\,\lambda \le -\frac{1}{n}M(|\mathrm{Im}\,\lambda|)\right\}$$

is *unbounded*.

Hence, one can choose a sequence of points in the complex plane $\{\lambda_n\}_{n=1}^{\infty}$ in the following way:

$$\lambda_n \in \sigma(A), \quad n = 1, 2, \ldots,$$

$$\mathrm{Re}\,\lambda_n > -\frac{1}{n}M(|\mathrm{Im}\,\lambda|), \quad n = 1, 2, \ldots,$$

$$\lambda_0 := 0, \; |\lambda_n| > \max\big[n, |\lambda_{n-1}|\big], \quad n = 1, 2, \ldots.$$

The latter, in particular, implies that the points λ_n are *distinct*:

$$\lambda_i \ne \lambda_j, \quad i \ne j.$$

Since the set

$$\left\{\lambda \in \mathbb{C} \;\middle|\; \mathrm{Re}\,\lambda > -\frac{1}{n}M(|\mathrm{Im}\,\lambda|), \;\; |\lambda| > \max\big[n, |\lambda_{n-1}|\big]\right\}$$

is *open* in $\mathbb{C}$ for any $n = 1, 2, \ldots$, there exists such an $\varepsilon_n > 0$ that this set contains together with the point λ_n the *open disk* centered at λ_n:

$$\Delta_n = \big\{\lambda \in \mathbb{C} \;\big|\; |\lambda - \lambda_n| < \varepsilon_n\big\},$$

i.e., for any $\lambda \in \Delta_n$:

$$\begin{gathered}\mathrm{Re}\,\lambda > -\frac{1}{n}M(|\mathrm{Im}\,\lambda|),\\ |\lambda| > \max\big[n, |\lambda_{n-1}|\big].\end{gathered} \tag{5.4}$$

Moreover, since the points λ_n are *distinct*, we can regard that the radii of the disks, ε_n, are chosen to be small enough so that

$$0 < \varepsilon_n < \frac{1}{n}, \quad n = 1, 2, \ldots, \tag{5.5}$$

and

$$\Delta_i \cap \Delta_j = \varnothing, \quad i \ne j \quad \text{(the disks are \textit{pairwise disjoint})}.$$

Note that, by the properties of the *s.m.*, the latter implies that the subspaces $E_A(\Delta_n)X$, $n = 1, 2, \ldots$, are *nontrivial* since $\Delta_n \cap \sigma(A) \ne \varnothing$ and Δ_n is *open* and

$$E_A(\Delta_i)E_A(\Delta_j) = 0, \quad i \neq j. \tag{5.6}$$

We can choose a unit vector e_n in each subspace $E_A(\Delta_n)X$ ($\|e_n\| = 1$) and thereby obtain a vector sequence such that

$$E_A(\Delta_i)e_j = \delta_{ij}e_i$$

(δ_{ij} is *Kronecker's delta symbol*).

The latter, in particular, implies that, the vectors $\{e_1, e_2, \dots\}$ are linearly independent. Moreover, there is an $\varepsilon > 0$ such that

$$d_n := \operatorname{dist}\left(e_n, \operatorname{span}(\{e_i \,|\, i \in \mathbb{N},\ i \neq n\})\right) \geq \varepsilon, \quad n = 1, 2, \dots. \tag{5.7}$$

Otherwise there is a subsequence $\left\{d_{n(k)}\right\}_{k=1}^{\infty}$ such that $d_{n(k)} \to 0$ as $k \to \infty$. Hence, for any $k = 1, 2, \dots$, we can find an $f_{n(k)} \in \operatorname{span}(\{e_i | i \in \mathbb{N},\ i \neq n\})$ such that $\|e_{n(k)} - f_{n(k)}\| < d_{n(k)} + 1/n(k)$, which immediately implies that

$$e_{n(k)} = E_A(\Delta_{n(k)})(e_{n(k)} - f_{n(k)}) \to 0 \quad \text{as} \quad k \to \infty.$$

Thus, the assumption that (5.7) doesn't hold leads to a contradiction.

As follows from the *Hahn–Banach Theorem*, for each $n = 1, 2, \dots$, there is a $e_n^* \in X^*$ such that

$$\|e_n^*\| = 1,$$

$$\langle e_i, e_j^* \rangle = \delta_{ij} d_i.$$

Let

$$g^* := \sum_{n=1}^{\infty} \frac{1}{n^2} e_n^*.$$

Hence,

$$\langle e_n, g^* \rangle = \frac{d_n}{n^2} \geq$$

by (5.7);

$$\geq \frac{\varepsilon}{n^2}. \tag{5.8}$$

Concerning the sequence of the real parts, $\{\operatorname{Re} \lambda_n\}_{n=1}^{\infty}$, there are two possibilities: it is either *bounded*, or not. Let's consider separately each of them.

First, assume that the sequence $\{\operatorname{Re} \lambda_n\}_{n=1}^{\infty}$ is *bounded*, i.e., there is such an $\omega > 0$ that

$$|\operatorname{Re} \lambda_n| \leq \omega, \quad n = 1, 2, \dots. \tag{5.9}$$

Let

$$f := \sum_{n=1}^{\infty} \frac{1}{n^2} e_n.$$

As can be easily deduced from the (5.6),

$$E_A(\Delta_n)f = \frac{1}{n^2}e_n, \quad n = 1, 2, \ldots,$$
$$E_A(\cup_{n=1}^{\infty}\Delta_n)f = f. \tag{5.10}$$

Also, for $n = 1, 2, \ldots,$

$$v(f, g^*, \Delta_n) \geq |\langle E_A(\Delta_n)f, g^*\rangle| =$$

by (5.10);

$$= \left|\left\langle \frac{1}{n^2}e_n, g^* \right\rangle\right| =$$

by (5.8);

$$= \frac{d_n}{n^4} \geq \frac{\varepsilon}{n^4}. \tag{5.11}$$

For an arbitrary $s > 0$, we have

$$\int\limits_{\sigma(A)} T(s|\lambda|)e^{\mathrm{Re}\,\lambda}\, dv(f, g^*, \lambda) =$$

by (5.10);

$$= \int\limits_{\sigma(A)} T(s|\lambda|)e^{\mathrm{Re}\,\lambda}\, dv(E_A(\cup_{n=1}^{\infty}\Delta_n)f, g^*, \lambda) =$$

by the properties of the *o.c.*;

$$= \int\limits_{\cup_{n=1}^{\infty}\Delta_n} T(s|\lambda|)e^{\mathrm{Re}\,\lambda}\, dv(f, g^*, \lambda) =$$

$$= \sum_{n=1}^{\infty} \int\limits_{\Delta_n} T(s|\lambda|)e^{\mathrm{Re}\,\lambda}\, dv(f, g^*, \lambda) \geq$$

for $\lambda \in \Delta_n$, by (5.4), (5.9), and (5.5): $|\lambda| \geq n$, and $\mathrm{Re}\,\lambda = \mathrm{Re}\,\lambda_n$

$$-(\mathrm{Re}\,\lambda_n - \mathrm{Re}\,\lambda) \geq \mathrm{Re}\,\lambda_n - |\lambda_n - \lambda| \geq -\omega - \varepsilon_n \geq -\omega - 1;$$

$$\geq \sum_{n=1}^{\infty} T(sn)e^{-(\omega+1)}v(f, g^*, \Delta_n) \geq$$

by (5.11);

$$\geq e^{-(\omega+1)} \sum_{n=1}^{\infty} \frac{\varepsilon T(sn)}{n^4} = \infty.$$

Indeed, by definition (3.1)

$$T(sn) \geq m_0 \frac{(sn)^4}{m_4}, \quad n = 1, 2, \ldots .$$

Thus, by [22], Proposition 3.1,

$$e^A f \notin \bigcup_{t>0} D(T(t|A|)).$$

Then, by (3.3),

$$e^A f \notin C_{\{m_n\}}(A).$$

Hence, according to Proposition 4.1, the orbit $e^{tA}f$, $t \geq 0$, does not belong to $C_{\{m_n\}}\big((0,\infty), X\big)$.

Now, suppose that the sequence $\{\text{Re}\ \lambda_n\}_{n=1}^{\infty}$ is *unbounded*. The sequence being *bounded above*, since A generates a C_0-semigroup [3] (see also [1]), this means there is a subsequence $\{\text{Re}\ \lambda_{n(k)}\}_{k=1}^{\infty}$ such that

$$\text{Re}\ \lambda_{n(k)} \leq -k, \quad k = 1, 2, \ldots . \tag{5.12}$$

Consider the vector

$$f := \sum_{k=1}^{\infty} \frac{1}{k^2} e_{n(k)}.$$

By (5.6),

$$E_A(\Delta_n(k))f = \frac{1}{k} e_{n(k)}, \quad k = 1, 2, \ldots ,$$

$$E_A(\cup_{k=1}^{\infty} \Delta_{n(k)})f = f.$$

For an arbitrary $s > 0$, we have similarly

$$\int\limits_{\sigma(A)} T(s|\lambda|)e^{\text{Re}\ \lambda}\, dv(f, g^*, \lambda) = \sum_{k=1}^{\infty} \int\limits_{\Delta_{n(k)}} T(s|\lambda|)e^{\text{Re}\ \lambda}\, dv(f, g^*, \lambda) = \infty.$$

Indeed, for all $\lambda \in \Delta_{n(k)}$, based on (5.5), (5.12), and (5.4), we have

$$\text{Re}\ \lambda = \text{Re}\ \lambda_{n(k)} - (\text{Re}\ \lambda_{n(k)} - \text{Re}\ \lambda) \leq \text{Re}\ \lambda_{n(k)} + |\lambda_{n(k)} - \lambda| \leq$$

$$\leq \text{Re}\ \lambda_{n(k)} + \varepsilon_{n(k)} \leq -k + 1 \leq 0$$

and

$$-\frac{1}{n(k)} M(|\text{ Im}\ \lambda|) < \text{Re}\ \lambda.$$

Therefore, for $\lambda \in \Delta_{n(k)}$:

$$-\frac{1}{n(k)} M(|\text{ Im}\ \lambda|) < \text{Re}\ \lambda \leq -k + 1 \leq 0.$$

Whence, for $\lambda \in \Delta_{n(k)}$,

$$\text{Re }\lambda \le -k+1 \le 0 \quad \text{and} \quad |\lambda| \ge |\text{ Im }\lambda| \ge M^{-1}(n(k)[-\text{ Re }\lambda]).$$

Using these estimates we have

$$\int\limits_{\Delta_{n(k)}} T(s|\lambda|)e^{\text{Re }\lambda}\,dv(f,g^*,\lambda) \ge \int\limits_{\Delta_{n(k)}} e^{M(s|\lambda|)}e^{\text{Re }\lambda}\,dv(f,g^*,\lambda) \ge$$

$$\ge \int\limits_{\Delta_{n(k)}} e^{M(sM^{-1}(n(k)[-\text{ Re }\lambda]))}e^{\text{Re }\lambda}\,dv(f,g^*,\lambda) \ge$$

by (3.5), $M(\lambda) \ge 2^{-n}M(h^n\lambda) + [1-2^{-n}]\ln(m_0 l)$, $\lambda \ge 0$, $n = 1, 2, \ldots;$

with some $h > 1$ and $l > 0$;

for a sufficiently large natural N so that $h^N s \le 1$;

$$\ge (m_0 l)^{[1-2^N]} \int\limits_{\Delta_{n(k)}} e^{2^{-N}M(h^N sM^{-1}(n(k)[-\text{ Re }\lambda]))}e^{\text{Re }\lambda}\,dv(f,g^*,\lambda) \ge$$

$$\ge (m_0 l)^{[1-2^N]} \int\limits_{\Delta_{n(k)}} e^{2^{-N}M(M^{-1}(n(k)[-\text{ Re }\lambda]))}e^{\text{Re }\lambda}\,dv(f,g^*,\lambda) \ge$$

$$\ge (m_0 l)^{[1-2^N]} \int\limits_{\Delta_{n(k)}} e^{(2^{-N}n(k)-1)[-\text{ Re }\lambda]}\,dv(f,g^*,\lambda) \ge$$

for all k's sufficiently large so that $2^{-N}n(k) - 1 > 0$ and $k - 1 \ge 1$;

$$\ge (m_0 l)^{[1-2^N]} e^{[2^{-N}n(k)-1](k-1)} v(f,g^*,\Delta_{n(k)}) \ge$$

by (5.11);

$$\ge (m_0 l)^{[1-2^N]} \frac{\varepsilon e^{2^{-N}n(k)-1}}{n(k)^4} \to \infty \quad \text{as} \quad k \to \infty.$$

Similarly to the above, we infer that the orbit $e^{tA}f$, $t \ge 0$, does not belong to the class $C_{\{m_n\}}\big((0,\infty),X\big)$.

Thus, all the possibilities concerning $\{\text{Re }\lambda_n\}_{n=1}^{\infty}$ analyzed, the "only if" part has been proved by *contrapositive*.

The theorem is proved.

In particular, for $m_n = [n!]^\beta$, $1 \le \beta < \infty$, we obtain Theorem 5.1 of [2].

As well as in (3.3), the function $T(\cdot)$ in Theorem 5.1 is replaceable by any *nonnegative*, *continuous*, and *increasing* function $L(\cdot)$ defined on $[0,\infty)$ such that

$$c_1 L(\gamma_1\lambda) \le T(\lambda) \le c_2 L(\gamma_2\lambda), \quad \lambda > R,$$

with some positive γ_1, γ_2, c_1, c_2, and a nonnegative R.

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