GENERALIZED WEYL’S THEOREM AND TENSOR PRODUCT
УЗАГАЛЬНЕНА ТЕОРЕМА ВЕЙЛЯ ТА ТЕНЗОРНИЙ ДОБУТОК

We give necessary and/or sufficient conditions ensuring the passage of generalized a-Weyl theorem and property (gw) from $A$ and $B$ to $A \otimes B$.

In the case in which $X$ and $Y$ are Hilbert spaces, Kubrusly and Duggal [15] proved that if $\sigma_b(A) = \sigma_w(A)$ and $\sigma_b(B) = \sigma_w(B)$, then $\sigma_b(A \otimes B) = \sigma_w(A \otimes B)$

if and only if $\sigma_w(A \otimes B) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B)$.

In other words, if $A$ and $B$ satisfy Browder’s theorem, then their tensor product satisfies Browder’s theorem if and only if the Weyl spectrum identity holds true. The same proof still holds in a Banach space setting.

For a bounded linear operator $S \in \mathcal{L}(X)$, let $\sigma(S), \sigma_p(S)$ and $\sigma_\delta(S)$ denote, respectively, the spectrum, the point spectrum and the approximate point spectrum of $S$ and if $G \subseteq \mathbb{C}$, then $G^{iso}$ denote the isolated points of $G$. Let $\alpha(S)$ and $\beta(S)$ denote the nullity and the deficiency of $S$, defined by $\alpha(S) = \dim \ker(S)$ and $\beta(S) = \text{codim} \mathcal{R}(S)$.

If the range $\mathcal{R}(S)$ of $S$ is closed and $\alpha(S) < \infty$ (respectively $\beta(S) < \infty$), then $S$ is called an upper semi-Fredholm (respectively a lower semi-Fredholm) operator. If $S \in \mathcal{L}(X)$ is either upper or lower semi-Fredholm, then $S$ is called a semi-Fredholm operator, and $\text{ind}(S)$, the index of $S$, is then defined by $\text{ind}(S) = \alpha(S) - \beta(S)$. If both $\alpha(S)$ and $\beta(S)$ are finite, then $S$ is a Fredholm operator. The ascent, denoted $\text{asc}(S)$, and the descent, denoted $\text{dsc}(S)$, of $S$ are given by $\text{asc}(S) = \inf \{ n \in \mathbb{N} : \ker(S^n) = \ker(S^{n+1}) \}$, $\text{dsc}(S) = \inf \{ n \in \mathbb{N} : \mathcal{R}(S)^n = \mathcal{R}(S^{n+1}) \}$ (where the infimum is taken over the set of non-negative integers); if no such integer $n$ exists, then $\text{asc}(S) = \infty$, respectively $\text{dsc}(S) = \infty$.

For $S \in \mathcal{L}(X)$ and a nonnegative integer $n$ define $S_{[n]}$ to be the restriction of $S$ to $\mathcal{R}(S^n)$ viewed as a map from $\mathcal{R}(S^n)$ into $\mathcal{R}(S^n)$ (in particular, $S_{[0]} = S$). If for some integer $n$ the range space $\mathcal{R}(S^n)$ is closed and $S_{[n]}$ is an upper (a lower) semi-Fredholm operator, then $S$ is called...
an upper (a lower) semi-B-Fredholm operator. In this case the index of $S$ is defined as the index of the semi-B-Fredholm operator $S_{[a]}$, see [8]. Moreover, if $S_{[a]}$ is a Fredholm operator, then $S$ is called a B-Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. An operator $S$ is said to be a B-Weyl operator [9] (Definition 1.1) if it is a B-Fredholm operator of index zero. The B-Weyl spectrum $\sigma_{BW}(S)$ of $S$ is defined by $\sigma_{BW}(S) = \{ \lambda \in \mathbb{C} : S - \lambda I \text{ is not a B-Weyl operator} \}.$

An operator $S \in \mathcal{L}(X)$ is called Drazin invertible if it has a finite ascent and descent. The Drazin spectrum $\sigma_D(S)$ of an operator $S$ is defined by $\sigma_D(S) = \{ \lambda \in \mathbb{C} : S - \lambda I \text{ is not Drazin invertible} \}.$

Define also the set $LD(X)$ by $LD(X) = \{ S \in \mathcal{L}(X) : a(S) < \infty \text{ and } \Re(T^a(S)+1) \text{ is closed} \}$ and $\sigma_{LD}(S) = \{ \lambda \in \mathbb{C} : S - \lambda \notin LD(X) \}.$ Following [10], an operator $S \in \mathcal{L}(X)$ is said to be left Drazin invertible if $S \in LD(X).$ We say that $\lambda \in \sigma_a(T)$ is a left pole of $S$ if $S - \lambda I \in LD(X),$ and that $\lambda \in \sigma_a(S)$ is a left pole of $S$ of finite rank if $\lambda$ is a left pole of $T$ and $a(S - \lambda I) < \infty.$ Let $\pi_a(S)$ denotes the set of all left poles of $S$ and let $\pi_a^0(S)$ denotes the set of all left poles of $S$ of finite rank. From [10] (Theorem 2.8) it follows that if $S \in \mathcal{L}(X)$ is left Drazin invertible, then $S$ is an upper semi-B-Fredholm operator of index less than or equal to 0. Note that $\pi_a(S) = \sigma_a(S) \setminus \sigma_{LD}(S)$ and hence $\lambda \in \pi_a(S)$ if and only if $\lambda \notin \sigma_{LD}(S)$.

Following [9], we say that generalized Weyl’s theorem holds for $S \in \mathcal{L}(X)$ (in symbol $S \in gW$) if $\Delta_\theta(S) = \sigma(S) \setminus \sigma_{BW}(S) = E(S),$ where $E(S) = \{ \lambda \in \sigma^{iso}(S) : 0 < a(S - \lambda I) \}$ is the set of all isolated eigenvalues of $S$, and that generalized Browder’s theorem holds for $S \in \mathcal{L}(X)$ (in symbol $S \in gB$) if $\Delta_\theta(S) = \pi(S),$ where $\pi(T)$ is the set of poles of the resolvent of $T.$ It is proved in [5] (Theorem 2.1) that generalized Browder’s theorem is equivalent to Browder’s theorem. In [10] (Theorem 3.9), it is shown that an operator satisfying generalized Weyl’s theorem satisfies also Weyl’s theorem, but the converse does not hold in general. Nonetheless and under the assumption $E(S) = \pi(S)$, it is proved in [11] (Theorem 2.9) that generalized Weyl’s theorem is equivalent to Weyl’s theorem. Let $\Psi_+(X)$ be the class of all upper semi-B-Fredholm operators, $\Psi_+(X) = \{ S \in \Psi_+(X) : \text{ind}(S) \leq 0 \}.$ The upper B-Weyl spectrum of $S$ is defined by $\sigma_{SBF^+}(S) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin \Psi_+(X) \}.$ We say that generalized a-Weyl’s theorem holds for $S \in \mathcal{L}(X)$ (in symbol $S \in gaW$) if $\Delta_\alpha^g(S) = \sigma_a(S) \setminus \sigma_{SBF^+}(S) = E_a(S),$ where $E_a(S) = \{ \lambda \in \sigma_{a}^{iso}(S) : \alpha(S - \lambda) > 0 \}$ is the set of all eigenvalues of $S$ which are isolated in $\sigma_a(S)$ and that $S \in \mathcal{L}(X)$ obeys generalized a-Browder’s theorem $(S \in gaB)$ if $\Delta_\alpha^g(S) = \pi_a(S).$ It is proved in [5] (Theorem 2.2) that generalized a-Browder’s theorem is equivalent to a-Browder’s theorem, and it is known from [10] (Theorem 3.11) that an operator satisfying generalized a-Weyl’s theorem satisfies a-Weyl’s theorem, but the converse does not hold in general and under the assumption $E_a(S) = \pi_a(S)$ it is proved in [11] (Theorem 2.10) that generalized a-Weyl’s theorem is equivalent to a-Weyl’s theorem.

The operator $T \in \mathcal{L}(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at $\lambda_0$) if for every open disc $\mathbb{D}$ centred at $\lambda_0$, the only analytic function $f : \mathbb{D} \to \mathbb{C}$ which satisfies the equation $(T-\lambda)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$ is the function $f \equiv 0.$ An operator $T \in \mathcal{L}(X)$ is said to have SVEP if $T$ has SVEP at every point $\lambda \in \mathbb{C}$.

Obviously, every $T \in \mathcal{L}(X)$ has SVEP at the points of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T).$ Moreover, from the identity theorem for analytic function, it easily follows that $T \in \mathcal{L}(X)$, as well as its dual.
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\( T^* \), has SVEP at every point of the boundary \( \partial \sigma(T) = \partial \sigma(T^*) \) of the spectrum \( \sigma(T) \). In particular, both \( T \) and \( T^* \) have SVEP at every isolated point of the spectrum, see [1, 4, 2, 3].

Let

\[ \Psi_+(S) = \{ \lambda \in \mathbb{C} : S - \lambda \text{ is upper semi-B-Fredholm} \} , \]
\[ \Psi(S) = \{ \lambda \in \mathbb{C} : S - \lambda \text{ is B-Fredholm} \} , \]
\[ \sigma_{SBF_+}(S) = \{ \lambda \in \sigma_a(S) : \lambda \notin \Psi_+(S) \} , \]
\[ \sigma_{SBF^+}(S) = \{ \lambda \in \sigma_a(S) : \lambda \in \sigma_{SBF_+}(S) \text{ or } \text{ind}(S - \lambda) > 0 \} , \]
\[ H_0(S) = \left\{ x \in \mathbb{X} : \lim_{n \to \infty} \| S^nx \|^{1/n} = 0 \right\} . \]

2. Main results. Let \( \sigma_a(S) = \{ \lambda \in \sigma(S) : S - \lambda \text{ is not surjective} \} \) denote, the surjectivity spectrum. Let \( \Psi_-(\mathbb{X}) \) be the class of all lower semi-B-Fredholm operators, \( \Psi_-(\mathbb{X}) = \{ S \in \Psi_-(\mathbb{X}) : \text{ind}(S - \lambda) > 0 \} \). The lower semi-B-Weyl spectrum of \( S \) is defined by \( \sigma_{SBF_+}(S) = \{ \lambda \in \mathbb{C} : S - \lambda \notin \Psi_+(S) \} \). Define \( RD(\mathbb{X}) = \{ S \in \mathbb{L}(\mathbb{X}) : dsc(S) = d < \infty \text{ and } \Re(S^{d+1}) \text{ is closed} \} \). The right Drazin invertible is defined by \( \sigma_{RD}(S) = \{ \lambda \in \mathbb{C} : S - \lambda \notin RD(\mathbb{X}) \} \). It is not difficult to see that \( \sigma_D(S) = \sigma_{LD}(S) \cup \sigma_{RD}(S) \). Moreover, \( \sigma_{LD}(S) = \sigma_{RD}(S^*) \) [7]. Then \( S \) satisfies generalized s-Browder’s theorem if \( \sigma_{SBF_+}(S) = \sigma_{RD}(S) \). Apparently, \( S \) satisfies generalized s-Browder’s theorem if and only if \( S^* \) satisfies generalized a-Browder’s theorem. A necessary and sufficient condition for \( S \) to satisfy generalized a-Browder’s theorem is that \( S \) has SVEP at every \( \lambda \in \Delta^a_0(S) \) [12] (Theorem 3.1); by duality, \( S \) satisfies generalized s-Browder’s theorem if and only if \( S^* \) has SVEP at every \( \lambda \in \sigma_a(S) \setminus \sigma_{SBF_+}(S) \). More generally, if either of \( S \) and \( S^* \) has SVEP, then \( S \) and \( S^* \) satisfy both generalized a-Browder’s theorem and generalized s-Browder’s theorem. Either of generalized a-Browder’s theorem and generalized s-Browder’s theorem implies generalized Browder’s theorem, but the converse is false. generalized a-Browder’s theorem fails to transfer from \( A \) and \( B \) to \( A \otimes B \) [13] (Example 1).

Lemma 2.1. Let \( A \in \mathbb{L}(\mathbb{X}) \) and \( B \in \mathbb{L}(\mathbb{Y}) \). Then \( 0 \notin \sigma_a(A \otimes B) \setminus \sigma_{SBF_+}(A \otimes B) \).

Proof. Suppose \( 0 \in \sigma_a(A \otimes B) \setminus \sigma_{SBF_+}(A \otimes B) \). Then \( 0 \in \sigma_a(A \otimes B) \cap \Psi_+(A \otimes B) \). So, there exists an integer \( n_0 \) such that for any \( n \geq n_0 \), \( A \otimes B - \frac{1}{n} I \) has closed range and \( 0 < \alpha \left( A \otimes B - \frac{1}{n} I \right) < \infty \). Since \( A \otimes B - \frac{1}{n} I \) is injective if and only if \( A \) and \( B \) are injective, we have \( \alpha(A) > 0 \) or \( \alpha(B) > 0 \). But then \( \alpha \left( A \otimes B - \frac{1}{n} I \right) = \infty \), and we have a contradiction.

Lemma 2.2. Let \( A \in \mathbb{L}(\mathbb{X}) \) and \( B \in \mathbb{L}(\mathbb{Y}) \). Then

\[ \sigma_{SBF_+}(A \otimes B) \subseteq \sigma_a(A) \sigma_{SBF_+}(B) \cup \sigma_{SBF_+}(A) \sigma_a(B) \subseteq \sigma_a(A) \sigma_{LD}(B) \cup \sigma_{LD}(A) \sigma_a(B) = \sigma_{LD}(A \otimes B) . \]

Proof. Since \( \sigma_{SBF_+}(S) \subseteq \sigma_{LD}(S) \) for every operator \( S \), it follows that the inclusion \( \sigma_a(A) \sigma_{SBF_+}(B) \cup \sigma_{SBF_+}(A) \sigma_a(B) \subseteq \sigma_a(A) \sigma_{LD}(B) \cup \sigma_{LD}(A) \sigma_a(B) \) is evident. To prove the
inclusion $\sigma_{SBF^-}(A \otimes B) \subseteq \sigma_{a}(A)\sigma_{SBF^-}(B) \cup \sigma_{SBF^-}(A)\sigma_{a}(B)$, take $\lambda \notin \sigma_{a}(A)\sigma_{SBF^-}(B) \cup \sigma_{SBF^-}(A)\sigma_{a}(B)$. Since

$$
\sigma_{SBF^+}(A \otimes B) \subseteq \sigma_{a}(A)\sigma_{SBF^+}(B) \cup \sigma_{SBF^+}(A)\sigma_{a}(B),
$$

Lemma 2.1 implies that $\lambda \neq 0$. For every factorization $\lambda = \mu \nu$ such that $\mu \in \sigma_{a}(A)$ and $\nu \in \sigma_{a}(B)$ we have that $\mu \in \sigma_{a}\setminus \sigma_{SBF^-}(A)$ and $\nu \in \sigma_{SBF^-}(B)$, i.e., $\mu \in \Psi_+(A)$, $\nu \in \Psi_+(B)$, $\text{ind}(A - \mu) \leq 0$ and $\text{ind}(B - \nu) \leq 0$. In particular, $\lambda \notin \sigma_{SBF^+}(A \otimes B)$.

We prove next that $\text{ind}(A \otimes B - \lambda) \leq 0$. Suppose $\text{ind}(A \otimes B - \lambda) > 0$. Then there exists an integer $n_0$ such that for any $n \geq n_0$ we have $\alpha \left( A \otimes B - \lambda - \frac{1}{n} I \right) < \infty$. But this implies that

$$
\beta \left( A \otimes B - \lambda - \frac{1}{n} I \right) < \infty,
$$

so that $A \otimes B - \lambda$ is B-Weyl. Let

$$
F = \left\{ (\mu_i, \nu_i)_{i=1}^k \in \sigma(A)\sigma(B); \mu_i\nu_i = \lambda \right\}.
$$

Then $F$ is a finite set. Furthermore

(i) if $m > 1$, then $\mu_i \in \sigma_{iso}(A)$ for $1 \leq i \leq m$;

(ii) if $k > m$, then $\nu_i \in \sigma_{iso}(B)$ for $m + 1 \leq i \leq k$;

(iii) $\text{ind}(A \otimes B - \lambda) = \sum_{j=m+1}^k \text{ind}(A - \mu_i) \dim H_0(B - \nu_i) + \sum_{j=1}^m \text{ind}(B - \nu_i) \dim H_0(A - \mu_i)$.

Since $\text{ind}(A - \mu_i)$ and $\text{ind}(B - \nu_i)$ are non-positive, we have a contradiction. Hence, $\text{ind}(A \otimes B - \lambda) \leq 0$, and consequently, $\lambda \notin \sigma_{SBF^-}(A \otimes B)$. This leaves us to prove the equality $\sigma_{LD}(A \otimes B) = \sigma_{a}(A)\sigma_{LD}(B) \cup \sigma_{LD}(A)\sigma_{a}(B)$.

Suppose that $\lambda \notin \sigma_{LD}(A \otimes B)$. Then $\lambda \neq 0$, $\lambda \in LD(A \otimes B)$, $a = \text{asc}(A \otimes B - \lambda) < \infty$ and $\Re(A \otimes B - \lambda)^{a+1}$ is closed and hence $\lambda \in \pi_{a}(A \otimes B)$. Observe that $\lambda \in \sigma_{iso}(A \otimes B)$. Let $\lambda = \mu \nu$ be any factorization of $\lambda$ such that $\mu \in \sigma_{a}(A)$ and $\nu \in \sigma_{a}(B)$; then $\mu \in LD(A)$ and $\nu \in LD(B)$. Furthermore, since $\sigma_{iso}(A \otimes B) \subseteq \sigma_{a}(A) \cup \sigma_{a}(B) \cup \{0\}$, $A$ has SVEP at $\mu$ and $B$ has SVEP at $\nu. Consequently, $\mu \notin \sigma_{LD}(A)$ and $\nu \notin \sigma_{LD}(B)$.

To prove the reverse inclusion we start by recalling the fact that if $\mu \in \sigma_{iso}(A)$ and $\mu \in \sigma_{iso}(B)$ for every factorization $\lambda = \mu \nu$ of $\lambda \neq 0$, then $\lambda = \mu \nu \in \sigma_{a}(A \otimes B)$. Let $\lambda \in \sigma_{a}(A)\sigma_{LD}(B) \cup \sigma_{LD}(A)\sigma_{a}(B)$. Then $\lambda \neq 0$. Furthermore, if $\lambda = \mu \nu$ is any factorization of $\lambda$ such that $\mu \in \sigma_{a}(A)$ and $\nu \in \sigma_{a}(B)$, then the following implications hold:

$$
\mu \notin \sigma_{LD}(A) \quad \text{and} \quad \nu \notin \sigma_{LD}(B) \Rightarrow \mu \in \pi_{a}(A) \quad \text{and} \quad \nu \in \pi_{a}(B) \Rightarrow
$$

$$
\Rightarrow \lambda \in \pi_{a}(A \otimes B), \mu \in \sigma_{iso}(A) \quad \text{and} \quad \nu \in \sigma_{iso}(B) \Rightarrow
$$

$$
\Rightarrow \lambda \in \pi_{a}(A \otimes B) \quad \text{and} \quad \lambda \in \sigma_{iso}(A \otimes B) \Rightarrow
$$

$$
\Rightarrow \lambda \notin \sigma_{LD}(A \otimes B).
$$

Hence $\sigma_{LD}(A \otimes B) \subseteq \sigma_{a}(A)\sigma_{LD}(B) \cup \sigma_{LD}(A)\sigma_{a}(B)$.

Lemma 2.2 is proved.
Lemma 2.3. Let $A \in \mathcal{L}(\mathcal{X})$ and $B \in \mathcal{L}(\mathcal{Y})$. If $A \otimes B$ satisfies generalized $a$-Browder’s theorem, then

$$\sigma_{SBF^-}(A \otimes B) = \sigma_a(A)\sigma_{SBF^-}(B) \cup \sigma_{SBF^-}(A)\sigma_a(B).$$

Proof. $A \otimes B$ satisfies generalized $a$-Browder’s theorem if and only if $\sigma_{SBF^-}(A \otimes B) = \sigma_{LD}(A \otimes B)$. Thus the stated result is an immediate consequence of Lemma 2.2.

The next theorem, our main result, proves that $A$ and $B$ satisfy generalized $a$-Browder’s theorem implies $A \otimes B$ satisfies generalized $a$-Browder’s theorem if and only if $\sigma_{SBF^-}(A \otimes B) = \sigma_a(A)\sigma_{SBF^-}(B) \cup \sigma_{SBF^-}(A)\sigma_a(B)$.

Theorem 2.1. Let $A \in \mathcal{L}(\mathcal{X})$ and $B \in \mathcal{L}(\mathcal{Y})$. If $A$ and $B$ satisfy generalized $a$-Browder’s theorem, then the following are equivalent:

(i) $A \otimes B$ satisfies generalized $a$-Browder’s theorem;

(ii) $\sigma_{SBF^-}(A \otimes B) = \sigma_a(A)\sigma_{SBF^-}(B) \cup \sigma_{SBF^-}(A)\sigma_a(B)$;

(iii) $A$ has SVEP at every $\mu \in \Psi_+(A)$ and $B$ has SVEP at every $\nu \in \Psi_+(B)$ such that $(0 \neq \lambda) = \mu\nu \in \sigma_a(A \otimes B) \setminus \sigma_{SBF^-}(A \otimes B)$.

Proof. If $A$ and $B$ satisfy generalized $a$-Browder’s theorem, then $\sigma_{LD}(A) = \sigma_{SBF^-}(A)$ and $\sigma_{LD}(B) = \sigma_{SBF^-}(B)$.

(i) $\Rightarrow$ (ii). By Lemma 2.3 we have, without any extra conditions.

(ii) $\Rightarrow$ (i). If (ii) is satisfied, then

$$\sigma_{SBF^-}(A \otimes B) = \sigma_a(A)\sigma_{SBF^-}(B) \cup \sigma_{SBF^-}(A)\sigma_a(B) = \sigma_a(A)\sigma_{LD}(B) \cup \sigma_{LD}(A)\sigma_a(B) = \sigma_{LD}(A \otimes B) \quad \text{(by Lemma 2.2)}.$$

Hence $A \otimes B$ satisfies generalized $a$-Browder’s theorem.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Let $\lambda \in \sigma_a(A \otimes B) \setminus \sigma_{SBF^-}(A \otimes B)$.

Suppose that (ii) is satisfied. Take $\lambda \in \sigma_a(A \otimes B) \setminus \sigma_{SBF^-}(A \otimes B)$ and $	ext{ind}(A \otimes B - \lambda) \leq 0$. The equality $\sigma_{SBF^-}(A \otimes B) = \sigma_a(A)\sigma_{SBF^-}(B) \cup \sigma_{SBF^-}(A)\sigma_a(B)$ implies that for any factorization $\lambda = \mu\nu$ (such that $\mu \in \sigma_a(A)$ and $\nu \in \sigma_a(B)$) we have that $\mu \in \Psi_+(A)$ and $\nu \in \Psi_+(B)$. The SVEP hypotheses on $A$ and $B$ implies that $\text{asc}(A - \mu I)$ and $\text{asc}(B - \lambda)$ are finite. Hence, $\mu \in \sigma_a^{iso}(A)$ and $\mu \in \sigma_a^{iso}(B)$. So, it follows from Theorem 2.8 of [10] that $\mu \in \pi_a(A)$ and $\nu \in \pi_a(B)$. Therefore, $\mu \notin \sigma_{LD}(A)$ and $\nu \notin \sigma_{LD}(B)$. But then $\lambda \notin \sigma_{LD}(A \otimes B)$. Hence $\sigma_{LD}(A \otimes B) \subseteq \sigma_{SBF^-}(A \otimes B)$.

Theorem 2.1 is proved.

The next theorem gives a sufficient condition for $A \otimes B$ to satisfy generalized a-Weyl theorem, given that $A$ and $B$ satisfy generalized a-Weyl theorem. But before that a couple of technical lemmas. Recall that an operator $S$ is said to be $a$-isoloid if $\lambda \in \sigma_a^{iso}(S)$ implies $\lambda \in \sigma_p(S)$. 

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Lemma 2.4. Suppose that $A, B$ and $A \otimes B$ satisfy generalized a-Browder’s theorem. If $\mu \in \pi_0(A)$ and $\nu \in \pi_0(B)$, then $\lambda = \mu \nu \in \pi_0(A \otimes B)$.

Proof. Since $\mu \in \sigma_a(A) \setminus \sigma_{SBF^-}(A)$, $\nu \in \sigma_a(B) \setminus \sigma_{SBF^-}(B)$ and $\sigma_{SBF^-}(A \otimes B) = \sigma_a(A)\sigma_{SBF^-}(B) \cup \sigma_{SBF^-}(A)\sigma_a(B)$. Hence, $\lambda = \mu \nu \in \sigma_a(A \otimes B) \setminus \sigma_{SBF^-}(A \otimes B) = \pi_0(A \otimes B)$.

Theorem 2.2. Suppose that $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ are a-isoloid which satisfy generalized a-Weyl theorem. If $\sigma_{SBF^-}(A \otimes B) = \sigma_a(A)\sigma_{SBF^-}(B) \cup \sigma_{SBF^-}(A)\sigma_a(B)$, then $A \otimes B$ satisfies generalized a-Weyl theorem.

Proof. The hypotheses imply that $A \otimes B$ satisfies generalized a-Browder’s theorem, that is, $\sigma_a(A \otimes B) \setminus \sigma_{SBF^-}(A \otimes B) = \pi_0(A \otimes B)$. Since $\pi_0(A \otimes B) \subseteq E_0(A \otimes B)$, we have to prove that $E_0(A \otimes B) \subseteq \pi_0(A \otimes B)$. Let $\lambda \in E_0(A \otimes B)$. Then $0 \neq \lambda = \mu \nu$ for some $\mu \in \sigma_{aiso}(A)$ and $\nu \in \sigma_{aiso}(B)$. The operators $A$ and $B$ being a-isoloid, it follows from $0 \neq \lambda = \mu \nu \in E_0(A \otimes B)$ that $\mu \in E_0(A) = \pi_0(A)$ and $\nu \in E_0(B) = \pi_0(B)$. By Lemma 2.4, $\lambda \in \pi_0(A \otimes B)$.

Theorem 2.2 is proved.

Following [16], we say that $S \in \mathcal{L}(X)$ satisfies property $(w)$ if $\sigma_a(S) \setminus \sigma_{aw}(S) = E^0(S)$. The property $(w)$ has been studied in [2, 3, 4, 16]. In [3] (Theorem 2.8), it is shown that property $(w)$ implies Weyl’s theorem, but the converse is not true in general. An operator $S \in \mathcal{L}(X)$ is said to be a-Browder’s theorem transfers from isoloid $A$ and $B$ to $A \otimes B$. But before that a lemma and some observations, which will often be used in the sequel. Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$. Then $\sigma_{aiso}(A \otimes B) \subseteq \sigma_{aiso}(A) \cup \sigma_{aiso}(B)$. If 0 is in the point spectrum of either of $A$ and $B$, then $\sigma_a(A \otimes B) = 0$; in particular, $0 \notin E(A \otimes B)$. It is easily seen, see the argument of the proof of [15] (Proposition 2), that $E(A \otimes B) \subseteq E(A)E(B)$.

Theorem 2.3. If $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ are isoloid operators which satisfy property $(gw)$, then the following conditions are equivalent:

(i) $A \otimes B$ satisfies property $(gw)$;

(ii) the generalized a-Weyl spectrum equality $\sigma_{SBF^-}(A \otimes B) = \sigma_a(A)\sigma_{SBF^-}(B) \cup \sigma_{SBF^-}(A)\sigma_a(B)$ is satisfied;

(iii) $A \otimes B$ satisfies generalized a-Browder’s theorem.

Proof. Since property $(gw)$ implies generalized a-Browder’s theorem, the equivalence (ii) \iff (iii) and (i) \implies (ii) follows from Theorem 2.2. We prove (iii) \implies (i). The hypothesis $A$ and $B$ satisfy property $(gw)$ implies $\sigma_a(A) \setminus \sigma_{SBF^-}(A) = E(A)$ and $\sigma_a(B) \setminus \sigma_{SBF^-}(B) = E(B)$.

Observe that (iii) implies generalized a-Browder’s theorem transfers from $A$ and $B$ to $A \otimes B$: hence $\sigma_{SBF^-}(A \otimes B) = \sigma_a(A)\sigma_{SBF^-}(B) \cup \sigma_{SBF^-}(A)\sigma_a(B)$. Let $\lambda \in E(A \otimes B)$; then $\lambda \neq 0$ and hence there exist $\mu \in \sigma_{aiso}(A)$ and $\nu \in \sigma_{aiso}(B)$ such that $\lambda = \mu \nu$. By hypotheses $A$ and $B$ are isoloid; hence $\mu$ is an eigenvalue of $A$ and $\nu$ is an eigenvalue of $B$. Hence $\mu \in E(A) = \sigma_a(A) \setminus \sigma_{SBF^-}(A)$ and $\nu \in E(B) = \sigma_a(B) \setminus \sigma_{SBF^-}(B)$. Consequently, $\lambda \in \sigma_a(A \otimes B) \setminus \sigma_{SBF^-}(A \otimes B)$; hence
$E(A \otimes B) \subseteq \sigma_a(A \otimes B) \setminus \sigma_{SBF^-}(A \otimes B)$. Conversely, if $\lambda \in \sigma_a(A \otimes B) \setminus \sigma_{SBF^-}(A \otimes B)$, then $\lambda \neq 0$. So, there exist $\mu \in \sigma_a(A) \setminus \sigma_{SBF^-}(A) = E(A)$ and $\nu \in \sigma_a(B) \setminus \sigma_{SBF^-}(B)$ such that $\lambda = \mu \nu$. But then $\lambda \in E(A \otimes B)$. Hence $\sigma_a(A \otimes B) \setminus \sigma_{SBF^-}(A \otimes B) \subseteq E(A \otimes B)$. Therefore, the proof is achieved.

An operator $S \in \mathcal{L}(\mathcal{X})$ is said to be polaroid (respectively, a-polaroid) if $\sigma^{iso}(S)$ (respectively, $\sigma^{iso}_a(S)$) is empty or every isolated point of $\sigma(S)$ (respectively, $\sigma_a(S)$) is a pole of the resolvent. $S \in \mathcal{L}(\mathcal{X})$ is polaroid implies $S^*$ polaroid. It is well known that if $S$ or $S^*$ has SVEP and $S$ is polaroid, then $S$ and $S^*$ satisfy generalized Weyl’s theorem. Not as well known is the fact [6] (Theorem 2.10), that if $S$ is polaroid and $S^*$ (respectively, $S$) has SVEP, then $S$ (respectively, $S^*$) satisfies property $(gw)$. Here the SVEP hypotheses on $S$ and $S^*$ can not be exchanged. The following theorem is the tensor product analogue of this result.

**Theorem 2.4.** Suppose that the operators $A \in \mathcal{L}(\mathcal{X})$ and $B \in \mathcal{L}(\mathcal{Y})$ are polaroid.

(i) If $A^*$ and $B^*$ have SVEP, then $A \otimes B$ satisfies property $(gw)$.

(ii) If $A$ and $B$ have SVEP, then $A^* \otimes B^*$ satisfies property $(gw)$.

**Proof.** (i) The hypothesis $A^*$ and $B^*$ have SVEP implies

$$\sigma(A) = \sigma_a(A), \quad \sigma(B) = \sigma_a(B), \quad \sigma_{SBF^+}(A) = \sigma_{SBF^+}(A), \quad \sigma_{SBF^-}(B) = \sigma_{BW}(B)$$

and

$$A^*, \ B^* \ and \ A^* \otimes B^* \ satisfy \ generalized \ s-Browder’s \ theorem.$$ 

Thus generalized s-Browder’s theorem and generalized Browder’s theorem transform from $A^*$ and $B^*$ to $A^* \otimes B^*$. Hence

$$\sigma_{SBF^+}(A \otimes B) = \sigma_{SBF^+}(A^* \otimes B^*) = \sigma_s(A^*)\sigma_{SBF^+}(B^*) \cup \sigma_{SBF^-}(A^*)\sigma_s(B^*) =$$

$$= \sigma_a(A)\sigma_{SBF^+}(B) \cup \sigma_{SBF^-}(A)\sigma_a(B) = \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B),$$

and

$$\sigma_{BW}(A \otimes B) = \sigma_{BW}(A^* \otimes B^*) = \sigma(A^*)\sigma_{BW}(B^*) \cup \sigma_{BW}(A^*)\sigma(B^*) =$$

$$= \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B).$$

Consequently,

$$\sigma_{SBF^-}(A \otimes B) = \sigma_{BW}(A \otimes B).$$

Already,

$$\sigma_a(A \otimes B) = \sigma_a(A)\sigma_a(B) = \sigma(A)\sigma(B) = \sigma(A \otimes B).$$

Evidently, $A \otimes B$ is polaroid by Lemma 2 of [14]; combining this with $A \otimes B$ satisfies generalized Browder’s theorem, it follows that $A \otimes B$ satisfies generalized Weyl’s theorem, i.e., $\sigma(A \otimes B) \setminus \sigma_{BW}(A \otimes B) = E(A \otimes B)$. It follows then

$$\sigma_a(A \otimes B) \setminus \sigma_{SBF^-}(A \otimes B) = \sigma(A \otimes B) \setminus \sigma_{BW}(A \otimes B) = E(A \otimes B),$$

that is, $A \otimes B$ satisfies property $(gw)$. 

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(ii) In this case \( \sigma(A) = \sigma_a(A^*), \sigma(B) = \sigma_a(B^*), \sigma_{BW}(A^*) = \sigma_{SBF_+}(A^*), \sigma_{BW}(B^*) = \sigma_{SBF_+}(B^*), \sigma(A^* \otimes B^*) = \sigma_a(A^* \otimes B^*), \) both generalized Browder’s theorem and generalized s-Browder’s theorem transfer from \( A \) and \( B \) to \( A \otimes B \). Hence

\[
\sigma_{SBF_+}(A^* \otimes B^*) = \sigma_{SBF_+}(A \otimes B) = \sigma_s(A)\sigma_{SBF_+}(B) \cup \sigma_{SBF_+}(A)\sigma_s(B) = \\
= \sigma_a(A^*)\sigma_{SBF_+}(B^*) \cup \sigma_{SBF_+}(A^*)\sigma_a(B^*) = \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B) = \\
= \sigma_{BW}(A \otimes B) = \sigma_{BW}(A^* \otimes B^*).
\]

Thus, since \( A^* \otimes B^* \) polaroid and \( A \otimes B \) satisfies generalized Browder’s theorem imply \( A^* \otimes B^* \) satisfy generalized Weyl’s theorem,

\[
\sigma_a(A^* \otimes B^*) \setminus \sigma_{SBF_+}(A^* \otimes B^*) = \sigma(A^* \otimes B^*) \setminus \sigma_{BW}(A^* \otimes B^*) = E(A^* \otimes B^*),
\]

that is, \( A^* \otimes B^* \) satisfies property (gw).

Theorem 2.4 is proved.


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