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ON THE STABILITY OF SEMILINEAR NONAUTONOMOUS EVOLUTION EQUATIONS IN BANACH SPACES AND ITS APPLICATION TO STRONGLY PARABOLIC EQUATIONS

ПРО СТІЙКІСТЬ НАПІВЛІНІЙНИХ НЕАВТОНОМНИХ ЕВОЛЮЦІЙНИХ РІВНЯНЬ У БАНАХОВИХ ПРОСТОРАХ ТА ЇЇ ЗАСТОСУВАННЯ ДО СИЛЬНО ПАРАБОЛІЧНИХ РІВНЯНЬ

The paper is concerned with the exponential stability of the zero solution of strongly nonautonomous parabolic equations. Conditions are found on time-dependent coefficients of a parabolic equation under which its solutions converge exponential to 0 as $t \to \infty$.

Розглядається експоненціальна стійкість нульового розв'язку сильно неавтономних параболічних рівнянь. Знайдено умови для залежних від часу коефіцієнтів параболічного рівняння, при яких його розв'язки експоненціально збігаються до 0 при $t \to \infty$.

1. Introduction and notations. Throughout the paper Ω is assumed to be an open bounded subset of $\mathbb{R}^n$ with the boundary $\partial\Omega$. As in [1], we will use the following standard notations: $\Omega_b = \Omega \times [0, b]$, $\Gamma_b = \partial\Omega \times [0, b]$, $\Omega_\infty = \Omega \times [0, +\infty)$, $\Gamma_\infty = \partial\Omega \times [0, \infty]$, $x = (x_1, \ldots, x_n) \in \Omega$. $u(x,t) = (u_1(x,t), \ldots, u_s(x,t))$ is a complex vector function; $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{N}$, $|\alpha| = \sum_{i=1}^n \alpha_i$; $D^\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$, $|D^\alpha u|^2 = \sum_{i=1}^s |D^\alpha u_i|^2$, $dx = dx_1, \ldots, dx_n$; $\overset{0}{C}{}^\infty(\Omega)$ is the space of infinitely differentiable functions which have compact support in Ω; $H^l(\Omega)$ is the space of function $u(x)$ having generalized derivatives $D^\alpha u_i$ in $L_2(\Omega)$, $|\alpha| \le l$, $1 \le i \le s$ and

$$\|u\|^2_{H^l(\Omega)} = \sum_{|\alpha|=0}^{l} \int_\Omega \sum_{i=1}^{s} |D^\alpha u_i|^2 dx < \infty;$$

$\overset{0}{H}{}^l(\Omega)$ is the closure of $\overset{0}{C}{}^\infty(\Omega)$ in $H^l(\Omega)$; $H^{l,k}(\Omega_T)$ is the space of function $u(x,t)$ such that $D^\alpha u_i \in L_2(\Omega_T)$, $\partial^j u_i / \partial t^j \in L_2(\Omega_T)$, $|\alpha| \le l$, $1 \le i \le s$, $1 \le j \le k$, with the norm

$$\|u\|^2_{H^{l,k}(\Omega_T)} = \sum_{|\alpha|=0}^{l} \int_{\Omega_T} |D^\alpha u|^2 dx\,dt + \sum_{j=1}^{k} \int_{\Omega_T} |u_{t^j}|^2 dx\,dt;$$

$H^{l,0}(\Omega_T)$ is the space of functions $u(x,t)$ with the norm

$$\|u\|^2_{H^{l,0}(\Omega_T)} = \sum_{|\alpha|=0}^{l} \int_{\Omega_T} |D^\alpha u|^2 dx\,dt;$$

$\overset{0}{H}{}^{l,k}(\Omega_T)$ is the space of functions $u_i \in H^{l,k}(\Omega_T)$, equal zero near Γ_T.

With the above notations let us consider the following differential operator

$$L(x,t,D) = \sum_{|p|,|q|=1}^{m} D^p a_{pq}(x,t) D^q + \sum_{|p|=1}^{m} a_p(x,t) D^p + a(x,t),$$

where a_{pq}, a_p, a are $s \times s$-matrices, $a_{pq} = (-1)^{|p|+|q|} a^*_{qp}$, whose entries are bounded complex functions on $\overline{\Omega}_\infty$. Moreover for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and $\eta \in C^s \setminus \{0\}$ the following holds for a constant $c_0 > 0$

$$\sum_{|p|,|q|=m} a_{pq}(x,t) \xi^p \xi^q \eta \overline{\eta} > c_0 |\xi|^{2m} |\eta|^2 \quad \forall (x,t) \in \overline{\Omega}_\infty, \tag{1}$$

where $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$.

By the same argument as in [2, p. 44], we can prove that if the functions a_{pq} are uniformly continuous on $\overline{\Omega}_\infty$ and satisfy (1) whenever $|p| = |q| = m$, then there exist positive numbers μ_0, λ_0, μ_1, λ_1 such that for all $u \in \overset{0}{H}{}^m(\Omega)$

$$(-1)^m \sum_{|p|,|q|=1}^{m} (-1)^{|p|} \int_\Omega a_{pq} D^q u \overline{D^p v} dx \geq \mu_0 \|u\|^2_{H^m(\Omega)} - \lambda_0 \|u\|^2_{L_2(\Omega)}, \tag{2}$$

$$(-1)^m \left[\sum_{|p|,|q|=1}^{m} (-1)^{|p|} \int_\Omega a_{pq} D^q u \overline{D^p v} dx + 2\,\mathrm{Re} \sum_{|p|=1}^{m} \int_\Omega a_p D^p u \overline{u}\, dx \right] \geq$$

$$\geq \mu_1 \|u\|^2_{H^m(\Omega)} - \lambda_1 \|u\|^2_{L_2(\Omega)}. \tag{3}$$

Consider the problem

$$(-1)^{m-1} L(x,t,D) u - u_t = 0, \tag{4}$$

$$u|_{t=0} = \varphi(x) \in \overset{0}{H}{}^m(\Omega), \qquad \left.\frac{\partial^j u}{\partial \nu^j}\right|_{\partial\Omega} = 0, \quad j = \overline{1, m-1}. \tag{5}$$

A function $u(x,t)$ is said to be *generalized solution* of the problem (4, (5) in the space $H^{m,1}(\Omega_\infty)$ if $u(x,t) \in \overset{0}{H}{}^{m,1}(\Omega_\infty)$, $u(x,0) = \varphi(x)$ and for all $T > 0$

$$(-1)^{m-1} \int_{\Omega_T} \left[\sum_{|p|,|q|=1}^{m} (-1)^{|p|} a_{pq} D^q u \overline{D^p \eta} + \sum_{|p|=1}^{m} a_p D^p u \overline{\eta} + a u \overline{\eta} \right] dx\,dt -$$

$$- \int_{\Omega_T} u_t \overline{\eta}\, dx\,dt = 0 \quad \forall \eta \in \overset{0}{H}{}^{m,0}(\Omega_T), \qquad \eta(x,T) = 0. \tag{6}$$

As is well known, the existence and asymptotic behavior of generalized solutions of (4), (5) are subjects of many studies. We refer the reader to [1, 3 – 6] and the references therein for more information on this direction. In this short note we will proceed also in this direction by proving in the next section Theorem 1 on the exponential stability of linear equations. Using the argument as in [3] the stability of solutions of a class of semilinear equations can be considered.

2. Main results. In this section we will prove the following main result of the note.

Theorem 1. *Suppose that* $\partial a_{pq}/\partial t$, $a_p \in L_2(\Omega_\infty)$, *and that there exists* $\lambda > 0$ *such that* $a - (-1)^m (\lambda + \lambda_0) I \in L_2(\Omega_\infty)$, *where* λ_0 *is determined from* (2). *If*

$$\left|\frac{\partial a_{pq}}{\partial t}, \frac{\partial a_p}{\partial t}\right| \leq \mu, \quad 1 \leq |p|, |q| \leq m, \quad \mu = \text{const},$$

then the problem (4), (5) *has a unique generalized solution* $u(x, t) \in H^{m,1}(\Omega_\infty)$. *Moreover* $\|u\|^2_{H^m(\Omega)} \leq Ce^{-2\lambda t}\|\varphi\|^2_{H^m(\Omega)}$.

Proof. 1. First, we prove the uniqueness. Suppose that the problem (4), (5) has two generalized solutions u^1, u^2. For all $T > 0$ and $b \in (0, T)$, let us set

$$\eta(x, t) = \int_b^t \left[u^1(x, \tau) - u^2(x, \tau)\right] d\tau,$$

for $0 \leq t \leq b$ and $\eta(x, t) = 0$ for $b \leq t < T$. From (6) we have

$$(-1)^{m-1} \int_{\Omega_b} \left[\sum_{|p|,|q|=1}^{m} (-1)^{|p|} a_{pq} D^q \eta_t \overline{D^p \eta} + (-1)^m \lambda_1 \eta_t \overline{\eta} + \right.$$

$$\left. + \sum_{|p|=1}^{m} a_p D^p \eta_t \overline{\eta} + a_1 \eta_t \overline{\eta} \right] dx\,dt - \int_{\Omega_b} \eta_{tt} \overline{\eta}\, dx\,dt = 0, \tag{7}$$

where $a_1 = a - (-1)^m \lambda_1 I$, λ_1 is determined from (3). By (7), (3) and the Cauchy inequality, from the boundedness of $\partial a_{pq}/\partial t$, $\partial a_p/\partial t$ it follows that

$$2 \int_{\Omega_b} |\eta_t|^2 dx\,dt + \mu_1 \|\eta(x, 0)\|^2_{H^m(\Omega)} \leq$$

$$\leq C(\varepsilon) \sum_{|p|=0}^{m} \int_{\Omega_b} \left|D^p \eta\right|^2 dx\,dt + \varepsilon \int_{\Omega_b} |\eta_t|^2 dx\,dt.$$

Therefore,

$$\|\eta(x, 0)\|^2_{H^m(\Omega)} \leq C \sum_{|p|=0}^{m} \int_{\Omega_b} \left|D^p \eta\right|^2 dx\,dt, \quad C = \text{const} > 0.$$

Put

$$J(t) = \sum_{|p|=0}^{m} \int_{\Omega} \left| \int_t^0 D^p u(x, \tau)\, d\tau \right|^2 dx.$$

Then we have

$$(1 - Cb) J(b) \leq C \int_0^b J(t)\, dt, \quad b \in [0, 1/2C].$$

By the Gronwall – Bellman inequality, $J(t) \equiv 0$. Thus, $u^1 \equiv u^2 \quad \forall t \in [0, 1/2C]$. Following the argument as for u^1, u^2 on $[1/2C, T]$, we can show that $u^1 \equiv u^2$ $\forall t \in [1/2C, 1/C]$. After a finitely many steps we get $u^1 \equiv u^2 \quad \forall t \in [0, T]$. Since $T > 0$ is arbitrary, $u^1 \equiv u^2 \quad \forall t \in [0, \infty)$.

2. We now prove the existence. Let $\{\varphi_k(x)\}_{k=1}^{\infty} \subset \overset{0}{C}{}^{\infty}(\Omega)$ be an orthonormal system in $L_2(\Omega)$ such that its linear closure in $H^m(\Omega)$ in the space $\overset{0}{H}{}^m(\Omega)$. Put

$$u^N(x,t) = \sum_{k=1}^{N} C_k^N(t)\varphi_k(x),$$

where $C_k^N(t)$ is the solution of the following system:

$$\int_\Omega \left[u_t^N \overline{\varphi}_l + \sum_{|p|,|q|=1}^{m} (-1)^{m+|p|} a_{pq} D^q u^N \overline{D^p \varphi_l} \right] dx +$$

$$+ (-1)^m \int_\Omega \left(\sum_{|p|=1}^{m} a_p D^p u^N + a u^N \right) \overline{\varphi}_l \, dx = 0, \tag{8}$$

satisfying $C_k^N(0) = \alpha_k^N$, $l = \overline{1, N}$, here α_k^N's are coefficients of the function

$$\varphi^N(x) = \sum_{k=1}^{N} \alpha_k^N \varphi_k(x), \qquad \varphi^N(x) \xrightarrow{N \to \infty} \varphi(x)$$

in the norm of the space $H^m(\Omega)$.

Put $v^N(x,t) = u^N(x,t)e^{\lambda t}$. Multiplying (8) by $e^{\lambda t} d\left(\overline{C_l^N(t)e^{\lambda t}}\right)/dt$, taking the sum in l from 1 to N and then integrating in t from 0 to t we get

$$\int_{\Omega_t} \left[u_t^N e^{\lambda t} \overline{v_t^N} \sum_{|p|,|q|=1}^{m} (-1)^{m+|p|} a_{pq} D^q v^N \overline{D^p v_t^N} + \lambda_0 v^N \overline{v_t^N} \right] dx\,dt +$$

$$+ (-1)^m \int_{\Omega_t} \left(\sum_{|p|=1}^{m} a_p D^p v^N + a_0 v^N \right) \overline{v_t^N} \, dx\,dt = 0 \quad (a_0 = a - (-1)^m \lambda_0 Id).$$

Hence, by (2)

$$2 \int_{\Omega_t} |v_t^N|^2 \, dx\,dt + \mu_0 \| v^N(x,t) \|_{H^m(\Omega)}^2 \le$$

$$\le C_1(\varepsilon) \int_0^t B(t) \| v^N(x,t) \|_{H^m(\Omega)}^2 dt + \varepsilon \int_{\Omega_t} |v_t^N|^2 \, dx\,dt + C_2 \| \varphi^N \|_{H^m(\Omega)}^2,$$

where

$$B(t) = \int_\Omega \left[\sum_{|p|,|q|=1}^{m} \left| \frac{\partial a_{pq}}{\partial t} \right|^2 + \sum_{|p|=1}^{m} |a_p|^2 + |a_0 - (-1)^m \lambda I|^2 \right] dx.$$

Thus,

$$\| v^N(x,t) \|_{H^m(\Omega)}^2 \le C_3 \int_0^t B(t) \| v^N(x,t) \|_{H^m(\Omega)}^2 dt + C_4 \| \varphi^N \|_{H^m(\Omega)}^2.$$

By the Gronwall – Bellman inequality, we get

$$\| u^N(x,t) \|_{H^m(\Omega)}^2 \le C_5 e^{-2\lambda t} \| \varphi^N \|_{H^m(\Omega)}^2. \tag{9}$$

Since $\| \varphi^N \|_{H^m(\Omega)} \le C$,

$$\left\| u^N(x,t) \right\|^2_{H^{m,0}(\Omega_\infty)} \leq C_6. \tag{10}$$

Multiplying (8) by $d\left(C_l^N(t)\right)/dt$, taking the sum in l from 1 to N and integrating in t from 0 to t one gets

$$\int_{\Omega_t} \left[u_t^N \overline{u_t^N} + \sum_{|p|,|q|=1}^{m} (-1)^{m+|p|} a_{pq} D^q u^N \overline{D^p u_t^N} + \lambda_0 u^N \overline{u_t^N} \right] dx\,dt +$$

$$+ (-1)^m \int_{\Omega_t} \left(\sum_{|p|=1}^{m} a_p D^p u^N \overline{u_t^N} + a_0 u^N \overline{u_t^N} \right) dx\,dt = 0. \tag{11}$$

By (11) and Cauchy inequality

$$\left\| u_t^N \right\|^2_{L_2(\Omega_t)} + \left\| u^N(x,t) \right\|^2_{H^m(\Omega)} \leq D_1 \left\| u^N \right\|^2_{H^{m,0}(\Omega_t)} + D_2 \left\| \varphi^N \right\|^2_{H^m(\Omega)}.$$

Therefore,

$$\left\| u_t^N \right\|^2_{L_2(\Omega_t)} \leq D_1 \left\| u^N \right\|^2_{H^{m,0}(\Omega_t)} + D_2 \left\| \varphi^N \right\|^2_{H^m(\Omega)}. \tag{12}$$

The inequalities (10) and (12) imply $\left\| u^N \right\|^2_{H^{m,1}(\Omega_\infty)} \leq C$, where C is independent of $N \in \mathbb{N}$. Since $\{u^N\}$ is uniformly bounded on $H^{m,1}(\Omega_\infty)$, we can pick up from $\left\{u^N\right\}_1^\infty$ a subsequence converging weakly on $H^{m,1}(\Omega_\infty)$ to $u(x,t)$. We will show that $u(x,t)$ is a solution to the problem (4), (5). In fact, since $u^N(x,0) = \varphi^N(x)$ $\forall x \in \Omega$, $u^N(x,t) \in \overset{0}{H}{}^{m,1}(\Omega_T)$, it suffices to show that $u(x,0) = \varphi(x)$, $u(x,t) \in \overset{0}{H}{}^{m,1}(\Omega_T)$. Multiplying (8) by $d_l(t) \in L_2(0,T)$, taking the sum in l from 1 to N, one gets

$$\int_{\Omega} u_t^N \overline{\eta}\, dx + \int_{\Omega} \left[\sum_{|p|,|q|=1}^{m} (-1)^{m+|p|} a_{pq} D^q u^N \overline{D^p \eta} \right] dx +$$

$$+ (-1)^m \int_{\Omega} \left[\sum_{|p|=1}^{m} a_p D^p u^N \overline{\eta} + a u^N \overline{\eta} \right] dx = 0. \tag{13}$$

The above equality should be true for any function $\eta \in M_N$, where M_N is the set of functions of the from

$$\sum_{i=1}^{N} d_i(t) \varphi_i(x), \quad d_i(t) \in H^1(0,T), \quad d_i(T) = 0.$$

Since $M = \bigcup_{N=1}^\infty M_N$ is dense on $\overset{0}{H}{}^{m,0}(\Omega_T)$, $u(x,t)$ is generalized solution to (4), (5). Moreover, from (9) we have

$$\| u(x,t) \|^2_{H^m(\Omega)} \leq C_5 e^{-2\lambda t} \| \varphi \|^2_{H^m(\Omega)}.$$

The theorem is proved.

In passing, we note that using the argument in [3] the stability of the zero solution of some class of semilinear equations can be considered. In fact, we can represent the problem (4), (5) in the form of an abstract Cauchy problem in the Hilbert space $H^m(\Omega)$:

$$\frac{du}{dt} = A(t)u,$$
$$u(0) = \varphi \in \overset{0}{H}{}^{m}(\Omega). \tag{14}$$

where the domain $D(A(t)) \equiv \overset{0}{H}{}^{m}(\Omega)$ is dense on $H^{m}(\Omega)$. By Theorem 1, (14) generates an evolution operator* $\{U(t, s)\}_{t \geq s \geq 0}$ such that $\|U(t, s)\| \leq Ke^{-\lambda(t-s)}$ $\forall t \geq s$, here K, λ are positive constants independent of t, s.

Therefore, if $f: \mathbb{R}_+ \times H^{m}(\Omega) \to H^{m}(\Omega)$ is a continuous operator, which satisfies the Lipschitz condition with respect to $u \in H^{m}(\Omega)$ uniformly in $t \in \mathbb{R}_+$ and $\|f(t, u)\| \leq \psi(t)\|u\|^{m}$, $m > 1$, $\limsup_{t\to\infty}(\ln|\psi(t)|/t) = 0$, then the following semilinear equation:

$$\frac{du}{dt} = A(t)u + f(t, u),$$
$$u(0) = \varphi \in \overset{0}{H}{}^{m}(\Omega) \tag{15}$$

has a unique generalized solution (see [3, 4]).

By the some argument as in [7], we can prove that, if in addition, the evolution operator $\{U(t, s)\}_{t \geq s \geq 0}$ generated by (14) is regular, i. e. there exists a generalized Lyapunov transformation $u = L(t)x$ transforms this evolution operator into semigroup $\{T(t)\}_{t \geq 0}$, where $T(t - s) = L^{-1}(t)U(t, s)L(t)$, then the zero solution of (15) is exponentially stable.

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1. *Нгуен Мань Хунг.* Асимптотика решений первой краевой задачи для сильно гиперболических систем вблизи канонической точки границы области // Мат. сб. – 1999. – **190**, № 7. – С. 103–126.
2. *Фикера Г.* Теоремы существования в теории упругости. – М.: Мир, 1974. – Ч.1. – 89 с.
3. *Aulbach B., Minh N. V.* Nonlinear semigroups and the existence, stability of semilinear nonautonomous evolution equations // Abstract and Appl. Analysis. – 1996. – **1**. – P. 351–380.
4. *Segal I.* Nonlinear semigroups // Ann. Math. – 1963. – **78**, № 2.
5. *Крейн С. Г.* Линейные дифференциальные уравнения в банаховом пространстве. – М.: Наука, 1967.
6. *Pazy A.* Semigroups of linear operators and applications to partial differential equations // Appl. Math. – 1983. – **44**.
7. *Tran Thi Loan.* Lyapunov transformation and stability of differential equations in Banach spaces // Ukr. Math. J. – 1999. – **51**, № 10.

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* For definition of an evolution operator (or evolutionary process), the reader is refered to [3, 4, 6].