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# BASIC PROPERTIES AND APPLICATIONS OF GRADED FRACTAL BUNDLES RELATED TO CLIFFORD STRUCTURES: AN INTRODUCTION*

# ОСНОВНІ ВЛАСТИВОСТІ ТА ЗАСТОСУВАННЯ СТУПІНЧАСТИХ ФРАКТАЛЬНИХ ЖМУТКІВ, ЩО ПОВ'ЯЗАНІ ЗІ СТРУКТУРАМИ КЛІФФОРДА. ВСТУП

Using the central extension of the Cuntz $C^*$-algebra, we study the periodicity for corresponding fractals.

З допомогою центрального розширення $C^*$-алгебри Кунца вивчається періодичність для відповідних фракталів.

**1. Introductory: dynamics of binary and ternary alloys.** The idea of fractal modelling of crystals comes back to Bethe [1] who observed its convenience when coming to first, second, third nearest neighbours of an atom. Taking into account that it is a neighbour of two or more other atoms, even in the case of one layer with a lattice formed by squares, one naturally comes to the notion of *cluster* [2, 3]. It is then natural to cut the plane of lattice correspondingly to the cluster involved and construct a Riemann surface or a *Bethe lattice — a fractal set of the branch type* [4, 5]. The construction is parallel to that related to the holomorphic function $f(z) = \exp z^2$ in $\mathbb{C}$ (Fig. 1). The example shows already the importance of the corresponding group $\Gamma$ of *cover symmetry transformations* (Decktransformationengruppe), *inoculation* (of the branch corresponding to no. 1 on the branch corresponding to no. 4), and *gradation* related to the points ■, •, □, ○.

The next important step was done by Kikuchi [6], who — within his theory of cooperative phenomena — developed a method of approximation for order-disorder phenomena.

In this context, Sukiennicki, Wojtczak, Zasada, and Castillo Alvarado [7] investigated an infinite thin film of an $AB_3$ alloy. As examples we may take $Ni_3Fe$ or $Cu_3Au$. They assumed the (111) orientation of the alloy. Let $z(j)$ denote the concentration of the $A$-atoms in the layer $j = 0, 1, \ldots$; $j = 0$ corresponding to the surface. Then the concentration of the $B$-atoms in that layer is $1 - z(j)$. If $U$ is the energy of interaction of the system, $T$ — the absolute temperature, and $g$ – the number of configurations possible, the entropy $S$ and the free energy $F$ of the system are given by

$$S = k \ln g \qquad \text{and} \qquad F = U - TS,$$

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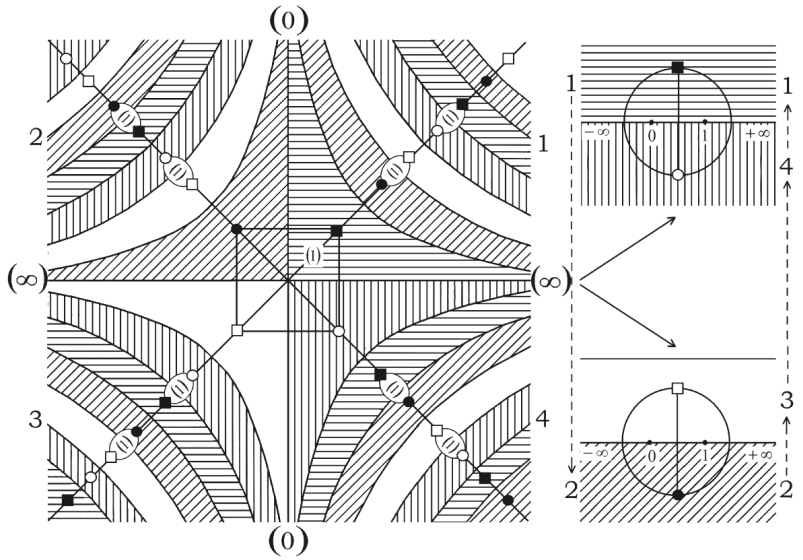



Fig. 1. Scheme showing the way of constructing the inoculated graded fractal $\Xi(f : c)$ of the branch type, where $f(z) = \exp z^2$, with gradation related to the points ■, ●, □, ○ corresponding to the Riemann surface of $\Gamma$.

respectively, where $k$ is the Boltzmann constant, and the conditions for thermodynamic equilibrium at a given temperature $T$ are

$$\left(\frac{\partial}{\partial\tau}\right)F\Big|_{\tau=\tau(j)} = 0, \qquad \left(\frac{\partial}{\partial z}\right)F\Big|_{z=z(j)} = 0 \quad \text{at each} \quad j,\ \lambda = \text{const}.$$

Thus

$$\frac{1}{T}\left(\frac{\partial}{\partial\tau}\right)U\Big|_{\tau=\tau(j)} = \left(\frac{\partial}{\partial z}\right)S\Big|_{\tau=\tau(j)}.$$

The authors of [7] have calculated that

$$\frac{\partial S}{\partial\tau} = \frac{3}{16}k\ln\frac{\left(z+\frac{3}{4}\tau\right)\left(1-z+\frac{1}{4}\tau\right)}{\left(z-\frac{1}{4}\tau\right)\left(1-z-\frac{3}{4}\tau\right)} \quad \text{for} \quad z = z(j) \quad \text{and} \quad \tau = \tau(j),$$

and thus, finally,

$$\frac{16}{3}kT\left(\frac{\partial}{\partial\tau}\right)U = \frac{z(1-z)+\frac{1}{4}(3-2z)+\frac{3}{16}\tau^2}{z(1-z)-\frac{1}{4}(1+2z)+\frac{3}{16}\tau^2} \quad \text{for} \quad z = z(j) \quad \text{and} \quad \tau = \tau(j).$$

Let us take the pseudometric

$$ds^2 = c^2dt^2 - dx_A^2 - dy_A^2 - dz_A^2 - dx_B^2 - dy_B^2 - dz_B^2 - \eta_\sigma c_\sigma d\sigma^2 - \eta_\tau c_\tau d\tau^2, \quad (1)$$

where $\eta_\sigma = 1$ or $-1$, $\eta_\tau = 1$ or $-1$, $c_\sigma$ and $c_\tau$ are positive constants, and $\sigma$ is the stochastic parameter (e.g., $\sigma = S$ or $\sigma = \sigma_0$, the short-range order parameter), and

consider a Lorentz-like deformation of the nine-dimensional system of the degrees of freedom:

$$\left\{\left(x_A,\, y_A,\, z_A,\, x_B,\, y_B,\, z_B,\, t,\, \sigma,\, \tau\right)\right\}$$

propagating itself as a wave-like perturbation. We assume, for the sake of simplicity, that

$$dx_A = dx'_A, \qquad dx_B = dx'_B, \qquad dy_A = dy'_A, \qquad dy_B = dy'_B, \qquad dt = dt',$$

$$dz_A = \left(1 + c_z^2\right)^{-1/2} c_z dz, \qquad dz_B = \left(1 + c_z^2\right)^{-1/2} dz,$$

$c_z$ being a positive constant. Then, thanks to [8], in that deformation:

$$\begin{aligned} dz &= \rho_z \left(dz' + v_\sigma^z d\sigma' + v_\tau^z d\tau'\right), \\ d\sigma &= \rho_\sigma \left(v_z^\sigma dz' + d\sigma' + v_\tau^\sigma d\tau'\right), \\ d\tau &= \rho_\tau \left(v_z^\tau dz' + v_\sigma^\tau d\sigma' + d\tau'\right), \end{aligned} \tag{2}$$

where $\rho_z$, $\rho_\sigma$, $\rho_\tau$ are positive constants and $v_\sigma^z$, $v_\tau^z$, $v_z^\sigma$, $v_\tau^\sigma$, $v_z^\tau$, $v_\sigma^\tau$ are real constants, *we already know the coefficients* $v_\tau^\sigma$ *and* $v_\sigma^\tau$. Indeed,

$$S = S\left(\sigma_0, \tau\right), \quad \sigma_0 = \sigma_0\left(S(\sigma_0, \tau), \tau\right); \qquad \left(\frac{\partial}{\partial \sigma_0}\right) S \approx 0, \quad \left(\frac{\partial}{\partial \tau}\right) S \approx 0.$$

Similarly, in the case of ternary alloys (e.g., Cd $S_x Te_{1-x}$; cf. [9]), we have to consider an additional order parameter and – in the simplest case — the deformation

$$\begin{aligned} dz &= \rho_z \left(dz' + v_\sigma^z d\sigma' + v_\tau^z d\tau' + v_\theta^z d\theta'\right), \\ d\sigma &= \rho_\sigma \left(v_z^\sigma dz' + d\sigma' + v_\tau^\sigma d\tau' + v_\theta^\sigma d\theta'\right), \\ d\tau &= \rho_\tau \left(v_z^\tau dz' + v_\sigma^\tau d\sigma' + d\tau' + v_\theta^\tau d\theta'\right), \\ d\theta &= \rho_\theta \left(v_z^\theta dz' + v_\sigma^\theta d\sigma' + v_\tau^\theta d\tau' + d\theta'\right), \end{aligned} \tag{3}$$

where also $\rho_\theta$ has to be a positive constant and $v_\theta^z$, $v_\theta^\sigma$, $v_\theta^\tau$, $v_z^\theta$, $v_\sigma^\theta$, $v_\tau^\theta$ — real constants. Here we arrive at a thirteen-dimensional system of the degrees of freedom.

This setting, especially formulae (1) – (3) provide one of possible motivations for studying fractals related to Clifford structures and their relationship with twistor-like structures. At the end of our outline we sketch, as an application, a geometrical model of the surface melting effect (five degrees of freedom). The research will be continued in the second part of the paper.

**2. From alloys to fractals related to Clifford structures.** Given generators $A_1^1 = = A_1$, $A_2^1 = A_2, \ldots, A_{2p-1}^1 = A_{p-1}$ of a Clifford algebra $Cl_{2p-1}(\mathbb{C})$, $p = 2, 3, \ldots$, in particular the generators

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

of the Pauli algebra [10, 11], consider the sequence

$$A_\alpha^{q+1} = \sigma_3 \otimes A_\alpha^q \equiv \begin{pmatrix} A_\alpha^q & 0 \\ 0 & -A_\alpha^q \end{pmatrix}, \quad \alpha = 1, 2, \ldots, 2p + 2q - 3,$$

$$A_{2p+2q-2}^{q+1} = \sigma_1 \otimes I_{p,q} \equiv \begin{pmatrix} 0 & I_{p,q} \\ I_{p,q} & 0 \end{pmatrix}, \tag{5}$$

$$A_{2p+2q-1}^{q+1} = -\sigma_2 \otimes I_{p,q} \equiv \begin{pmatrix} 0 & iI_{p,q} \\ -iI_{p,q} & 0 \end{pmatrix},$$

of generators of Clifford algebras $Cl_{2p+2q-1}(\mathbb{C})$, $q = 1, 2, \ldots$, and the sequence of the corresponding systems of closed squares $Q_{q+1}^\alpha$ (of diameter 1, centred at the origin of $\mathbb{C}$, where $I_{p,q} = I_{2^{p+q-2}}$, the unit matrix of order $2^{p+q-2}$) together with their $4^{p+q-3}$ subsquares $Q_{q+1,k}^{\alpha,j}$ with sides parallel to the sides of $Q_{q+1}^\alpha$ for $\alpha \leq 2p-1$, and into $4^{p+q-2}$ analogous subsquares for $\alpha \geq 2p$. We endow the squares $Q_q^\alpha$ with the *gradating function* (*colour*) $g_q^\alpha$ equal $a_{\alpha j}^{qk}$ within $Q_{qk}^{\alpha j}$ and 0 otherwise. We call (5) a *basic construction* [12].

It is convenient to start with $q$ always from 1, i.e., to shift $q$ for $\alpha \geq 2p$ correspondingly. This means that, in the case of the latter two generators in (5), we have to shift $q$ by $p - 1$. For

$$q = 5 - p, \qquad p = 2, 3, 4,$$

the sequence (5) gives the expected fractal model for a binary alloy. For

$$q = 7 - p, \qquad p = 2, 3, 4, 5, 6,$$

(5) gives the expected fractal model for a ternary alloy.

If, for fixed $\alpha$ in (5), we now consider

$$\left(Q_n^\alpha, (Q_{nh}^{\alpha\gamma})\right), \quad n = 2, 3, \ldots,$$

we obtain a *graded* (*coloured*) *Clifford-type fractal* $\Sigma_\alpha$, $\alpha = 1, 2, \ldots$, *of the flower type* [4, 5]. The *fractal set* of each $\Sigma_\alpha$ is a dense subset of the diagonal of the corresponding squares, running

$$\text{from} \quad \frac{1}{2\sqrt{2}}(-1 + i) \quad \text{to} \quad \frac{1}{2\sqrt{2}}(1 - i);$$

namely, it consists of points whose distance from the begining of the diagonal is an integral multiple $m$ of $1/2$ to some power (a positive integer) times the length 1 of the diagonal of the matrix represented by the unit square. The *graded* (*coloured*) *Clifford-type fractal bundle* $\Sigma_\alpha$, $\alpha = 1, 2, \ldots$) is well defined which can be proved [13] using the Cuntz algebra $\mathcal{O}(4)$ [14], Kakutani dichotomy theorem [15] and properties of *petals* [13] being suitable pairs of (ordered) quadruples of neighbouring, sufficiently small subsets of $Q_q^\alpha$ and of (ordered) quadruples of the corresponding matrix entries.

It appears that, independently which generator is taken into account in the basic construction (5), we get precisely two kinds of periods (dependent on the algebra and generator), related to

$$m = 1, 2, 4, 7, 8, 11, 13, 14, 16, 19, 21, 22, 25, 26, 28, 31, 32, 35, \ldots \tag{6}$$

and

$$m = 3, 5, 6, 9, 10, 12, 15, 17, 18, 20, 23, 24, 27, 29, 30, 33, 34, 36, \ldots \tag{7}$$

respectively, independently of $\alpha$ in $\Sigma_\alpha$. The result is precisely formulated and proved in [13, 16, 17].

Clearly, (5) and (6) are not formal definitions of the sequences. Even if one writes $n = 1, 2, \ldots$, it is possible to imagine the sequence $(1, 2, 4, 5, 7, 8, \ldots)$. Formally, we have to consider two double sequences:

$$(2m-1)2^{2p-1}, \quad m, p = 1, 2, \ldots, \tag{8}$$

and

$$(2m-1)2^{2p} - 1, \quad m, p = 1, 2, \ldots. \tag{9}$$

If we order the set of numbers (8) to make the sequence

$$(a_n), a_n < a_{n+1}, \quad n = 1, 2, \ldots, \tag{10}$$

then $a_n$ with $n$ odd give a subsequence of (6), and $a_n$ with $n$ even give a subsequence of (7). (Explicitly, 2, 8, 14, 22, 26, 32, ... correspond to (6), and 6, 10, 18, 24, 30, 34, ... correspond to (7).) If we order the set of numbers (9) to make the sequence

$$(b_n), b_n < b_{n+1}, \quad n = 1, 2, \ldots, \tag{11}$$

then $b_n$ with $n$ odd give the subsequence of (7), complementary to that previously given, and $b_n$ with $n$ even give the subsequence of (6), complementary to that previously given. (Explicitly, 3, 15, 27, 43, 51, 63, ... correspond to (7), and 11, 19, 35, 47, 59, 67, ... correspond to (6).)

It is natural to ask whether the sequences (6) and (7) really reflect some properties of generators (4), in particular of $\sigma_1$, $\sigma_2$, $\sigma_3$, or they are of more general, perhaps trivial character like decomposition of an arbitrary integer into a linear combination of different powers of 2. The role of 2 (more precisely, of 0, 1 and $-1$) is here replaced by $0$, $1$, $-1$, $i$, $-i$, the entries constituting the Pauli matrices $\sigma_1$, $\sigma_2$, $\sigma_3$ because of the crucial role of the Pauli algebra and of the Cuntz algebra $\mathcal{O}(4)$generated by four isometries. At present we cannot answer this question in the form of a theorem, but we can see the following three important facts.

1. For each $\Sigma_\alpha$ the corresponding fractal set is as described above, i.e., it is naturally related to the sequences (6) and (7).

2. The fact that the Clifford product of two vectors in $\mathbb{R}^3$ has a symmetric part and an antisymmetric part is trivial, but the fact that the symmetric part is the scalar product of those vectors, whereas the antisymmetric part is the wedge product is quite important [18].

3. It seems that the construction of $\Sigma_\alpha$ provides a successful quick search of interesting geometrical objects involving Clifford algebras in the following sense (P. Jordan, von Neumann, and Wigner [19]): Consider the family of algebras $\mathcal{A}$ with addition $+$ and multiplication $\circ$ such that

$$\lambda A \in \mathcal{A} \quad \text{for} \quad A \in \mathcal{A}, \quad \lambda \in \mathbb{R},$$

$$((A \circ A) \circ B) \circ A = (A \circ A) \circ (B \circ A) \quad \text{for} \quad A, B \in \mathcal{A},$$

$$\text{if} \quad (A \circ A) + (B \circ B) + (C \circ C) + \ldots = 0 \quad \text{for} \quad A, B, C \in \mathcal{A},$$

$$\text{then} \quad A = B = C = \ldots = 0.$$

The only irreducible such algebras are the following:

(i) the algebra of real numbers with $A + B$, $\lambda A$ and $A \circ B$ defined in the usual way;

(ii) $\mathcal{C}_n$, $n = 3, 4, \ldots$; $\mathcal{C}_n$ being the algebra with the linear basis $1, s_1, \ldots, s_{n-1}$, where $A + B$ and $\lambda A$ are defined in the usual way, but $A \circ B$ is defined by

$$1 \circ 1 = 1, \quad 1 \circ s_j = s_j \quad \text{and} \quad s_j \circ s_k = \delta_{jk} \circ 1, \quad j,\ k = 1, \ldots, n-1,$$

and $\delta_{jk}$ denoting the Kronecker delta;

(iii) $\mathcal{H}_q^p$, $p = 1, 2, 4, 8$, and $q = 3$, or $p = 1, 2, 4$ and $q = 4, 5, \ldots$; $\mathcal{H}_q^p$ being the algebra of Hermitian matrices of order $q$ whose elements are real numbers for $p = 1$, complex numbers for $p = 2$, quaternions for $p = 4$, and octonions for $p = 8$, $A + B$ and $\lambda A$ are defined in the usual way, but for $\circ$ we have

$$A \circ B = \frac{1}{2}(AB + BA), \tag{12}$$

where $AB$ represents the usual matrix multiplication.

This means that the subfamily of $\{\mathcal{H}_q^p\}$ including octonions $(p = 8)$ is quite marginal. In contrast those corresponding to complex numbers $(p = 2)$ and quaternions $(p = 4)$ have an infinite number of members. As far as our graded fractal bundles are concerned,

$$\Sigma_1 \quad \text{corresponds to} \quad \mathcal{H}_6^2, \qquad \Sigma_2 \quad \text{corr. to} \quad \mathcal{H}_{40}^2, \qquad \Sigma_3 \quad \text{corr. to} \quad \mathcal{H}_{224}^2, \ldots$$

and

$$\Sigma_1 \quad \text{corresponds to} \quad \mathcal{H}_3^4, \qquad \Sigma_2 \quad \text{corr. to} \quad \mathcal{H}_{20}^4, \qquad \Sigma_3 \quad \text{corr. to} \quad \mathcal{H}_{112}^4, \ldots.$$

Generally, we have

$$\Sigma_\alpha \quad \text{corr. to} \quad \mathcal{H}^2_{2^{2\alpha-1}(2\alpha+1)} \qquad \text{and} \qquad \Sigma_\alpha \quad \text{corr. to} \quad \mathcal{H}^4_{2^{2\alpha-2}(2\alpha+1)}.$$

Now, let us come back to the relatioship between the gradation and inoculation of fractals [20, 21]. Let us recall [22, 23] that each fractal $\Sigma$ of the flower type has its *dual* $\Xi$ of the branch type and vice versa. In the case of the Sierpiński gasket this is illustrated by Fig. 2 a), b). We consider there two copies of $\Xi$ : $\Xi_1$ and $\Xi_2$ in Fig. 2 b), c), differing in gradation (colour) and *inoculate* $\Xi_1$ *of the first kind* on its $n$-th embranchment by a branch of $\Xi_2$ (Fig. 2 d)). *The second kind of inoculation* appears when, together with changing gradation at the embranchment we change the number of branches (Fig. 4 vs.

Fig. 3). If, for some $n$, the $n$-th branch of a fractal $\Xi_1$ within the graded fractal bundle in question is considered in the bundle together with the 1-st embranchment of a fractal $\Xi_2$ within the fractal bundle in question, we say thet $\Xi_1$ is *inoculated of the third kind* at its $n$-th embranchment by $\Xi_2$.

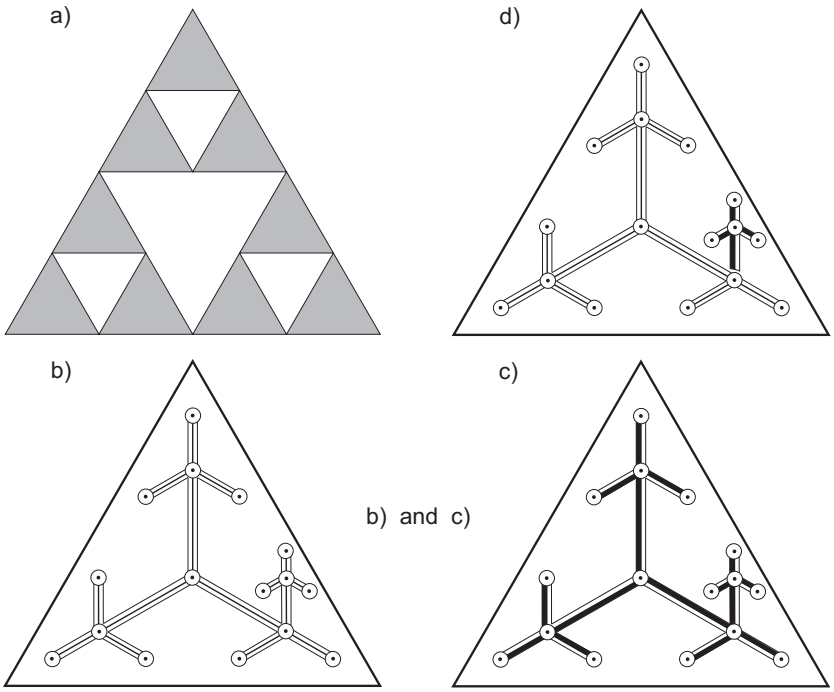



Fig. 2. a) The Sierpiński gasket $\Sigma^1$ (of the flower type) after the 2nd step of construction.
b) The fractal $\Xi_1$, dual to $\Sigma_1$ (of the branch type) after the 2nd step of construction.
c) The fractal $\Xi_2$ differing from $\Xi_1$ in gradation.
b) and c) The bundle $(\Xi_1, \Xi_2)$.
d) A graded fractal obtained from $(\Xi_1, \Xi_2)$ by inoculation of the first kind.

The corresponding definitions for fractals of the flower type are similar. For instance in the case of inoculation of the third kind we just replace "embranchment" by "growing step". As far as the graded Clifford-type fractal bundle $(\Sigma_\alpha)$ is concerned, $\Sigma_1$, $\Sigma_2, \ldots$ $\ldots$, $\Sigma_{2p-1}$ are inoculated of the third kind at their first growing step by $\Sigma_{2p}$ and $\Sigma_{2p+1}$; the fractals $\Sigma_{2p}$ and $\Sigma_{2p+1}$ are inoculated of the third kind at their first growing step by $\Sigma_{2p+2}$ and $\Sigma_{2p+3}$, also $\Sigma_1, \Sigma_2, \ldots, \Sigma_{2p-1}$ are inoculated of the third kind at their second growing step by $\Sigma_{2p+2}$ and $\Sigma_{2p+3}$, etc.

Now, following [20] we define the graded fractal $\Xi$ of the branch type, inoculated of the first kind (Fig. 3), related to the sequences (6) and (7), in six steps:

(i) At the $n$-th embranchment we have the numbers $1, 2, \ldots, 2^{n-1}$ in the growing order. Then from 1 we get 1 and 2, from 2 we get 3 and 4; finally, from $2^{n-1}$ we get $2^n - 1$ and $2^n$.

(ii) Inoculation concerns the sequences (6) and (7), and we have to determine all the related numbers preceding and following the inoculation.

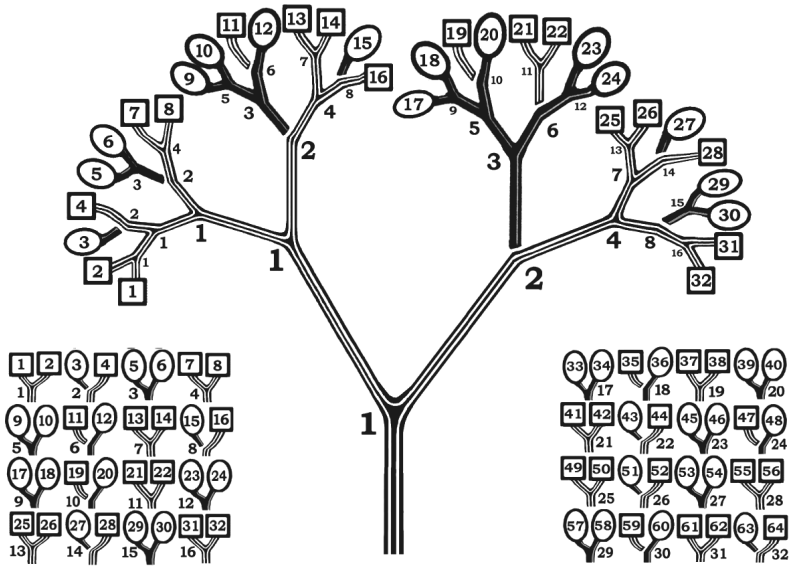


Fig. 3. The inoculated graded fractal $\Xi$ of the branch type, corresponding to the sequences (6) and (7) responsible for the types of periods related to (5).

(iii) The numbers preceding the inoculation are of the form (8).

(iv) The numbers following the inoculation are of the form (9).

(v) If we order the set of numbers (8) to make the sequence (10), we realize that $a_n$ with $n$ odd correspond to (6), and $a_n$ with $n$ even correspond to (7).

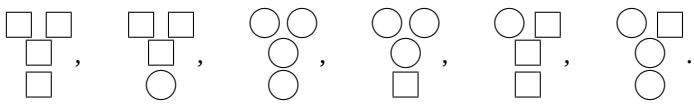
(vi) If we order the set of numbers (9) to make the sequence (11), we realize that $b_n$ with $n$ odd correspond to (7), and $(b_n)$ with $n$ even correspond to (6).

The whole construction leading to the fractal $\Xi$ is visualized on Fig. 3. The sixth embranchment is drawn separately in the lower left and lower right parts of the figure. We have the following *Fractal Inoculation Theorem*:

**Theorem 1.** *The inoculated graded fractal $\Xi$ of the branch type can be decomposed to the bundle of inoculated fractals $\Xi_1$ and $\Xi_2$ without gradation, where $\Xi_2$ is repeated infinitely many times. Here $\Xi_1$ corresponds to* (6) *and $\Xi_2$ corresponds to* (7). *Precisely, the embranchments of $\Xi_1$ are renumbered according to the scheme $n \mapsto n+1$. At the new first embranchment of $\Xi_1$ this fractal is inoculated of the second kind by the first copy $\Xi_2^1$ of $\Xi_2$. At the first embranchment of $\Xi_2^1$ this fractal is inoculated of the second kind by the second copy $\Xi_2^2$ of $\Xi_2$. At the first embranchment of $\Xi_2^2$ this fractal is inoculated of the second kind by the third copy $\Xi_2^3$ of $\Xi_2$, etc.*

The whole construction is visualized on Fig. 4. A detailed proof will be given in Part II of the paper.

**3. Atomization theorem for fractals and Hurwitz twistor-like structures.** The fractal $\Xi$ (shown on Fig. 3) has six types of embranchments:



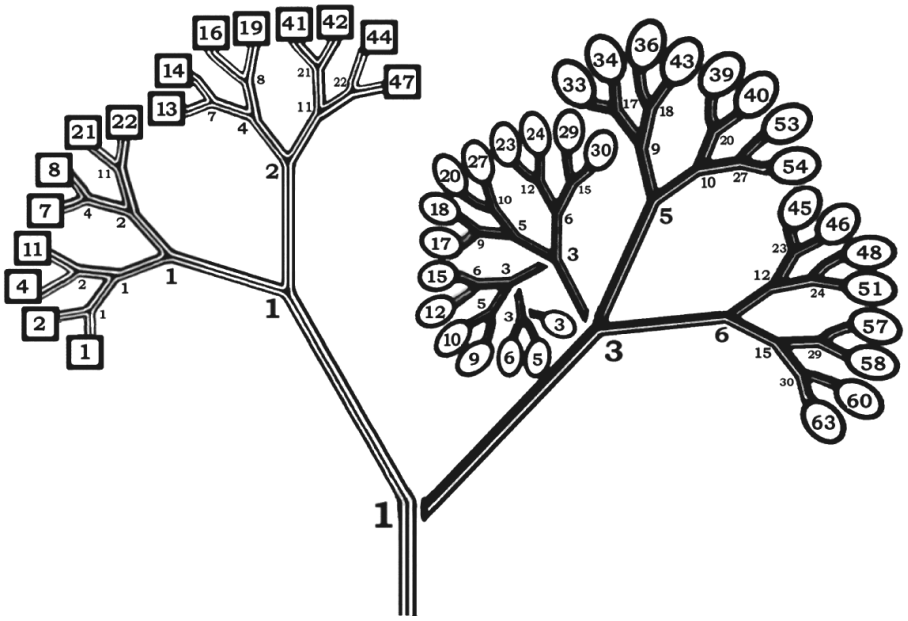


Fig. 4. The bundle $(\Xi_1, \Xi_2^1, \Xi_2^2, \ldots)$ of inoculated fractals without gradation, corresponding to the sequences (6) and (7) responsible for the types of periods related to (5).

Here $\square$ resp. $\bigcirc$ represent members of (6) resp. (7). Only two latter structures correspond to inoculation at the embranchment. They will be called *gemmae* of $\Xi$. Extending our Atomization Theorem (on isometric embeddings) [24, 25] we are going to state the following *Atomization Theorem for Fractals* [21]:

**Theorem 2.** *Suppose that the pseudometric corresponding to the $p$-dimensional real vector space appearing in the definition of an Hermitian Hurwitz pair of bidimension $(p, n)$ has the form*

$$\begin{aligned} \langle dx,\, dx\rangle \quad &= \quad dx_1^2 - dx_2^2 - \ldots - dx_5^2, \\ &\textit{or} \quad dx_1^2 + \ldots + dx_3^2 - dx_4^2 - dx_5^2, \\ &\textit{for} \;\; p = 5, \end{aligned} \tag{13}$$

*respectively:*

$$\begin{aligned} \langle dx,\, dx\rangle \quad &= \quad dx_1^2 - dx_2^2 - \ldots - dx_9^2, \\ &\textit{or} \quad dx_1^2 + \ldots + dx_5^2 - dx_6^2 - \ldots - dx_9^2, \\ &\textit{or} \quad dx_1^2 + \ldots + dx_3^2 - dx_4^2 - \ldots - dx_9^2, \\ &\textit{or} \quad dx_1^2 + \ldots + dx_7^2 - dx_8^2 - dx_9^2, \\ &\textit{for} \;\; p = 9, \end{aligned} \tag{14}$$

*respectively:*

$$\begin{aligned}
\langle dx,\, dx\rangle \;&=\; dx_1^2 - dx_2^2 - \ldots - dx_{13}^2,\\
&\quad or \quad dx_1^2 + \ldots + dx_7^2 - dx_8^2 - \ldots - dx_{13}^2,\\
&\quad or \quad dx_1^2 + \ldots + dx_3^2 - dx_4^2 - \ldots - dx_{13}^2,\\
&\quad or \quad dx_1^2 + \ldots + dx_{11}^2 - dx_{12}^2 - dx_{13}^2,\\
&\quad or \quad dx_1^2 + \ldots + dx_5^2 - dx_6^2 - \ldots - dx_{13}^2,\\
&\quad or \quad dx_1^2 + \ldots + dx_9^2 - dx_{10}^2 - \ldots - dx_{13}^2,\\
&\quad for \;\; p = 13,
\end{aligned} \tag{15}$$

*Then in each of these cases there exists a finite subfractal* $\Xi_0 \subset \Xi$, *where all gemmae can be obtained with the help of type-changing transformations listed in Table* 1.

*Table* 1. Fractal atoms

| No. | Atom | No. | Atom |
|---|---|---|---|
| 1 | $s^1\colon \left(\begin{smallmatrix}\boxed{0}\\ (0)\end{smallmatrix}\right) \Longleftrightarrow \left(\begin{smallmatrix}(3)\\ \boxed{0}\end{smallmatrix}\right)$ | 7 | $s_1^4\colon \left(\begin{smallmatrix}\boxed{35}\\ (27)\end{smallmatrix}\right) \Longleftrightarrow \left(\begin{smallmatrix}(43)\\ \boxed{35}\end{smallmatrix}\right)$ |
| 2 | $r^1\colon \left(\begin{smallmatrix}(3)\\ \boxed{0}\end{smallmatrix}\right) \Longleftrightarrow \left(\begin{smallmatrix}\boxed{11}\\ (3)\end{smallmatrix}\right)$ | 8 | $r_1^3\colon \left(\begin{smallmatrix}(43)\\ \boxed{35}\end{smallmatrix}\right) \Longleftrightarrow \left(\begin{smallmatrix}\boxed{47}\\ (43)\end{smallmatrix}\right)$ |
| 3 | $s_1^3\colon \left(\begin{smallmatrix}\boxed{11}\\ (3)\end{smallmatrix}\right) \Longleftrightarrow \left(\begin{smallmatrix}(15)\\ \boxed{11}\end{smallmatrix}\right)$ | 9 | $s_1^2\colon \left(\begin{smallmatrix}\boxed{47}\\ (43)\end{smallmatrix}\right) \Longleftrightarrow \left(\begin{smallmatrix}(51)\\ \boxed{47}\end{smallmatrix}\right)$ |
| 4 | $r_1^2\colon \left(\begin{smallmatrix}(15)\\ \boxed{11}\end{smallmatrix}\right) \Longleftrightarrow \left(\begin{smallmatrix}\boxed{19}\\ (15)\end{smallmatrix}\right)$ | 10 | $r_2^3\colon \left(\begin{smallmatrix}(51)\\ \boxed{47}\end{smallmatrix}\right) \Longleftrightarrow \left(\begin{smallmatrix}\boxed{59}\\ (51)\end{smallmatrix}\right)$ |
| 5 | $s_2^3\colon \left(\begin{smallmatrix}\boxed{19}\\ (15)\end{smallmatrix}\right) \Longleftrightarrow \left(\begin{smallmatrix}(27)\\ \boxed{19}\end{smallmatrix}\right)$ | 11 | $s_3^3\colon \left(\begin{smallmatrix}\boxed{59}\\ (51)\end{smallmatrix}\right) \Longleftrightarrow \left(\begin{smallmatrix}(63)\\ \boxed{59}\end{smallmatrix}\right)$ |
| 6 | $r_1^4\colon \left(\begin{smallmatrix}(27)\\ \boxed{19}\end{smallmatrix}\right) \Longleftrightarrow \left(\begin{smallmatrix}\boxed{35}\\ (27)\end{smallmatrix}\right)$ | | |

In the above table $\left(\begin{smallmatrix}\boxed{11}\\ (3)\end{smallmatrix}\right)$ etc. stays for the pair of gemmae

$$\begin{smallmatrix}\boxed{21}\;\boxed{22}\\ \boxed{11}\\ (6)\end{smallmatrix} \quad \text{and} \quad \begin{smallmatrix}(5)\;(6)\\ (3)\\ \boxed{2}\end{smallmatrix}$$

with (6) and $\boxed{2}$ taken away. The other symbols and numbers can correspondingly be deduced from Fig. 5 illustrating the (constructive) proof. For the full transformations listed in the table, the gemmae indicated have to be extended to finite sequences of petals corresponding to these gemmae and related to the generators of the Clifford algebra concerned. Under the *type* of gemmae we mean the class of abstraction of all the structures

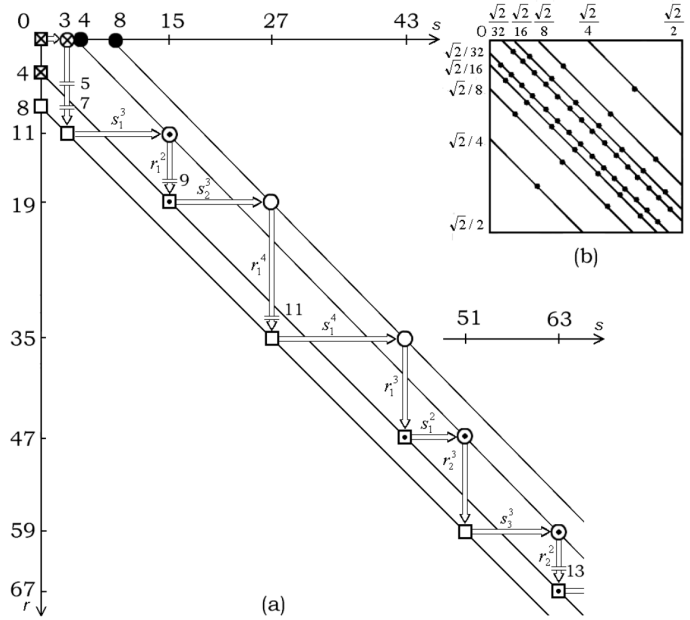



Fig. 5. Type-changing transformations $(\square) \Longleftrightarrow (\bigcirc)$:
(a) decomposed as stigmas of the pistil, (b) compared with the stamens.

$\binom{\square}{\bigcirc}$ resp. $\binom{\bigcirc}{\square}$. A transformation of gemmae is *type-changing* if it sends $\binom{\square}{\bigcirc}$ to $\binom{\bigcirc}{\square}$ or vice versa. In analogy to the fact that $0 \in \mathbb{C}$ lies in the real and imaginary axis as well, we complete the both sequences (6) and (7) by 0 at the beginning, so that we have objects $\boxed{0}$ and $\textcircled{0}$. This is caused by the fact that we have no pairing for $\textcircled{3}$ [21], so we need this extension.

The table is illustrated by Fig. 5. The choice of coordinates is motivated by the notions of pistil and stamens introduced and discussed in [13]. The choice of symbols $r$ and $s$ for particular basic type-changing transformations, called *fractal atoms*, as well as the corresponding lower indices seem natural. Except for $r^1$, the choice of upper indices refers to the length of the corresponding vectors. The notation $\shortparallel 5$, $\shortparallel 7$, etc. informs on the end of the sequence

$$\big(m/2^{p-1}, \quad m = 0, 1, \ldots, 2^{p-1} - 1\big) \quad \text{for} \quad 2p - 1 = 5, 7, \quad \text{etc.}$$

A detailed proof of Theorem 2 will be given in Part II of the paper.

The Hurwitz-twistor counterpart of the Penrose theorem in the semiglobal version states a one-to-one correspondence of the space of holomorphic solutions of the above mentioned spinor equations of spin $\dfrac{1}{2}n$ with the one-dimensional Dolbeault cohomology group $H^1$ depending on $\mathcal{O}(n-2) = \mathcal{O}([e]^{n-2})$, where $[e]$ is the canonical effective divisor of $\mathbb{P}^3(\mathbb{C})$. On the other side the analogous pseudotwistor counterpart of the Penrose theorem states a one-to-one correspondence of the respective space of holomorphic solutions with the group $H^1$ depending on $\mathcal{O}(-\alpha n - \beta)$, where $\alpha$ and $\beta$, $\beta \geq 2$, are some positive integers. Therefore, again, the both structures have to be linked by a proper type-changing transformation.

Let us analyze the situation more closely. In [26] we can find such dualities, expressed as type-changing transformations, relating manifolds $M$ with the following pseudometrics.

*Case* I. $dx_1^2 + \ldots + dx_3^2 - dx_4^2 - dx_5^2$ and $dx_1^2 - dx_2^2 - \ldots - dx_5^2$ for $p = 5$.

In analogy to Penrose twistors [27] we arrive at the structure of Hurwitz twistors (H for short) [28], determined by a system of $\binom{5}{4} = 5$ algebraic equations, and at their anti-objects (aH), corresponding to the 5-dimensional Kałuża – Klein theory [29, 30].

*Case* II. $dx_1^2 - dx_2^2 - \ldots - dx_9^2$ and $dx_1^2 + \ldots + dx_7^2 - dx_8^2 + dx_9^2$ for $p = 9$.

Here we arrive at the structure of pseudotwistors (p) [24, 25, 31], determined by a system of $\binom{9}{4} = 126$ algebraic equations, and at their anti-objects (ap).

*Case* III. $dx_1^2 + \ldots + dx_7^2 - dx_8^2 - \ldots - dx_{13}^2$ and $dx_1^2 + \ldots + dx_5^2 - dx_6^2 - \ldots - dx_{13}^2$ for $p = 13$.

Here we arrive at the structure of bitwistors (b) [24, 25, 31, 32], determined by a system of $\binom{13}{4} = 715$ algebraic equations, and at their anti-objects (ab).

*Case* IV. $dx_1^2 + \ldots + dx_5^2 - dx_6^2 - \ldots - dx_9^2$ and $dx_1^2 + \ldots + dx_3^2 - dx_4^2 - \ldots - dx_9^2$ for $p = 9$.

Here we arrive at the structure of pseudobitwistors (pb) [31, 32], determined by a system of 126 algebraic equations, and at their anti-objects (apb).

The other cases appearing in (15) were not discussed in [21] because of the (8,8)-periodicity of the Clifford structure.

The above demands can be fulfilled with the help of Atomization Theorem for fractals. Namely, we have, as a corollary to that theorem, the following *Atomization Theorem for Twistor-like Structures*:

**Theorem 3.** *Suppose that the pseudometric corresponding to the p-dimensional real vector space appearing in the definition of an Hermitian Hurwitz pair of bidimension* $(p, n)$ *has the form given in cases* I – IV. *Then in each of these cases there exists a type-changing transformation of the form listed in Table* 2.

*Table* 2. Basic type-changing transformations for Hurwitz twistor-like structures

| No. | Transformation | No. | Transformation |
|---|---|---|---|
| 1 | H $\Longleftrightarrow$ ap | 7 | p $\Longleftrightarrow$ apb |
| 2 | aH $\Longleftrightarrow$ p | 8 | ap $\Longleftrightarrow$ pb |
| 3 | H $\Longleftrightarrow$ aH | 9 | p $\Longleftrightarrow$ ap |
| 4 | p $\Longleftrightarrow$ ab | 10 | H $\Longleftrightarrow$ ab |
| 5 | ap $\Longleftrightarrow$ b | 11 | aH $\Longleftrightarrow$ b |
| 6 | b $\Longleftrightarrow$ ab | | |

The theorem may be regarded as a further contribution to the so-called double Cartan-like triality of Hermitian Hurwitz pairs [18, 33].

**4. Geometrical model of the surface melting effect.** A good example of physical application of Theorem 2 is provided by the geometrical model of the surface melting effect [8, 34 – 36], related to the formulae (13).

If a very thin layer at the crystal surface is the solid-vapour interface whose appearance defines the surface melting effects, it can be described by its tangent bundle with the arc-length element $dx^2 + dy^2$. Let us embed each tangent space of the bundle in a three-dimensional space-time with the arc-length element $c^2dt^2 - dx^2 - dy^2$, with the usual meaning of *c* and *t*. We consider the further embedding in a four-dimensional space time $\mathbb{M}_4$ with the arc-length element

$$ds^2 := c^2dt^2 - dx^2 - dy^2 - \eta d\tau^2 \tag{16}$$

where $\eta = 1$, or its modification $\mathbb{M}_4^*$ with arc-length element (16), where $\eta = -1$. Here each atom at the surface is treated as a small oscillator and the stochastical character of its behaviour is introduced by the *stochastical dimension* $\tau$ related to entropy, while $\eta = 1$ or $-1$ corresponds to two different kinds of stochastic nature. This means that $\eta = 1$ stands for the *Kałuża – Klein-type* of differential equations governing the motion. Correspondingly, $\eta = -1$ stands for the temporal character of the stochasticity which leads to the *Penrose-type* of differential equations governing the motion. Thereafter $\mathbb{M}_0 = \mathbb{M}_4$ or $\mathbb{M}_4^*$ will be called the *base space* of our construction.

Fix now a point 0 in $\mathbb{M}_0$, consider a family of the analogues of $\mathbb{M}_0$, corresponding to different further layers of the crystal, and take into account the uniquely determined curve $\mathbb{M}_\#$ starting from 0, passing through all the layers in question, and normal to each of them. Denote by $\mathbb{M}_0 = \mathbb{M}_5$ or $\mathbb{M}_5^*$ the bundle of all those layers, with the arc-length element

$$ds^2 := c^2dt^2 - dx^2 - dy^2 - dz^2 - \eta(z)d\tau^2, \quad \eta(z) = 1 \text{ or } -1,$$

and this corresponds to the formulae (13) in Theorem 2 and Case I in Theorem 3.

The notation $\mathbb{M}_5$ or $\mathbb{M}_5^*$ is justified by the theorem formulated in [37], Sect. 5.6, which assures that: (i) an arbitrary system of particles governed by equations of the Kałuża – Klein-type, completed by an even number of particles governed by equations of the Penrose type, is again a system of particles governed by equations of the Kałuża – Klein-type, (ii) an arbitrary system of particles governed by equations of the Kałuża – Klein-type, completed by an odd number of particles governed by equations of the Penrose type, is a system of particles governed by equations of the Penrose type.

In order to find the surface potential $f_s(\rho_s)$, where $\rho_s$ stands for the value of the order parameter $\rho$ at the surface, we take into account the general idea that the inhomogeneous behaviour of thermodynamic parameters can be connected with a deformation of the space-time in which a considered system is embedded. This idea introduced in analogy to the considerations reported by Ruppeiner [38] in order to describe the thermodynamical curvature in terms of the pseudoriemannian geometry was preliminarily applied to the model of the surface melting description [36]. In the present paper we outline a self-consistent approach determining the surface energy characteristics.

Our purpose requires that the surface energy is proportional to the inverse of the space-time curvature in the neighbourhood of the surface, i.e., in the region of inhomogeneity, in analogy to the Ruppeiner's hypothesis originally introduced for the thermodynamic space. Our condition can be then expressed as follows:

$$F_s(T = T_m) = k_B T_m \mathbb{R}^2,$$

where $F_s$ denotes the *surface free energy* at the phase transition solid-liquid temperature $T_m$ and $R$ is the curvature describing a deformation of a very thin layer whose thickness is given by

$$\breve{z} = b/R,$$

where $b$ is a positive parameter determined in [8]. This is the self-consistent condition connecting the physical deformation close to the surface with its geometrical description in terms of the Clifford-algebraic structure.

From the methodological point of view the deformation can be considered fibre bundles, which are very appropriate not only to global topological problems but also for local problems of differential geometry and field theory. The concept of induced representations of Lie groups and algebras may be most easily explained using the language of bundles as reported by Trautman [39].

The algebraical content of this staff is strictly related to the idea of one of us and Rembieliński [29, 30] to consider two vector spaces, the so-called Hurwitz pair $(S,\ V)$ restricted by the Hurwitz-type condition:

$$\langle a,\ a\rangle_S \langle x,\ y\rangle_V\ = \langle a\ \circ\ x,\ a\ \circ\ y\rangle_V, \qquad a \in S, \quad x,\ y \in V,$$

where $\langle a,\ a\rangle_S$ and $\langle x,\ y\rangle_V$ are the corresponding pseudoscalar products and $\circ$ is the multiplication in the Clifford algebra involved.

In the same manner the methodological aspect can be extended to the case of surface physics. In particular, the surface energy characteristics can be described in terms of such structures where the behaviour of atoms within the crystallographic lattice is characterized by a standard equation $\mathcal{H}\psi = E \circ \psi$ where $\psi$ is a spinor defined in a domain in $S$ with values in $V$; $S, V$ and $\circ$ being properly chosen. Practically, this choice is governed by the so-called Clifford constant.

From the physical point of view we consider surface of a sample treated as an *inhomogeneity* of the space. We choose the simplest idealized situation which is sufficient to be a proper example showing the usefullness of the proposed methodology.

The investigations in the field of surface phenomena, in particular, surface melting, show that there are at least two different configurations which are observed at the surface. One of them represents the surface melting where a liquid-like layer is formed at the surface. The other configuration is of solid-like layer character, but its structure is different from the structure inside a sample.

Two different configurations at the surface are described by means of the density whose profile corresponds to the distribution in the superficial region and contributes to the surface energy. A comparison between two surface energies shows that the phase transition characterized by the change of the density can be expected.

Explicit calculations are left to Part II of the paper.

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