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FIRST EIGENVALUE OF THE LAPLACE OPERATOR AND MEAN CURVATURE*

ПЕРШЕ ВЛАСНЕ ЗНАЧЕННЯ ОПЕРАТОРА ЛАПЛАСА ТА СЕРЕДНЯ КРИВИНА

The main theorem of this paper states a relation between the first nonzero eigenvalue of Laplace operator and the squared norm of mean curvature in irreducible compact homogeneous manifolds under spatial conditions. This statement has some results that states in the remainder of paper.

Основна теорема цієї статті встановлює зв'язок між першим ненульовим власним значенням оператора Лапласа та нормою середньої кривини у квадраті у незвідних компактних однорідних многовидах під дією просторових умов. Одержано також деякі інші результати.

1. Introduction. Let M be an n-dimensional Riemannian manifold and $p \in M$. For an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of the tangent space T_pM, the scalar curvature S of p is defined to be $S(p) = \sum_{i<j} K(e_i, e_j)$, where $K(e_i, e_j)$ is sectional curvature of M associated with tangent plane generated by e_j and e_j at P. Let $S(L)$ be the scalar curvature of L, where L is a subspace of T_pM of dimension $r < n$. Thus, the scalar curvature $S(M)$ of M at p is nothing but the scalar curvature tangent space of M at p and if L is a 2-plane section, $S(L)$ is nothing but the sectional curvature of L.

For an integer $k \geq 0$, denote by $\gamma(n, k)$ the finite set consisting of k-tuples $(n_1, n_2, \ldots, n_k)$ of integers grater than 1 satisfying $n_1 \geq 2$ and $n_1 + \ldots + n_k \leq n$. Denote by $\gamma(n)$ the set of k-tuples with $k \geq 0$ for fixed n.

The cardinal number $\#\gamma(n)$ of $\gamma(n)$ increases quite rapidly with n. For each k-tuples $(n_1, n_2, \ldots, n_k)$, we define an invariant $\delta(n_1, \ldots, n_k)$ by

$$\delta(n_1, \ldots, n_k)(p) = S(p) - \inf(S(L_1) + \ldots + S(L_k)),$$

where $L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_i = n_i$, $i = 1, \ldots, k$. In particular, we have $\delta(\phi) = S$, $\delta(2) = S - \inf K$. The invariants $\delta(n_1, \ldots, n_k)$ with $k > 0$ and the scalar curvature S are very different in nature.

For each k-tuples $(n_1, n_2, \ldots, n_k)$ with in $\gamma(n)$, we define following two positive numbers that are used in the following:

$$c(n_1, n_2, \ldots, n_k) = \frac{n^2(n + k - 1 - \sum_j n_j)}{2(n + k - \sum_j n_j)},$$

$$b(n_1, n_2, \ldots, n_k) = \frac{1}{2}\left(n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)\right).$$

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Now an isometric immersion $f\colon M^n \to R^m(c)$ from a Riemannian n-manifold into a Riemannian space form of constant curvature c is called an ideal immersion if

$$\delta(n_1, n_2, \ldots, n_k) = c(n_1, \ldots, n_k)|H|^2 + b(n_1, \ldots, n_k)c$$

for some k-tuples $(n_1, n_2, \ldots, n_k)$ in $\gamma(n)$.

2. Main theorem. In this section, the main theorem is stated. This theorem demonstrates a relation between the first nonzero eigenvalue of Laplace operator and squared norm of mean curvature in a manifold with certain properties.

We define an ideal submanifold as a submanifold whose inclusion map is an ideal immersion.

In the following, suppose that the first nonzero eigenvalue of Laplacian operator of a manifold is denoted by λ_1 and the tensor of mean curvature is denoted by H.

Theorem 2.1. *Let M be an n-dimensional irreducible compact homogeneous Riemannian manifold that is also an ideal submanifold of a space form $R^m(c)$ with $c \geq 0$. Then*

$$\lambda_1 \geq n|H|^2,$$

where $|H|^2$ is the squared norm of the mean curvature of M.

Proof. Since M is an n-dimensional submanifold in a Riemannian space form of constant curvature c, by [1], at every point $p \in M$ and for each of k-tuples $(n_1, n_2, \ldots, n_k)$ in $\gamma(n)$, we have

$$\delta(n_1, n_2, \ldots, n_k) \leq c(n_1, n_2, \ldots, n_k)|H|^2 + b(n_1, n_2, \ldots, n_k)c.$$

Since M is an irreducible compact homogeneous Riemannian n-manifold, again by [1], for any of k-tuples $(n_1, n_2, \ldots, n_k)$ in $\gamma(n)$, λ_1 satisfies

$$\lambda_1 \geq n\Delta(n_1, \ldots, n_k),$$

where

$$\Delta(n_1, \ldots, n_k) = \frac{\delta(n_1, \ldots, n_k)}{c(n_1, \ldots, n_k)}.$$

Therefore, for a Riemannian n-manifold with an ideal immersion into a space form with nonnegative constant curvature, we have $\Delta(n_1, \ldots, n_k) \geq |H|^2$ for some k-tuples $(n_1, n_2, \ldots, n_k)$ in $\gamma(n)$. This enables us to have result.

From now on, let us denote an n-dimensional Riemannian irreducible compact and homogeneous manifold by *nRich*-manifold.

Other direct applications of this theorem can be found in the following corollaries:

***Corollary* 2.1.** *Let M be a nRich-manifold that is also an ideal submanifold of a space form $R^m(c)$ with $c \geq 0$. Then*

$$\int\limits_M |H|^2 dV \leq \frac{\lambda_1 V}{n}, \tag{2.1}$$

where V is the volume of M. If $c = 0$, we have an equality in (2.1).

Proof. The first part is clear by Theorem 2.1. For the second part, by the Reilly formula (see [2]), λ_1 has also the upper bound $\dfrac{n}{V(M)}\displaystyle\int_M |H|^2 dV$, so the equality is obtained.

The Remark 2.1 as follows is needed for the next corollary and other results in the rest.

***Remark* 2.1** [1]. Every totally umbilical submanifold of a real space form (Euclidean spaces, real hyperbolic spaces, spheres and real projective spaces) is an ideal immersion.

We also denote a k-dimensional Euclidean space and a k-dimensional Euclidean sphere by R^k and S^k, respectively.

***Corollary* 2.2.** *Let M be an irreducible compact homogeneous surface of unit sphere S^3 such that the length of its mean curvature is constant the H_0. If M is a topological sphere, then*

$$\lambda_1 \geq nH_0^2.$$

Proof. By [3], with this conditions M is totally umbilic, so by Remark 2.1, M is an ideal hypersurface of S^{n+1}. Thus, this result is a spacial case of Theorem 2.1.

Using Theorem 2.1, each of following results states a relation between the nonzero eigenvalue of Laplace operator and circumradius of certain submanifolds of R^{n+1} and S^{n+1}.

First, we need the definition of a circumradius for a manifold. For a given immersion $x\colon M^n \to R^{n+p}$ or S^{n+p}, where M^n is an n-dimensional manifold, the circumradius of M^n denoted by $r = r(M)$ is the radius of the smallest closed ball containing $x(M)$.

Theorem 2.2. *Let M be an nRich-manifold that is also an ideal hypersurface of R^{n+1}, $n \geq 2$. In this case,*

$$\lambda_1 \geq \frac{n}{r^2}.$$

Proof. Let $x\colon M^n \to R^{n+1}$ be an ideal (inclusion) immersion. Then by [4] for x, we have

$$\int\limits_M |H|^2 dV \geq \frac{V}{r^2}. \tag{2.3}$$

So, the result may be obtained by Theorem 2.1.

Theorem 2.3. *Suppose that M is an nRich hypersurface of the unit sphere S^{n+1}, $n \geq 2$, with following conditions*:

i) *M is of constant scalar curvature $S \geq 0$ and, in the case $S = 0$, the sign of mean curvature is unchanged,*

ii) *Gaussian image of M is in a closed hemisphere of S^{n+1}, whence $\lambda_1 \geq \dfrac{n}{r^2}$.*

Proof. By [5], conditions i) and ii) deduce that M is totally umbilic. So, by Remark 2.1, M is an ideal hypersurface of S^{n+1} and, therefore, the result follows from Theorem 2.1.

***Corollary* 2.3.** *Let M be an nRich hypersurface of R^{n+1} such that, for a positive scalar a, the scalar curvature of M satisfies in $S = a|H|$. In this case,*

$$\lambda_1 \geq \frac{n}{r^2}.$$

Proof. By Corollary 2.5 [6, p. 13], M is totally umbilic and so, by Remark 1.3, M is an ideal hypersurface of R^{n+1}. Therefore, the result follows from Theorem 2.1.

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