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COMPARISON THEOREMS AND NECESSARY/SUFFICIENT CONDITIONS FOR EXISTENCE OF NONOSCILLATORY SOLUTIONS OF FORCED IMPULSIVE DELAY DIFFERENTIAL EQUATIONS

ТЕОРЕМИ ПОРІВНЯННЯ ТА НЕОБХІДНІ/ДОСТАТНІ УМОВИ ІСНУВАННЯ НЕОСЦИЛЯЦІЙНИХ РОЗВ'ЯЗКІВ ЗБУРЕНИХ ІМПУЛЬСНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ІЗ ЗАПІЗНЕННЯМ

In 1997, A. H. Nasr provided necessary and sufficient conditions for the oscillation of the equation

$$x''(t) + p(t)\,|x(g(t))|^{\eta}\,\mathrm{sgn}\,(x(g(t))) = e(t),$$

where $\eta > 0$, p, and g are continuous functions on $[0, \infty)$ such that $p(t) \geq 0$, $g(t) \leq t$, $g'(t) \geq \alpha > 0$, and $\lim_{t\to\infty} g(t) = \infty$. It is important to note that the condition $g'(t) \geq \alpha > 0$ is required. In this paper, we remove this restriction under the superlinear assumption $\eta > 1$. Infact, we can do even better by considering impulsive differential equations with delay and obtain necessary and sufficient conditions for the existence of nonoscillatory solutions and also a comparison theorem that enables us to apply known oscillation results for impulsive equations without forcing terms to yield oscillation criteria for our equations.

У 1997 році, А. Х. Наср отримав необхідні та достатні осциляційні умови для рівняння

$$x''(t) + p(t)\,|x(g(t))|^{\eta}\,\mathrm{sgn}\,(x(g(t))) = e(t),$$

де $\eta > 0$, p та g — неперервні функції на $[0, \infty)$ такі, що $p(t) \geq 0$, $g(t) \leq t$, $g'(t) \geq \alpha > 0$ та $\lim_{t\to\infty} g(t) = \infty$. Слід зауважити, що необхідною тут є умова $g'(t) \geq \alpha > 0$. У даній статті ми усуваємо це обмеження при суперлінійному припущенні $\eta > 1$. Насправді, можна отримати навіть кращий результат, розглядаючи імпульсні диференціальні рівняння з запізненням, і встановити необхідні та достатні умови існування неосциляційних розв'язків, а також теорему порівняння, яка дає змогу застосувати відомі осциляційні результати для імпульсних рівнянь без збурюючих членів, щоб отримати осциляційні критерії для наших рівнянь.

1. Introduction. In 1997, A. H. Nasr in [1] provided necessary and sufficient conditions for the oscillation of the equation

$$x''(t) + p(t)\,|x(g(t))|^{\eta}\,\mathrm{sgn}\,(x(g(t))) = e(t), \tag{1}$$

where $\eta > 0$, p and g are continuous functions on $[0, \infty)$ such that $p(t) \geq 0$, $g(t) \leq t$, $g'(t) \geq \alpha > 0$ and $\lim_{t\to\infty} g(t) = \infty$. Under a nice assumption on the function e (that the solution z of (5) is oscillatory and (22) holds), it is stated in reference [1] that the following conclusions hold: for $\eta > 1$, equation (1) is oscillatory if, and only if, $\int_0^{\infty} tp(t)dt = \infty$; and for $0 < \eta < 1$, equation (1) is oscillatory if, and only if $\int_0^{\infty} t^{r}p(t)dt = \infty$. These conclusions extend those in [2] in which the well known Emden – Fowler equation without delay is studied.

It is important to note that the condition "$g'(t) \geq \alpha > 0$" is needed in [1]. However, in [3], the author removes the restriction for the sublinear case $0 < \eta < 1$. In this paper, we intend to improve the same restriction for the superlinear case $\eta > 1$ (see Corollary 3, or Theorems 1 and 4 below).

Indeed, we can do even better by considering impulsive differential equations with delay. More specifically, we obtain necessary and sufficient conditions for the existence of nonoscillatory solutions and also a comparison theorem which enables us to apply known oscillation results (see, e.g., [4]) for impulsive equations without forcing terms to yield oscillation criteria for our equations (an example is illustrated in the last section).

To this end, we first recall some usual notations. $\mathbf{R}$ and $\mathbf{N}$ denote the set of real numbers and positive integers respectively. $\mathbf{R}^+$ denotes the interval $(0, +\infty)$. Assume I_1 and I_2 are any two intervals in $\mathbf{R}$, we define

$$ALC(I_1, I_2) = \big\{\varphi\colon I_1 \to I_2\colon \varphi \ \text{ is continuous almost everywhere (a.e.) in } \ I_1$$

$$\text{with discontinuities of first kind}\big\},$$

$$PC(I_1, I_2) = \big\{\varphi \in ALC(I_1, I_2)\colon \varphi \ \text{ is continuous in each interval } \ I_1 \cap (t_k, t_{k+1}], \quad k \in \mathbf{N}_0\big\},$$

and

$$PC'(I_1, I_2) = \big\{\varphi \in PC(I_1, I_2)\colon \varphi \text{ is continuously differentiable a.e. in } \ I_1\big\}.$$

We let

$$\Upsilon = \{t_1, t_2, \ldots\}$$

be a set of real numbers such that $0 = t_0 < t_1 < t_2 < \ldots$ and $\lim_{k\to\infty} t_k = +\infty$. Also, $x'(t)$ will be used to denote the left derivative of the function $x(t)$ at t. We investigate the following nonlinear delay differential systems 'with impulsive effects'

$$\big(r(t)x'(t)\big)' + F(t, x(g(t))) = e(t), \quad t \in [0, \infty)\backslash\Upsilon, \tag{2}$$

$$x(t_k^+) = a_k x(t_k), \quad k \in \mathbf{N}, \tag{3}$$

$$x'(t_k^+) = b_k x'(t_k), \quad k \in \mathbf{N}, \tag{4}$$

under some of the following conditions:

(A1) For $t \geq 0$, the function $F(t, \mu)$ is continuous on $\mathbf{R}$ with $\mu F(t, \mu) \geq 0$ for $\mu \neq 0$, and for $\mu \in \mathbf{R}$, the function $F(t, \mu)$ belongs to $ALC([0, \infty), \mathbf{R})$. Furthermore, $F(t, \mu_2) \geq F(t, \mu_1)$ for $t \geq 0$ and $\mu_2 \geq \mu_1$;

(A2) g is a continuous function on $[0, \infty)$ with $g(t) \leq t$ for $t \geq 0$ and $\lim_{t\to\infty} g(t) = +\infty$;

(A3) $0 < t_1 < t_2 < \ldots$ are fixed numbers with $\lim_{k\to\infty} t_k = +\infty$;

(A4) for each $k \in \mathbf{N}$, $a_k > 0$ and $b_k > 0$;

(A5) r is a positive and differentiable function on $[0, \infty)$;

(A6) e is a function on $[0, \infty)$ continuous a.e.;

(A7) there are $M > 0$ and $m > 0$ such that $m \leq A(s, t) \leq M$ for $t \geq s \geq 0$ where

$$A(s, t) = \begin{cases} \prod_{s \leq t_k < t} a_k & \text{if } \ [s, t) \cap \Upsilon \neq \varnothing, \\ 1 & \text{if } \ [s, t) \cap \Upsilon = \varnothing. \end{cases}$$

Let $\sigma \geq 0$ be given. We define $r_\sigma = \min_{t\geq\sigma} g(t)$ and

$$B(s,t) = \begin{cases} \prod_{s\leq t_k<t} b_k & \text{if } [s,t)\cap\Upsilon \neq \varnothing, \\ 1 & \text{if } [s,t)\cap\Upsilon = \varnothing \end{cases}$$

for $t \geq s \geq 0$.

Definition 1. *Let $\sigma \geq 0$. For any $\phi \in PC'([r_\sigma,\sigma],\mathbf{R})$, a function $x \in PC'([r_\sigma,\infty),\mathbf{R})$ is said to be a solution of system* (2)–(4) *on $[\sigma,\infty)$ satisfying the initial value condition*

$$x(t) = \phi(t), \quad t \in [r_\sigma,\sigma],$$

if the following conditions are satisfied:

(i) $x' \in PC'([\sigma,\infty),\mathbf{R})$;

(ii) *x satisfies* (2) *for a.e. $t \geq \sigma$*;

(iii) *x satisfies* (3) *and* (4) *for $t \geq \sigma$.*

Definition 2. *Let $x = x(t)$ be a real function defined for all sufficiently large t. We say that x is eventually positive* (*or negative*) *if there exists a number T such that $x(t) > 0$* (*respectively $x(t) < 0$*) *for every $t \geq T$. We say that x is nonoscillatory if $x(t)$ is eventually positive or eventually negative. Otherwise, x is said to be oscillatory.*

In the subsequent discussions, we assume that there exists a solution z of the system

$$\begin{aligned} (r(t)z'(t))' &= e(t), \quad t \in [0,\infty)\setminus\Upsilon, \\ z(t_k^+) &= a_k z(t_k), \quad k \in \mathbf{N}, \\ z'(t_k^+) &= b_k z'(t_k), \quad k \in \mathbf{N}, \end{aligned} \tag{5}$$

on $[\tau,\infty)$ for some $\tau \geq 0$. Let $T \geq 0$ and $\varphi \in PC([r_T,T],\mathbf{R})$. For $\delta \in PC([T,\infty),\mathbf{R})$, we define a function $w_\varphi(\delta)$ by

$$w_\varphi(\delta)(t) = \begin{cases} \delta(g(t)) & \text{if } g(t) > T, \\ \varphi(g(t)) & \text{if } r_T \leq g(t) \leq T \end{cases}$$

for $t \geq T$.

This paper is mainly concerned with oscillation of impulsive differential equations, but for more general background material, the reader is referred to [5–8].

2. Main results. We begin with a simple comparison principle.

Lemma 1. *Assume that* (A1)–(A6) *hold, that the solution z of* (5) *is oscillatory*, *and*

$$\int_{t_0}^{\infty} \frac{B(t_0,s)}{A(t_0,s)r(s)}ds = \infty \tag{6}$$

for any $t_0 \geq 0$. Let x be an eventually positive solution of system (2)–(4). *Then $x(t) > z(t)$ and $x'(t) \geq z'(t)$ eventually.*

Proof. Without loss of generality, we may assume that $\tau = 0$, and $x(t) > 0$ for $t \geq r_0$. Let $y(t) = x(t) - z(t)$ for $t \geq 0$. So the function y satisfies

$$\big(r(t)y'(t)\big)' + F(t, x(g(t))) = 0, \quad \text{a.e.} \quad t \geq 0, \tag{7}$$

and

$$y(t_k^+) = a_k y(t_k) \qquad \text{and} \qquad y'(t_k^+) = b_k y'(t_k), \quad k \in \mathbf{N}. \tag{8}$$

By (7), we see that $(r(t)y'(t))' \leq 0$ for a.e. $t \geq 0$. Assume that there exists $T > 0$ such that $(r(T)y'(T))'$ exists and $y'(T) < 0$. By (8),

$$r(t)y'(t) \leq B(T,t)r(T)y'(T) < 0, \quad \text{a.e.} \quad t \geq T. \tag{9}$$

Dividing (9) by $r(t)A(T,t)$, and then integrating the subsequent inequalities from T to t, we obtain

$$y(t) \leq A(T,t)\left(y(T) + r(T)'y'(T)\int\limits_T^t \frac{B(T,s)}{A(T,s)r(s)}ds\right), \quad t \geq T.$$

In view of (6), $y(t) < 0$ eventually. This is a contradiction since $x(t) > 0$ eventually. So $y'(t) \geq 0$ eventually. We note that it is impossible that $y(t) \leq 0$ eventually because of $x(t) > 0$ eventually. So there exists sufficiently large T_2 such that $y(T_2) > 0$, then

$$y(t) \geq A(T_2, t)y(T_2) > 0, \quad t \geq T_2,$$

which implies that $y(t) > 0$ eventually.

Lemma 1 is proved.

***Remark* 1.** If $e(t) = 0$ eventually, we may assume without loss of generality that the function z is the trivial function. By Lemma 1, we may see that the derivative of any eventually positive solution of system (2)–(4) is eventually nonnegative.

Theorem 1. *Assume that* (A1)–(A7) *hold, that the solution z of* (5) *is bounded, and that*

$$\int\limits_\varepsilon^\infty \frac{1}{r(s)}\int\limits_s^\infty \frac{F(v,c)}{B(s,v)}dvds < \infty \tag{10}$$

for any $c > 0$ and some $\varepsilon \geq 0$. Then system (2)–(4) *has an eventually positive solution x which is bounded.*

Proof. Without loss of generality, we may assume that $|z(t)| \leq M$ for $t > \tau$. Let

$$d = M\left(\frac{M+1}{m} + 2\right).$$

Clearly, $d > 1$. By (10), there exists $T \in \Upsilon$ such that $T > \max\{\tau, \varepsilon\}$ and

$$\int\limits_T^\infty \frac{1}{r(s)}\int\limits_s^\infty \frac{F(v,d)}{B(s,v)}dvds \leq 1. \tag{11}$$

Let

$$X_1 = \big\{\delta \in PC\,([T,\infty),\mathbf{R})\colon 1 \le \delta(t) \le d \ \text{ for } \ t \ge T\big\}. \tag{12}$$

Let $\varphi(t) = 1$ for $r_T \le t \le T$. We define the operator H_1 on X_1 by

$$H_1(\delta)(t) = A(T,t)\frac{M+1}{m} + z(t) + \int\limits_T^t \frac{A(s,t)}{r(s)} \int\limits_s^\infty \frac{F(v, w_\varphi(\delta)(v))}{B(s,v)} dvds \tag{13}$$

for $t \ge T$. Obviously, $H_1(\delta) \in PC([T,\infty),\mathbf{R})$. By the definition of $w_\varphi(\delta)$, we may see that $1 \le w(\delta)(t) \le d$ for $t \ge T$. By (11),

$$H_1(\delta)(t) \ge m\frac{M+1}{m} - M = 1$$

and

$$H_1(\delta)(t) \le M\frac{M+1}{m} + M + M\int\limits_T^t \frac{1}{r(s)}\int\limits_s^\infty \frac{F(v,d)}{B(s,v)}dvds \le M\left(\frac{M+1}{m} + 2\right) = d \tag{14}$$

for $t \ge T$. So $H_1(X_1) \subseteq X_1$. We shall use the Knaster – Tarski fixed point theorem to prove that H_1 has a fixed point in X_1. We first define a relation in X_1. If δ_1 and δ_2 belong to X_1, let us say that $\delta_1 \le \delta_2$ if and only if $\delta_1(t) \le \delta_2(t)$ a.e. on $[T,\infty)$. Clearly, X_1 is a complete lattice. Given $\delta_1, \delta_2 \in X_1$ with $\delta_1 \le \delta_2$. Then $w_\varphi(\delta_1)(t) \le w_\varphi(\delta_2)(t)$ for a.e. $t \ge T$, which follows that $F(t, w_\varphi(\delta_1)(t)) \le F(t, w_\varphi(\delta_2)(t))$ for a.e. $t \ge T$. Then $H_1(\delta_1)(t) \le H_1(\delta_2)(t)$ for a.e. $t \ge T$. So H_1 is increasing in X_1. By the Knaster – Tarski fixed point theorem, there exists $\theta_1 \in X_1$ such that $H_1(\theta_1) = \theta_1$. Let

$$x(t) = \begin{cases} \theta_1(t) & \text{if} \quad t > T, \\ 1 & \text{if} \quad T \ge t \ge r_T. \end{cases}$$

Clearly, $x \in PC'([T,\infty),[1,d])$. Let $T_1 > 0$ such that $r_{T_1} > T$. We note that $x(g(t)) = w_\varphi(\theta_1)(t)$ for $t \ge T_1$. In view of $H_1(\theta_1) = \theta_1$, we have

$$x'(t) = z'(t) + \frac{B(T_1,t)}{r(t)}\int\limits_t^\infty \frac{F(s, x(g(s)))}{B(T_1,s)}ds, \quad t \ge T_1, \tag{15}$$

which leads us to $x' \in PC'([T_1,\infty),\mathbf{R})$ and

$$\big(r(t)x'(t)\big)' + F\big(t, x(g(t))\big) = e(t), \quad \text{a.e.} \quad t > T_1.$$

For any $t_k \ge T_1$, by $H_1(\theta_1) = \theta_1$ and (15),

$$x(t_k^+) = a_k x(t_k) \qquad \text{and} \qquad x'(t_k^+) = b_k x'(t_k).$$

Then x is a bounded and positive solution of system (2) – (4) on $[T_1,\infty)$.

Theorem 1 is proved.

As a direct consequence, we have the following dual conclusion.

***Corollary* 1.** *Assume that* (A1) – (A7) *hold, that the solution* z *of* (5) *is bounded, and that*

$$\int_{\varepsilon}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} \frac{F(v,\tau)}{B(s,v)} dvds > -\infty \tag{16}$$

for any $\tau < 0$ *and some* $\varepsilon \geq 0$. *Then system* (2) – (4) *has an eventually negative solution* x *which is bounded.*

Lemma 2. *Assume that* (A1) – (A6) *hold. If the system*

$$\big(r(t)u'(t)\big)' + F(t, u(g(t))) \leq 0, \quad t \in [0,\infty)\backslash\Upsilon,$$

$$u(t_k^+) = a_k u(t_k), \quad k \in \mathbf{N}, \tag{17}$$

$$u'(t_k^+) = b_k u'(t_k), \quad k \in \mathbf{N},$$

has an eventually positive solution u *with* $u(t)u'(t) \geq 0$ *eventually, then*

$$\big(r(t)u'(t)\big)' + F(t, u(g(t))) = 0, \quad t \in [0,\infty)\backslash\Upsilon, \tag{18}$$

$$u(t_k^+) = a_k u(t_k), \quad k \in \mathbf{N}, \tag{19}$$

$$u'(t_k^+) = b_k u'(t_k), \quad k \in \mathbf{N}, \tag{20}$$

has an eventually positive solution solution $\widetilde{u}$ *with* $\widetilde{u}(t)\,\widetilde{u}'(t) \geq 0$ *eventually.*

Proof. There is $T > 0$ such that $u(t) > 0$ and $u'(t) \geq 0$ for $t \geq r_T$. For any $d \geq t \geq T$, we divide (17) by $B(T,t)$, and then integrate from t to d. We have

$$\frac{r(d)u'(d)}{B(T,d)} - \frac{r(t)u'(t)}{B(T,t)} + \int_{t}^{d} \frac{F(s,u(g(s)))}{B(T,s)} ds \leq 0.$$

Since $u'(d) \geq 0$ and d is arbitrary, we may see that

$$u'(t) \geq \frac{1}{r(t)} \int_{t}^{\infty} \frac{F(s,u(g(s)))}{B(t,s)} ds$$

for $t \geq T$. Again, we divide the above inequality by $A(T,t)$ and then integrate from T to t. Then we have

$$u(t) \geq A(T,t) \left(u(T) + \int_{T}^{t} \frac{1}{A(T,s)r(s)} \int_{s}^{\infty} \frac{F(v,u(g(v)))}{B(s,v)} dvds \right) \tag{21}$$

for $t \geq T$. Let

$$X_2 = \big\{ \delta \in PC([T,\infty),\mathbf{R}) \colon A(T,t)u(T) \leq \delta(t) \leq u(t) \ \text{ for } \ t \geq T \big\}$$

and we define an operator H_2 on X_2 by

$$H_2(\delta)(t) = A(T,t)\left(u(T) + \int\limits_T^t \frac{1}{A(T,s)r(s)} \int\limits_s^\infty \frac{F(v, w_u(\delta)(v)))}{B(s,v)} dvds\right)$$

for $t \geq T$. Clearly, X_2 is nonempty because $u \in X_2$. We impose in X_2 the same order relation imposed in the set X_1. Then X_2 is a complete lattice. Given $\delta_1, \delta_2 \in X_2$ with $\delta_1 \leq \delta_2$. We note that

$$0 < w_u(\delta_1)(t) \leq w_u(\delta_2)(t) \leq u(g(t))$$

for $t \geq T$. By assumption,

$$F(t, w_u(\delta_1)(t)) \leq F(t, w_u(\delta_2)(t)) \leq F(t, u(g(t))),$$

where $t \geq T$. By (21),

$$A(T,t)u(T) \leq H_2(\delta_1)(t) \leq H_2(\delta_2)(t) \leq u(t)$$

for $t \geq T$. So $H_2(X_2) \subseteq X_2$ and H_2 is increasing on X_2. By the Knaster – Tarski fixed point Theorem, there exists $\theta_2 \in X_2$ such that $H_2(\theta_2) = \theta_2$. Let

$$\widetilde{\mu}(t) = \begin{cases} \theta_2(t) & \text{if} \quad t > T, \\ u(t) & \text{if} \quad T \geq t \geq r_T. \end{cases}$$

Similar to the proof of Theorem 1, we may check that $\widetilde{\mu}$ is an eventually positive solution of system (18) – (20) with $\widetilde{\mu}(t)\widetilde{\mu}'(t) \geq 0$ eventually.

Lemma 2 is proved.

We now compare forced impulsive equations and unforced impulsive equations.

Theorem 2. *Assume that* (A1) – (A6) *and* (6) *hold and the solution z of* (5) *is oscillatory. Assume that there exist two sequences $\{s_n\}_{n\in\mathbf{N}}$ and $\{\widetilde{s}_n\}_{n\in\mathbf{N}}$ such that*

$$z(s_n) = \inf\left\{\frac{z(t)}{A(s_n,t)} : t \geq s_n\right\} \qquad \textit{and} \qquad z(\widetilde{s}_n) = \sup\left\{\frac{z(t)}{A(\widetilde{s}_n,t)} : t \geq \widetilde{s}_n\right\}. \tag{22}$$

If the system (2) – (4) *has a nonoscillatory solution x, then the system*

$$\big(r(t)u'(t)\big)' + F(t, u(g(t))) = 0, \quad t \in [0,\infty)\backslash\Upsilon, \tag{23}$$

$$u(t_k^+) = a_k u(t_k), \quad k \in \mathbf{N}, \tag{24}$$

$$u'(t_k^+) = b_k u'(t_k), \quad k \in \mathbf{N}, \tag{25}$$

has a nonoscillatory solution u such that $u(t)u'(t) \geq 0$ eventually. Furthermore, $u(t)$ is bounded if $x(t)$ is bounded and (A7) *holds.*

Proof. We first assume that x is an eventually positive solution of (2) – (4). By Lemma 1, $x(t) > z(t)$ and $x'(t) \geq z'(t)$ eventually. Without loss of generality, we may assume that $\tau = 0$, $x(t) > 0$, $x(t) > z(t)$ and $x'(t) \geq z'(t)$ for $t \geq s_1$. Let $y(t) = x(t) - z(t)$ and $v(t) = y(t) + A(s_1,t)z(s_1)$ for $t \geq s_1$. Clearly, $v, v' \in PC'([s_1,\infty),\mathbf{R})$. By Lemma 1, we see that $y'(t) \geq 0$ for $t \geq s_1$. So $y(t) \geq A(s_1,t)y(s_1)$ for $t \geq s_1$, from which it follows that

$$v(t) = y(t) + A(s_1, t)z(s_1) \geq A(s_1, t)\left(y(s_1) + z(s_1)\right) = A(s_1, t)x(s_1) > 0 \tag{26}$$

for $t \geq s_1$. We note that

$$v'(t) = y'(t) \geq 0, \tag{27}$$

and by (22)

$$x(t) = y(t) + z(t) \geq y(t) + A(s_1, t)z(s_1) = v(t) \tag{28}$$

for $t \geq s_1$. In view of (A3), there exists $T > s_1$ such that $r_T > s_1$. By (28), then $x(g(t)) \geq v(g(t)) >$ > 0 and

$$F(t, x(g(t))) \geq F(t, v(g(t)))$$

for $t \geq T$. By (27) and (28), we see that

$$\left(r(t)v'(t)\right)' + F(t, v(g(t))) \leq \left(r(t)y'(t)\right)' + F(t, x(g(t))) = 0, \quad \text{a.e.} \quad t \geq T.$$

We observe that $v(t_k^+) = a_k v(t_k)$ and $v'(t_k^+) = b_k v'(t_k)$ for $t_k \geq T$. In view of (26), the function v is an eventually positive solution of system

$$\left(r(t)u'(t)\right)' + F(t, u(g(t))) \leq 0, \quad t \in [0, \infty)\backslash\Upsilon,$$
$$u(t_k^+) = a_k u(t_k), \quad k \in \mathbf{N}, \tag{29}$$
$$u'(t_k^+) = b_k u'(t_k), \quad k \in \mathbf{N},$$

such that $v'(t) \geq 0$ for $t \geq T$. By Lemma 2, the system (23)–(25) has an eventually positive solution u such that $v(t) \geq u(t)$ and $u'(t) \geq 0$ eventually. Assume that x is bounded and (A7) holds. By (22), we note that the function z is bounded. Then the function v is bounded. So the function u is also bounded.

Lastly, we assume that x is an eventually negative solution of (2)–(4). Let $G(t, x) = -F(t, -x)$ for $x \in \mathbf{R}$. Let $\widetilde{x}(t) = -x(t)$ and $\widetilde{z}(t) = -z(t)$ for sufficiently large t. Then $\widetilde{x}$ is an eventually positive solution

$$\left(r(t)x'(t)\right)' + G(t, x(g(t))) = -e(t), \quad t \in [0, \infty)\backslash\Upsilon,$$
$$x(t_k^+) = a_k x(t_k), \quad k \in \mathbf{N},$$
$$x'(t_k^+) = b_k x'(t_k), \quad k \in \mathbf{N}.$$

By (22), we note that

$$\widetilde{z}(\widetilde{s}_n) = -z(\widetilde{s}_n) = -\sup\left\{\frac{z(t)}{A(\widetilde{s}_n, t)} : t \geq \widetilde{s}_n\right\} = \inf\left\{\frac{\widetilde{z}(t)}{A(\widetilde{s}_n, t)} : t \geq \widetilde{s}_n\right\},$$

for $n \in \mathbf{N}$. By the former case, we see that the system

$$\left(r(t)u'(t)\right)' + G(t, u(g(t))) = 0, \quad t \in [0, \infty)\backslash\Upsilon,$$

$$u(t_k^+) = a_k u(t_k), \quad k \in \mathbf{N},$$

$$u'(t_k^+) = b_k u'(t_k), \quad k \in \mathbf{N},$$

has an eventually positive solution $\widetilde{u}(t)$ such that $\widetilde{u}'(t) \geq 0$ eventually. Furthermore, $\widetilde{u}$ is bounded if $\widetilde{x}$ is bounded and (A7) holds. Let $u(t) = -\widetilde{u}(t)$ for sufficiently large t. So the system (23)–(25) has an eventually negative solution u such that $u'(t) \leq 0$ eventually, and u is bounded if x is bounded and (A7) holds.

Theorem 2 is proved.

Theorem 3. *Assume that* (A1)–(A7) *hold and the solution* z *of* (5) *satisfies* $\lim_{t\to\infty} z(t) = 0$. *If the system* (23)–(25) *has a nonoscillatory solution* u *with* $u(t)u'(t) \geq 0$ *eventually*, *then the system* (2)–(4) *has a nonoscillatory solution* x.

Proof. Assume that u is an eventually positive solution. Without loss of generality, we may assume that $u(t) > 0$ and $u'(t) \geq 0$ for $t \geq r_0$. Then

$$u(\overline{T}) \geq A(0,\overline{T})u(0) \geq mu(0) \qquad \text{for any} \quad \overline{T} \geq 0. \tag{30}$$

Since $\lim_{t\to\infty} z(t) = 0$, there exists $T > 0$ such that $|z(t)| < m^2 u(0)/3$ for $t \geq T$. By the same reasoning for obtaining the inequality (21), we have

$$u(t) \geq A(T,t)\left(u(T) + \int_T^t \frac{1}{r(s)} \int_s^\infty \frac{F(v, u(g(v)))}{B(s,v)} dvds\right) \geq m^2 u(0) \tag{31}$$

for $t \geq 0$. Let

$$X_3 = \left\{\delta \in PC\left([T,\infty), \mathbf{R}\right) \colon m^2 \frac{u(0)}{3} \leq \delta(t) \leq u(t) \ \text{ for } \ t \geq T\right\}.$$

Clearly, X_3 is nonempty because $u \in X_3$. We impose in X_3 the same order relation imposed in X_1. Then X_3 is a complete lattice. We define an operator H_3 on X_3 by

$$H_3(\delta)(t) = A(T,t)\frac{2u(T)}{3} + z(t) + \int_T^t \frac{A(s,t)}{r(s)} \int_s^\infty \frac{F(v, w_u(\delta)(v))}{B(s,v)} dvds$$

for $t \geq T$. In view of (30),

$$H_3(\delta)(t) \geq m\frac{2u(T)}{3} - |z(t)| \geq m^2 \frac{u(0)}{3}, \quad t \geq T,$$

for any $\delta \in X_3$. Given $\delta_1, \delta_2 \in X_3$ with $\delta_1 \leq \delta_2$. We note that

$$0 \leq w_u(\delta_1)(t) \leq w_u(\delta_2)(t) \leq u(g(t))$$

for $t \geq T$. By assumption, (30) and (31), we have

$$H_3(\delta_1)(t) \leq H_3(\delta_2)(t) \leq$$

$$\leq A(T,t)\frac{2u(T)}{3} + |z(t)| + \int_T^t \frac{A(s,t)}{r(s)} \int_s^\infty \frac{F(v, u(g(v)))}{B(s,v)} dvds \leq$$

$$\leq A(T,t)\frac{2u(T)}{3} + m^2\frac{u(0)}{3} + \int\limits_T^t \frac{A(s,t)}{r(s)} \int\limits_s^\infty \frac{F(v,u(g(v)))}{B(s,v)} dvds <$$

$$< A(T,t)\frac{2u(T)}{3} + m\frac{u(T)}{3} + \int\limits_T^t \frac{A(s,t)}{r(s)} \int\limits_s^\infty \frac{F(v,u(g(v)))}{B(s,v)} dvds \leq u(t)$$

for $t \geq T$. So $H_3(X_3) \subseteq X_3$ and H_3 is increasing on X_3. By the Knaster – Tarski fixed point theorem, there exists $\theta_3 \in X_3$ such that $H_3(\theta_3) = \theta_3$. Let

$$x(t) = \begin{cases} \theta_3(t), & t > T, \\ u(t), & T \geq t \geq r_T. \end{cases}$$

Let $T_1 > 0$ such that $r_{T_1} > T$. We note that $x(g(t)) = w_u(\theta_3)(t)$ for $t \geq T_1$. In view of $H_3(\theta_3) = \theta_3$, we have

$$x'(t) = z'(t) + \frac{B(T_2,t)}{r(t)} \int\limits_t^\infty \frac{F(s,x(g(s)))}{B(T_2,s)} ds$$

from which it follows that $x' \in PC'([T_1,\infty),\mathbf{R})$ and

$$\big(r(t)x'(t)\big)' + F(t,x(g(t))) = e(t), \quad \text{a.e.} \quad t \geq T_1.$$

For any $t_k \geq T_2$, by $H_3(\theta_3) = \theta_3$ and (15),

$$x(t_k^+) = a_k x(t_k) \qquad \text{and} \qquad x'(t_k^+) = b_k x'(t_k).$$

Then x is a positive solution of system (2) – (4) on $[T_1,\infty)$.

Theorem 3 is proved.

We remark that in the proof of Theorem 3, we see that in the condition (A7), we only need "$A(s,t) \geq m$ for $t \geq s \geq 0$".

The following result offers a *necessary and sufficient* oscillation theorem for (2) – (4).

Corollary 2. *Assume that* (A1) – (A7) *and* (6) *hold, and the solution* z *of* (5) *is oscillatory and* $\lim_{t\to\infty} z(t) = 0$. *Then the system* (2) – (4) *has a nonoscillatory solution* x *if, and only if, system* (23) – (25) *has a nonoscillatory solution* u.

Proof. We first show that there exist two sequences $\{s_n\}_{n\in\mathbf{N}}$ and $\{\widetilde{s}_n\}_{n\in\mathbf{N}}$ such that (22) holds. Let $\widetilde{z}(t) = z(t)/A(\tau,t)$ for $t \geq \tau$. Clearly, $\widetilde{z}(t)$ is continuous on $[\tau,\infty)$ and is oscillatory. By (A7) and $\lim_{t\to\infty} z(t) = 0$, we see that $\lim_{t\to\infty} \widetilde{z}(t) = 0$. In view of oscillation, there exists $s_1 > \tau$ such that $\widetilde{z}(s_1) \leq \widetilde{z}(t)$ for $t \geq s_1$, which implies that

$$z(s_1) \leq \frac{z(t)}{A(s_1,t)} \quad \text{for} \quad t \geq s_1.$$

There exists $s_2 > s_1$ such that $\widetilde{z}(s_2) \leq \widetilde{z}(t)$ for $t \geq s_1$, which implies that

$$z(s_2) \leq \frac{z(t)}{A(s_2,t)} \quad \text{for} \quad t \geq s_2.$$

By induction, we can take sequence $\{s_n\}_{n\in\mathbf{N}}$ such that (22) holds. Similarly, we can also take the sequence $\{\widetilde{s}_n\}_{n\in\mathbf{N}}$ such that (22) holds. By Theorem 2, the necessary condition holds. Conversely, assume that u is nonoscillatory solution of system (23)–(25). By Lemma 1, we see that $u(t)u'(t) \geq 0$ eventually. By Theorem 3, the sufficient condition holds.

Corollary 2 is proved.

We have another *necessary and sufficient* condition for the existence of bounded nonoscillatory solutions of (2)–(4).

***Corollary* 3.** *Assume that* (A1)–(A7) *and* (6) *hold and* $F(t,\mu) = p(t)f(\mu)$ *where* $p \in ALC([0,\infty),[0,\infty))$ *and* f *is nondecreasing on* $\mathbf{R}$ *with* $\mu f(\mu) > 0$ *for* $\mu \neq 0$. *Assume that the solution* z *of* (5) *is oscillatory, and there exist two sequences* $\{s_n\}_{n\in\mathbf{N}}$ *and* $\{\widetilde{s}_n\}_{n\in\mathbf{N}}$ *such that* (22) *hold. Then the system* (2)–(4) *has a bounded and nonoscillatory solution if, and only if,*

$$\int_{\varepsilon}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} \frac{p(v)}{B(s,v)} dv ds < \infty \tag{32}$$

for some $\varepsilon \geq 0$.

Proof. Assume that (32) holds. Clearly, (10) and (16) holds. By (A7) and (22), we note that the function z is bounded. By Theorem 1 and Corollary 1, the necessary condition holds. Conversely, we see that by Theorem 2, the system (23)–(25) has a nonoscillatory solution u such that $u(t)u'(t) \geq 0$ eventually, and u is bounded. Assume that u is eventually positive with $u'(t) \geq 0$ eventually. Then there exists $T > 0$ such that $u(t) > 0$ and $u'(t) \geq 0$ for $t \geq T$. We observe that

$$M_u \geq u(t) \geq A(T,t)u(T) \geq mu(T) > 0$$

for $t \geq T$ and some $M_u > 0$. So the function u has a positive lower bound. Since f is nondecreasing on $\mathbf{R}$, there is $m_f > 0$ such that $f(u(t)) \geq m_f$ for $t \geq T$. We observe that

$$\left(\frac{r(t)u'(t)}{u(t)}\right)' = \left(\frac{r(t)u'(t)}{u(t)}\right)' \leq -\frac{p(t)f(u(g(t)))}{u(t)} \leq -\frac{m_f}{M_u}p(t)$$

for $t \geq T$. We divide the above inequality by $B(T,t)/A(T,t)$ and then integrate from t to d where $d \geq t \geq T$. We have

$$\frac{A(T,d)r(d)u'(d)}{B(T,d)u(d)} - \frac{A(T,t)r(t)u'(t)}{B(T,t)u(t)} \leq -\frac{m_f}{M_u} \int_{t}^{d} \frac{A(T,s)p(s)}{B(T,s)} ds$$

for $d \geq t \geq T$. Since $u'(d) > 0$ and d is arbitrary, we see that

$$\frac{u'(t)}{u(t)} \geq \frac{m_f}{M_u r(t)} \int_{t}^{\infty} \frac{A(t,s)p(s)}{B(t,s)} ds$$

for $t \geq T$, from which it follows that

$$u'(t) \geq u'(t)\frac{mu(T)}{u(t)} \geq \frac{m^2 u(T) m_f}{M_u} \frac{1}{r(t)} \int_{t}^{\infty} \frac{p(s)}{B(t,s)} ds.$$

We divide the above inequality by $A(T,t)$, and then integrate from T to t. Then

$$u(t) \geq A(T,t)u(T) + \frac{m^2 u(T) m_f}{M_u} \int_T^t \frac{A(s,t)}{r(s)} \int_t^\infty \frac{p(s)}{B(t,s)} ds \geq$$

$$\geq m \left(u(T) + \frac{m u(T) m_f}{M_u} \int_T^t \frac{1}{r(s)} \int_t^\infty \frac{p(s)}{B(t,s)} ds \right) \tag{33}$$

for $t \geq T$. Since u is bounded, we may easily see that (32) holds. Assume that u is eventually negative with $u'(t) \leq 0$ eventually. Then $-u$ is an eventually positive solution of system

$$\big(r(t)u'(t)\big)' + p(t)\widetilde{f}(u(g(t))) = 0, \quad t \in [0,\infty)\backslash\Upsilon, \tag{34}$$

$$u(t_k^+) = a_k u(t_k), \quad k \in \mathbf{N}, \tag{35}$$

$$u'(t_k^+) = b_k u'(t_k), \quad k \in \mathbf{N}, \tag{36}$$

where $\widetilde{f}(\mu) = -f(-\mu)$ for $\mu \in \mathbf{R}$. Similarly, the function $-u$ has a positive lower bound. We note that the function $\widetilde{f}$ satisfies all assumption of f. So the condition (32) holds.

Corollary 3 is proved.

Theorem 4. *Assume that the hypotheses of Corollary* 3 *hold except for the condition* (A7), *and that* $g'(t) > 0$ *for* $t \geq 0$, $a_k \geq 1$ *for* $k \in \mathbf{N}$, *and*

$$\int_\varepsilon^\infty \frac{1}{f(\mu)} d\mu < \infty \quad \textit{for any} \quad \varepsilon > 0. \tag{37}$$

If the system (2)–(4) *has a nonoscillatory solution, then* (32) *holds.*

Proof. Since the system (2)–(4) has a nonoscillatory solution, by Theorem 2, we see that system (23)–(25) has a nonoscillatory solution u with $u(t)u'(t) \geq 0$ eventually. Assume that u is eventually positive with $u'(t) \geq 0$ eventually. There exists $T > 0$ such that $u(t) > 0$ and $u'(t) \geq 0$ for $t \geq t_T$. For any $T_1 > t \geq T$, we integrate (23) from t to T_1, and we obtain

$$\frac{r(T_1)u'(T_1)}{B(T,T_1)} - \frac{r(t)u'(t)}{B(T,t)} + \int_t^{T_1} \frac{p(s)f(u(g(s)))}{B(T,s)} ds = 0. \tag{38}$$

Since T_1 is arbitrary, by (38), we see that

$$\frac{r(t)u'(t)}{B(T,t)} \geq \int_t^\infty \frac{p(s)f(u(g(s)))}{B(T,s)} ds, \quad t \geq T. \tag{39}$$

Since $g'(t) > 0$ for $t \geq 0$ and $a_k \geq 1$ for $k \in \mathbf{N}$, we may see that $g(s) \geq g(t)$ and $A(t,s) \geq 1$ for $s \geq t \geq T$, from which it follows that

$$u(g(s)) \geq A(g(t), g(s))u(g(t)) \geq u(g(t)) \tag{40}$$

for $s \ge t \ge T$. Since f is nondecreasing on $\mathbf{R}$, we may further see that

$$f(u(g(s))) \ge f(u(g(t))) > 0 \tag{41}$$

for $s \ge t \ge T$. We divided (39) by $f(u(g(t)))$. Then

$$\frac{u'(g(t))}{f(u(g(t)))} \ge \frac{1}{r(g(t))} \int\limits_{g(t)}^{\infty} \frac{p(s) f(u(g(s)))}{B(g(t), s) f(u(g(t)))} ds,$$

from which it follows that, by (41),

$$\frac{u'(g(t))}{f(u(g(t)))} \ge \frac{1}{r(g(t))} \int\limits_{g(t)}^{\infty} \frac{p(s)}{B(g(t), s)} ds \tag{42}$$

for $t \ge T$. We multiply (42) by $g'(t)$, and then integrate from T to ∞. We obtain

$$\int\limits_{T}^{\infty} \frac{(u(g(s)))'}{f(u(g(s)))} ds \ge \int\limits_{T}^{\infty} \frac{g'(s)}{r(g(s))} \left(\int\limits_{g(s)}^{\infty} \frac{p(v)}{B(g(s), v)} dv \right) ds = \int\limits_{g(T)}^{\infty} \frac{1}{r(s)} \left(\int\limits_{s}^{\infty} \frac{p(v)}{B(s, v)} dv \right) ds. \tag{43}$$

By (37), (40) and (43), we see that

$$\int\limits_{g(T)}^{\infty} \frac{1}{r(s)} \left(\int\limits_{s}^{\infty} \frac{p(v)}{B(s, v)} dv \right) ds \le \int\limits_{T}^{\infty} \frac{(u(g(s)))'}{f(u(g(s)))} ds \le \int\limits_{u(g(T))}^{\infty} \frac{1}{f(\mu)} d\mu < \infty.$$

So (32) holds. Assume that $u(t)$ is eventually negative. Let $\widetilde{f}(\mu) = -f(-\mu)$ for $\mu \in \mathbf{R}$. By (37), we have

$$\int\limits_{\varepsilon}^{\infty} \frac{1}{\widetilde{f}(\mu)} d\mu < \infty \quad \text{for any} \quad \varepsilon > 0.$$

We note that $-u(t)$ is eventually positive solution of system (34) – (36). Similarly, we may verify (32).

Theorem 4 is proved.

Recall now the equation (1) under the condition $\eta > 1$. For the ease of discussion, we state the result in [1].

***Corollary* 4** [1]. *Let $\eta > 1$ be given and let e, p and g be continuous functions on $[0, \infty)$ such that $\lim_{t \to \infty} g(t) = \infty$, $p(t) \ge 0$, $g(t) \le t$ and $g'(t) \ge \alpha > 0$ for $t \ge 0$. Assume that there exists a bounded function $z(t)$ on $[0, \infty)$ such that $z''(t) = e(t)$ for all sufficient large t, and that z is oscillatory and* (22) *holds. Then*

$$\int\limits_{0}^{\infty} s p(s) ds < \infty \tag{44}$$

if, and only if, the equation (1) *has a nonoscillatory solution.*

Because the equation (1) has no impulsive effects, $a_k = b_k = 1$ for all k. That is, $A(s,t) = B(s,t) = 1$ for $t \geq s \geq 0$. Let $r(t) = 1$ for $t \geq 0$. Clearly, (6) holds and

$$\int\limits_0^\infty \frac{1}{r(s)} \int\limits_s^\infty \frac{p(v)}{B(s,v)} dvds = \int\limits_0^\infty sp(s)ds.$$

So either condition (10) or (32) is equivalent to (44). We have the following conclusions.

(1) Assume that we replace the condition "$g'(t) \geq \alpha > 0$ for $t \geq 0$" by $g'(t) > 0$ for $t \geq 0$. By Theorem 1 and 4, we can obtain the same Corollary in [1].

(2) Assume that we remove the condition "$g'(t) \geq \alpha > 0$ for $t \geq 0$". By Corollary 3, we can see that (44) holds if, and only if, the equation (1) has a nonoscillatory and bounded solution. We note that if the equation (1) has a nonoscillatory and unbounded solution, the condition (44) may not be true. So this result is sharp without the condition "$g'(t) > 0$ for $t \geq 0$." We give an example to illustrate it. Let $\varepsilon_1(t) = t^{1/3}\,(2 + \sin t)$ for $t \geq 0$. Clearly, there exists $a > 0$ such that $\varepsilon_1(t) \leq t$ for $t \geq a$. Let

$$g(t) = \begin{cases} \varepsilon_1(t), & t \geq a, \\ \varepsilon_2(t), & a > t \geq 0, \end{cases}$$

where ε_2 is a nonnegative and continuous function on $[0, a]$ with $\varepsilon_2(t) \leq t$ for $0 \leq t \leq a$, and $\varepsilon_1(a) = \varepsilon_2(a)$. By simple computation, we can see that $\lim_{t\to\infty} g(t) = \infty$ and $g(t) \leq t$ for $t \geq 0$, and it is impossible that $g'(t) > 0$ for sufficiently large t. We consider a special equation

$$x''(t) + \frac{1}{4t^{3/2}g(t)} x^2(g(t)) = 0, \quad t \geq 0. \tag{45}$$

Then the function $x(t) = \sqrt{t}$ is an eventually positive solution of (45) and is unbounded. But we can see that

$$\int\limits_a^\infty \frac{t}{4t^{3/2}g(t)} dt = \int\limits_a^\infty \frac{1}{4t^{5/6}\,(2 + \sin t)} dt \geq \int\limits_a^\infty \frac{1}{12t^{5/6}} dt = \infty.$$

Hence we have indeed made an improvement by avoiding the condition "$g'(t) \geq \alpha > 0$".

3. Example. Assume that (A2) and (A3) hold. Let $\alpha \in \mathbf{R}$ and the function

$$p(t) = \begin{cases} t^\alpha & \text{if } \ t > 0, \\ 0 & \text{if } \ t = 0. \end{cases}$$

Consider the Klein – Gordon equation (c.f. Example 2.6.3 in [5])

$$x''(t) + p(t)\,|x(g(t))| \exp(|x(g(t))|^2) = e^{-t} \sin t, \quad t \in [0, \infty) \backslash \Upsilon, \tag{46}$$

$$x(t_k^+) = a_k x(t_k), \quad k \in \mathbf{N}, \tag{47}$$

$$x'(t_k^+) = a_k x'(t_k), \quad k \in \mathbf{N}, \tag{48}$$

where a_k are positive constants for $k \in \mathbf{N}$ such that (A7) holds. We can easily check that

$$z(t) = A(0,t) \int\limits_{\frac{3\pi}{2}}^{t} \int\limits_{\frac{3\pi}{4}}^{s} \frac{e^{-v}\sin v}{A(0,v)} dvds, \quad t \geq \frac{3\pi}{2},$$

is a solution of the system

$$z''(t) = e^{-t}\sin t, \quad t \in [0,\infty)\backslash\Upsilon,$$

$$z(t_k^+) = a_k z(t_k), \quad k \in \mathbf{N},$$

$$z'(t_k^+) = a_k z'(t_k), \quad k \in \mathbf{N}.$$

Since

$$z(t) \geq m \int\limits_{\frac{3\pi}{2}}^{t} \int\limits_{\frac{3\pi}{4}}^{s} e^{-v}\sin v dvds \geq m \left(\frac{e^{-t}}{2}\cos t\right)$$

and

$$z(t) \leq M \int\limits_{\frac{3\pi}{2}}^{t} \int\limits_{\frac{3\pi}{4}}^{s} e^{-v}\sin v dvds = M \left(\frac{e^{-t}}{2}\cos t\right)$$

for $t \geq 3\pi/2$, we may see that $z(t)$ is oscillatory and $\lim_{t\to\infty} z(t) = 0$. Before stating the following conclusions, recall that a function φ defined for all sufficiently large t is oscillatory if φ is neither eventually positive nor eventually negative.

(1) It is easy to check that all hypotheses of Corollary 3 are satisfied. We first note that $\alpha < -2$ if, and only if,

$$\int\limits_{1}^{\infty} \int\limits_{s}^{\infty} \frac{v^{\alpha}}{A(s,v)} dvds < \infty.$$

By Corollary 3, we can see that $\alpha < -2$ if, and only if, the system (46) – (48) has a nonoscillatory solution x which is bounded.

(2) It is easy to check that all hypotheses of Corollary 2 are satisfied. Assume that $m \geq 1$ and $\alpha \geq -1$. Then

$$\int\limits_{1}^{\infty} \frac{p(t)}{A(0,t)} dt = \int\limits_{1}^{\infty} \frac{t^{\alpha}}{A(0,t)} dt \geq \int\limits_{1}^{\infty} \frac{t^{\alpha}}{M} dt = \infty.$$

Since

$$\sum_{i=1}^{\infty} \left(\prod_{1 \leq t_j < t_i} \frac{1}{a_j} \right) \int\limits_{t_{i-1}}^{t_i} p(t)dt = \int\limits_{0}^{\infty} \frac{t^{\alpha}}{A(0,t)} dt = \infty.$$

By Theorem 1 in [4], then every solution of system

$$x''(t) + p(t)\,|x(g(t))|\exp(|x(g(t))|^2) = 0, \quad t \in [0,\infty)\backslash\Upsilon, \tag{49}$$

$$x(t_k^+) = a_k x(t_k), \quad k \in \mathbf{N}, \tag{50}$$

$$x'(t_k^+) = a_k x'(t_k), \quad k \in \mathbf{N}, \tag{51}$$

is oscillatory. By Corollary 3, we can further see that every solution of system (46) – (48) is oscillatory.

(3) Assume that $g'(t) > 0$ for $t \geq 0$, $a_k \geq 1$ for $k \in \mathbf{N}$. Since

$$\int\limits_{\varepsilon}^{\infty} \frac{e^{-\mu}}{\mu} d\mu \leq \frac{1}{\varepsilon} \int\limits_{\varepsilon}^{\infty} e^{-\mu} d\mu < \infty \quad \text{for any} \quad \varepsilon > 0,$$

by Theorem 4, we see that if $\alpha \geq -2$, then every solution of system (46) – (48) is oscillatory.

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