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DISTRIBUTED ORDER CALCULUS: AN OPERATOR-THEORETIC INTERPRETATION *

ЧИСЛЕННЯ РОЗПОДІЛЕНОГО ПОРЯДКУ: ТЕОРЕТИКО-ОПЕРАТОРНА ІНТЕРПРЕТАЦІЯ

Within the Bochner – Phillips functional calculus and the Hirsch functional calculus, we describe the operators of distributed order differentiation and integration as functions of the classical differentiation and integration operators, respectively.

У межах функціональних числень Бохнера – Філліпса та Хірша наведено опис операторів диференціювання та інтегрування розподіленого порядку як функцій від класичних операторів диференціювання та інтегрування.

1. Introduction and preliminaries. In the distributed order calculus [1], used in physics for modeling ultraslow diffusion and relaxation phenomena, we consider derivatives and integrals of distributed order. The definitions are as follows.

Let μ be a continuous non-negative function on [0, 1]. The *distributed order derivative* $\mathbb{D}^{(\mu)}$ of weight μ for a function φ on $[0, T]$ is

$$(\mathbb{D}^{(\mu)}\varphi)(t) = \int_0^1 (\mathbb{D}^{(\alpha)}\varphi)(t)\mu(\alpha)\,d\alpha \tag{1}$$

where $D^{(\alpha)}$ is the Caputo – Dzhrbashyan regularized fractional derivative of order α, that is

$$(\mathbb{D}^{(\alpha)}\varphi)(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{d}{dt}\int_0^t (t-\tau)^{-\alpha}\varphi(\tau)\,d\tau - t^{-\alpha}\varphi(0)\right], \qquad 0 < t < T. \tag{2}$$

Denote

$$k(s) = \int_0^1 \frac{s^{-\alpha}}{\Gamma(1-\alpha)}\mu(\alpha)\,d\alpha, \qquad s > 0. \tag{3}$$

It is obvious that k is a positive decreasing function. The definition (1), (2) can be rewritten as

$$(\mathbb{D}^{(\mu)}\varphi)(t) = \frac{d}{dt}\int_0^t k(t-\tau)\varphi(\tau)\,d\tau - k(t)\varphi(0). \tag{4}$$

The right-hand side of (4) makes sense for a continuous function φ, for which the derivative $\frac{d}{dt}\int_0^t k(t-\tau)\varphi(\tau)\,d\tau$ exists.

If a function φ is absolutely continuous, then

$$(\mathbb{D}^{(\mu)}\varphi)(t) = \int_0^t k(t-\tau)\varphi'(\tau)\,d\tau. \tag{5}$$

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Below we always assume that $\mu \in C^3[0,1]$, $\mu(1) \neq 0$, and either $\mu(0) \neq 0$, or $\mu(\alpha) \sim a\alpha^\nu$, a, $\nu > 0$, as $\alpha \to 0$. Under these assumptions (see [1]),

$$\left.\begin{aligned} k(s) &\sim s^{-1}(\log s)^{-2}\mu(1), \\ k'(s) &\sim -s^{-2}(\log s)^{-2}\mu(1), \end{aligned}\right. \qquad s \to 0,$$

so that $k \in L_1(0,T)$ and k does not belong to any L_p, $p > 1$. We cannot differentiate under the integral in (4), since k' has a non-integrable singularity.

It is instructive to give also the asymptotics of the Laplace transform

$$\mathcal{K}(z) = \int_0^\infty k(s)e^{-zs}\,ds.$$

Using (4) we find that

$$\mathcal{K}(z) = \int_0^1 z^{\alpha-1}\mu(\alpha)\,d\alpha,$$

so that $\mathcal{K}(z)$ can be extended analytically to an analytic function on $\mathbb{C}\setminus\mathbb{R}_-$, $\mathbb{R}_- = \{z \in \mathbb{C} : \operatorname{Im} z = 0,\ \operatorname{Re} z \leq 0\}$. If $z \in \mathbb{C}\setminus\mathbb{R}_-$, $|z| \to \infty$, then [1]

$$\mathcal{K}(z) = \frac{\mu(1)}{\log z} + O\left((\log|z|)^{-2}\right); \tag{6}$$

see [1] for further properties of $\mathcal{K}$.

The *distributed order integral* $\mathbb{I}^{(\mu)}$ is defined as the convolution operator

$$(\mathbb{I}^{(\mu)}f)(t) = \int_0^t \kappa(t-s)f(s)\,ds, \qquad 0 \leq t \leq T, \tag{7}$$

where $\kappa(t)$ is the inverse Laplace transform of the function $z \mapsto \dfrac{1}{z\mathcal{K}(z)}$,

$$\kappa(t) = \frac{d}{dt}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{zt}}{z}\frac{1}{z\mathcal{K}(z)}\,dz, \qquad \gamma > 0. \tag{8}$$

It was proved in [1] that $\kappa \in C^\infty(0,\infty)$ and κ is completely monotone; for small values of t,

$$\kappa(t) \leq C\log\frac{1}{t}, \qquad |\kappa'(t)| \leq Ct^{-1}\log\frac{1}{t}. \tag{9}$$

If $f \in L_1(0,T)$, then $\mathbb{D}^{(\mu)}\mathbb{I}^{(\mu)}f = f$.

The aim of this paper is to clarify the operator-theoretic meaning of the above constructions. It is well known that fractional derivatives and integrals can be interpreted as fractional powers of the differentiation and integration operators in various Banach spaces; see, for example, [2 – 5].

Let A be the differential operator $Au = -\dfrac{du}{dx}$ in $L_p(0,T)$, $1 \le p < \infty$, with the boundary condition $u(0) = 0$. Its domain $D(A)$ consists of absolutely continuous functions $u \in L_p(0,T)$, such that $u(0) = 0$ and $u' \in L_p(0,T)$. We show that on $D(A)$ the distributed order differentiation coincides with the function $\mathcal{L}(-A)$ of the operator $-A$, where $\mathcal{L}(z) = z\mathcal{K}(z)$, and the function of an operator is understood in the sense of the Bochner – Phillips functional calculus (see [6 – 8]).

Moreover, if $p = 2$ then the distributed order integration operator $\mathbb{I}^{(\mu)}$ equals $\mathcal{N}(J)$, where $\mathcal{N}(x) = \dfrac{1}{\mathcal{L}(x)}$, J is the integration operator, $(Ju)(t) = \int_0^t u(\tau)\,d\tau$. This result is obtained within Hirsch's functional calculus [9, 10] giving more detailed results for a more narrow class of functions. As by-products, we obtain an estimate of the semigroup generated by $-\mathcal{L}(-A)$, and an expression for the resolvent of the operator $\mathbb{I}^{(\mu)}$.

2. Functions of the differentiation operator. The semigroup U_t of operators on the Banach space $X = L_p(0,T)$ generated by the operator A has the form

$$(U_t f)(x) = \begin{cases} f(x-t), & \text{if } 0 \le t \le x < T, \\ 0, & \text{if } 0 < x < t, \end{cases}$$

$x \in (0,T)$, $t \ge 0$. This follows from the easily verified formula for the resolvent $R(\lambda, A) = (A - \lambda I)^{-1}$ of the operator A:

$$(R(\lambda, A)u)(x) = -\int_0^x e^{-\lambda(x-y)} u(y)\,dy; \tag{10}$$

see [11] for a similar reasoning for operators on $L_p(0,\infty)$. The semigroup U_t is nilpotent, $U_t = 0$ for $t > T$; compare Sect. 19.4 in [12]. It follows from the expression (10) and the Young inequality that $\|R(\lambda, A)\| \le \lambda^{-1}$, $\lambda > 0$, so that U_t is a C_0-semigroup of contractions.

In the Bochner – Phillips functional calculus, for the operator A, as a generator of a contraction semigroup, and any function f of the form

$$f(x) = \int_0^\infty (1 - e^{-tx})\sigma(dt) + a + bx, \qquad a, b \ge 0, \tag{11}$$

where σ is a measure on $(0,\infty)$, such that

$$\int_0^\infty \frac{t}{1+t}\sigma(dt) < \infty,$$

the subordinate C_0-semigroup U_t^f is defined by the Bochner integral

$$U_t^f = \int_0^\infty (U_s u)\sigma_t(ds)$$

where the measures σ_t are defined by their Laplace transforms,

$$\int_0^\infty e^{-sx}\,\sigma_t(ds) \;=\; e^{-tf(x)}.$$

The class $\mathcal{B}$ of functions (11) coincides with the class of Bernstein functions, that is functions $f \in C[0,\infty) \bigcap C^\infty(0,\infty)$, for which f' is completely monotone. Below we show that $\mathcal{L} \in \mathcal{B}$.

The generator A^f of the semigroup U_t^f is identified with $-f(-A)$. On the domain $D(A)$,

$$A^f u \;=\; -\,au + bAu + \int_0^\infty (U_t u - u)\sigma(dt), \quad u \in D(A). \tag{12}$$

Theorem 1. (i) *If* $u \in D(A)$, *then* $A^{\mathcal{L}} u = -\,\mathbb{D}^{(\mu)} u$.

(ii) *The semigroup* $U_t^{\mathcal{L}}$ *decays at infinity faster than any exponential function*:

$$\left\| U_t^{\mathcal{L}} \right\| \;\le\; C_r e^{-rt} \quad \textit{for any} \quad r > 0. \tag{13}$$

The operator $A^{\mathcal{L}}$ *has no spectrum.*

(iii) *The resolvent* $R(\lambda, -A^{\mathcal{L}})$ *of the operator* $-A^{\mathcal{L}}$ *has the form*

$$(R(\lambda, -A^{\mathcal{L}})u)(x) \;=\; \int_0^x r_\lambda(x-s)u(s)\,ds, \quad u \in X, \tag{14}$$

where

$$r_\lambda(s) \;=\; \frac{1}{\lambda}\frac{d}{ds}u_\lambda(s), \tag{15}$$

and u_λ *is the solution of the Cauchy problem*

$$\mathbb{D}^{(\mu)} u_\lambda \;=\; \lambda u_\lambda, \qquad u_\lambda(0) \;=\; 1. \tag{16}$$

(iv) *The inverse* $(-A^{\mathcal{L}})^{-1}$ *coincides with the distributed order integration operator* $\mathbb{I}^{(\mu)}$.

(v) *The resolvent of* $\mathbb{I}^{(\mu)}$ *has the form*

$$(\mathbb{I}^{(\mu)} - \lambda I)^{-1} u \;=\; -\frac{1}{\lambda}u - \frac{1}{\lambda^2} r_{1/\lambda} * u, \quad \lambda \ne 0. \tag{17}$$

Proof. Let $\sigma(dt) = -\,k'(t)\,dt$. By (3),

$$k'(t) \;=\; -\int_0^1 \frac{\alpha t^{-\alpha-1}}{\Gamma(1-\alpha)}\mu(\alpha)\,d\alpha,$$

so that

$$\int_0^\infty \frac{t}{1+t}\,\sigma(dt) \;=\; \int_0^1 \frac{\alpha\mu(\alpha)}{\Gamma(1-\alpha)}\,d\alpha \int_0^\infty \frac{t^{-\alpha}}{1+t}\,\sigma(dt).$$

Using the integral formula 2.2.5.25 from [13] we find that

$$\int_0^\infty \frac{t}{1+t}\,\sigma(dt) \;=\; \pi\int_0^1 \frac{\alpha\mu(\alpha)}{(\sin\alpha\pi)\,\Gamma(1-\alpha)}\,d\alpha \;<\; \infty.$$

Let us compute the function (11) with $a = b = 0$. We have

$$f(x) = -\int_0^\infty (1-e^{-tx})k'(t)\,dt = x\int_0^\infty e^{-tx}k(t)\,dt = x\mathcal{K}(x) = \mathcal{L}(x).$$

The corresponding expression (12) for $A^{\mathcal{L}}u$, $u \in D(A)$, is as follows:

$$(A^{\mathcal{L}}u)(x) = -\int_0^\infty [(U_t u)(x) - u(x)]k'(t)\,dt =$$

$$= -\int_0^x [u(x-t) - u(x)]k'(t)\,dt + u(x)\int_x^\infty k'(t)\,dt =$$

$$= -k(x)u(x) - \int_0^x [u(x-t) - u(x)]k'(t)\,dt.$$

By (4), we find that $A^{\mathcal{L}}u = -\mathbb{D}^{(\mu)}u$, $u \in D(A)$.

The function $\mathcal{L}(z)$ is holomorphic for $\operatorname{Re} z > 0$. We will need a detailed information (refining (6)) on the behavior of $\operatorname{Re}\mathcal{L}(\sigma + i\tau)$, $\sigma, \tau \in \mathbb{R}$, $\sigma > 0$, when $|\tau| \to \to \infty$. We have

$$\operatorname{Re}\mathcal{L}(\sigma + i\tau) = \int_0^1 \varphi(\alpha, \sigma, \tau)\mu(\alpha)\,d\alpha$$

where

$$\varphi(\alpha, \sigma, \tau) = (\sigma^2 + \tau^2)^{\alpha/2}\cos\left(\alpha \arctan\frac{\tau}{\sigma}\right).$$

We check directly that $\varphi(\alpha, \sigma, 0) = \sigma^\alpha$,

$$\frac{\partial\varphi(\alpha, \sigma, \tau)}{\partial\tau} = \alpha(\sigma^2 + \tau^2)^{\alpha/2-1}\cos\left(\alpha \arctan\frac{\tau}{\sigma}\right)\left[\tau - \sigma\tan\left(\alpha \arctan\frac{\tau}{\sigma}\right)\right] \ge 0,$$

and $\dfrac{\partial\varphi(\alpha, \sigma, \tau)}{\partial\tau} > 0$ for $\alpha < 1$. This means that the function $g_\sigma(\tau) = \operatorname{Re}\mathcal{L}(\sigma + i\tau)$ (which is even in τ) is strictly monotone increasing in τ for $\tau > 0$. Its minimal value is

$$g_\sigma(0) = \int_0^1 \sigma^\alpha \mu(\alpha)\,d\alpha.$$

On the other hand,

$$\operatorname{Re}\mathcal{L}(\sigma + i\tau) \ge \int_0^1 (\sigma^2 + \tau^2)^{\alpha/2}\cos\frac{\alpha\pi}{2}\mu(\alpha)\,d\alpha = \frac{2}{\pi}\int_0^{\pi/2} (\sigma^2 + \tau^2)^{t/\pi}\mu\left(\frac{2t}{\pi}\right)\cos t\,dt =$$

$$= \frac{2}{\pi}\int_0^{\pi/2} e^{qt}\mu\left(\frac{2t}{\pi}\right)\cos t\,dt = \frac{2}{\pi}e^{q\pi/2}\int_0^{\pi/2} e^{-qs}\mu\left(1 - \frac{2}{\pi}s\right)\sin s\,ds$$

where $q = \dfrac{1}{\pi}\log(\sigma^2 + \tau^2)$. By Watson's asymptotic lemma (see [14]), since $\mu(1) \ne \ne 0$, we have

$$\int_0^{\pi/2} e^{-qs}\mu\left(1-\frac{2}{\pi}s\right)\sin s\, ds \sim Cq^{-2}$$

where C does not depend on σ, τ. Roughening the estimate a little we find that

$$e^{-t\operatorname{Re}\mathcal{L}(\sigma+i\tau)} \le Ce^{-t\rho|\tau|^{1/2-\varepsilon}} \tag{18}$$

where $0 < \varepsilon < 1/2$ can be taken arbitrarily, and the positive constants C and ρ do not depend on σ and τ.

It follows from (18) (see [15]) that for each $t > 0$ the function $x \mapsto e^{-t\mathcal{L}(x)}$ is represented by an absolutely convergent Laplace integral. This means that the measure $\sigma_t(ds)$ has a density $m(t, s)$ with respect to the Lebesgue measure. Moreover,

$$m(t, s) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{zs}e^{-t\mathcal{L}(z)}dz, \qquad \gamma > 0. \tag{19}$$

Since $U_t = 0$ for $t > T$, we have

$$U_t^f u = \int_0^T (U_s u)m(t, s)ds. \tag{20}$$

The representation (19) yields the expression

$$m(t, s) = \frac{e^{\gamma s}}{\pi}\int_0^{\infty} e^{i\tau}e^{-t\mathcal{L}(\gamma+i\tau)}d\tau,$$

$$\mathcal{L}(\gamma + i\tau) = \int_0^1 (\gamma + i\tau)^{\alpha}\mu(\alpha)d\alpha, \qquad 0 \le \tau < \infty.$$

We have

$$|m(t, s)| \le \frac{e^{\gamma s}}{\pi}\int_0^{\infty} e^{-tg_\gamma(\tau)}d\tau.$$

The above monotonicity property of g_γ makes it possible to apply to the last integral the Laplace asymptotic method [14]. We obtain that, for large values of t,

$$|m(t, s)| \le Ct^{-1}e^{\gamma s}e^{-tg_\gamma(0)}.$$

Changing γ and C we can make the coefficient $g_\gamma(0)$ arbitrarily big. By (20), this leads to the estimate (13).

Due to (13), the resolvent

$$R(\lambda, A^{\mathcal{L}}) = -\int_0^T e^{-\lambda t}U_t^{\mathcal{L}}\, dt, \tag{21}$$

is an entire function, so that $A^{\mathcal{L}}$ has no spectrum.

It follows from (21) that

$$R(\lambda, -A^{\mathcal{L}}) = \int_0^T e^{\lambda t}U_t^{\mathcal{L}}\, dt,$$

and if $u \in X$, $\operatorname{Re}\lambda \le 0$, then

$$(R(\lambda, -A^{\mathcal{L}})u)(x) = \int_0^\infty e^{\lambda t} dt \int_0^x u(x-s)m(t,s)\,ds = \int_0^x r_\lambda(x-s)u(s)\,ds$$

where

$$r_\lambda(s) = \int_0^\infty e^{\lambda t} m(t,s)\,dt. \tag{22}$$

For a fixed $\omega \in (1/2, 1)$, let us deform the contour of integration in (19) from the vertical line to the contour $S_{\gamma,\omega}$ consisting of the arc

$$T_{\gamma,\omega} = \{z \in \mathbb{C} : |z| = \gamma,\ |\arg z| \le \omega\pi\},$$

and two rays

$$\Gamma^{\pm}_{\gamma,\omega} = \{z \in \mathbb{C} : |\arg z| = \pm\omega\pi,\ |z| \ge \gamma\}.$$

The contour $S_{\gamma,\omega}$ is oriented in the direction of growth of $\arg z$. By Jordan's lemma,

$$m(t,s) = \frac{1}{2\pi i} \int_{S_{\gamma,\omega}} e^{zs} e^{-t\mathcal{L}(z)}\,dz.$$

Under this integral, we may integrate in t, as required in (22). We find that

$$r_\lambda(s) = \frac{1}{2\pi i} \int_{S_{\gamma,\omega}} \frac{e^{zs}}{\mathcal{L}(z) - \lambda}\,dz, \qquad s > 0 \tag{23}$$

(for $\operatorname{Re}\lambda > 0$, γ should be taken big enough).

If $\lambda = 0$, the right-hand side of (23) coincides with that of (8) (see also the formula (3.4) in [1], and we prove that $(-A^{\mathcal{L}})^{-1} = \mathbb{I}^{(\mu)}$.

For $\lambda \ne 0$, we rewrite (23) as

$$r_\lambda(s) = \frac{1}{2\pi i\lambda} \int_{S_{\gamma,\omega}} e^{zs} \frac{\mathcal{L}(z)}{\mathcal{L}(z) - \lambda}\,dz - \frac{1}{2\pi i\lambda} \int_{S_{\gamma,\omega}} e^{zs}\,dz. \tag{24}$$

For $0 < s < T$, we have

$$\int_{S_{\gamma,\omega}} e^{zs}\,dz = -\lim_{R\to\infty} \int_{\substack{|z|=R \\ \omega\pi<|\arg z|<\pi}} e^{zs}\,dz,$$

$$\left| \int_{\substack{|z|=R \\ \omega\pi<|\arg z|<\pi}} e^{zs}\,dz \right| \le 2R \int_{\omega\pi}^{\pi} e^{Rs\cos\varphi}\,d\varphi \le 2R\pi(1-\omega)e^{Rs\cos\omega\pi} \to 0,$$

as $R \to \infty$.

Thus, the second integral in (24) equals zero, and it remains to compare (24) with the formula (2.15) of [1] giving an integral representation of the function u_λ.

The formula (17) follows from (15) and the general connection between the resolvents of an operator and its inverse ([11], Chapter 3, formula (6.18)).

The theorem is proved.

Note that the expression (17) for the resolvent of a distributed order integration operator is quite similar to the Hille – Tamarkin formula for the resolvent of a fractional integration operator (see [12], Sect. 23.16). In our case, the function u_λ is a counterpart of the function $z \mapsto E_\alpha(\lambda z^\alpha)$ (for the order α case). However, in our situation no analog of the entire function E_α (the Mittag-Leffler function) has been identified so far. Accordingly, our proof of (17) is different from the reasoning in [12].

3. Functions of the integration operator. In this section we assume that $p = 2$.

Hirsch's functional calsulus deals with the class $\mathcal{R}$ of functions which are continuous on $\mathbb{C}\setminus(-\infty,0)$, holomorphic on $\mathbb{C}\setminus(-\infty,0]$, transform the upper half-plane into itself, and transform the semi-axis $(0,\infty)$ into itself. The class $\mathcal{R}$ is a subclass of $\mathcal{B}$.

Another important class of functions is the class $\mathcal{S}$ of Stieltjes functions

$$f(z) = a + \int_0^\infty \frac{d\rho(\lambda)}{z+\lambda}, \quad z \in \mathbb{C}\setminus(-\infty,0],$$

where $a \geq 0$, ρ is a non-decreasing right-continuous function, such that $\int_0^\infty \frac{d\rho(t)}{1+t} < \infty$.

If f is a nonzero function from $\mathcal{S}$, then the function

$$\tilde{f}(z) = \frac{1}{f(z^{-1})}$$

also belongs to $\mathcal{S}$.

If $f \in \mathcal{S}$, then the function $H_f(z) = f(z^{-1})$ belongs to $\mathcal{R}$. It has the form

$$H_f(z) = a + \int_0^\infty \frac{z}{1+\lambda z} d\rho(\lambda), \quad z \in \mathbb{C}\setminus(-\infty,0].$$

For some classes of linear operators V, the function $H_f(V)$ is defined as a closure of the operator

$$Wx = ax + \int_0^\infty V(I+\lambda V)^{-1} x\, d\rho(\lambda), \quad x \in D(V).$$

In particular, this definition makes sense if $-V$ is a generator of a contraction C_0-semigroup, and in this case the above construction is equivalent to the Bochner – Phillips functional calculus [3, 16]. In addition, by Theorem 2 of [9], if $(-V)^{-1}$ is also a generator of a contraction C_0-semigroup, then

$$[H_f(V)]^{-1} = H_{\tilde{f}}(V^{-1}). \tag{25}$$

In order to apply the above theory to our situation, note that [9]

$$z^\alpha = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \frac{z}{1+\lambda z} \lambda^{-\alpha} d\lambda, \quad 0 < \alpha < 1,$$

whence

$$\mathcal{L}(z) = \int_0^\infty \frac{z}{1+\lambda z} \beta(\lambda)\, d\lambda$$

where

$$\beta(\lambda) = \int_0^1 \frac{\lambda^{-\alpha}\mu(\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)} d\alpha.$$

Thus $\mathcal{L}(z) = H_f(z)$, with

$$f(z) = \int_0^{\infty} \frac{1}{z+\lambda} \beta(\lambda) d\lambda .$$

It follows from Watson's lemma [14] that $\beta(\lambda) \leq C(\log\lambda)^{-2}$ for large values of λ. Therefore

$$\int_0^{\infty} \frac{\beta(\lambda)}{1+\lambda} d\lambda < \infty .$$

Denote $\mathcal{N}(z) = H_{\tilde{f}}(z) = \dfrac{1}{\mathcal{L}(z)}$.

If $V = -A$, then $(-V)^{-1} = -J$, where J is the integration operator. It is easy to check that $\langle (J+J^*)u, u\rangle \geq 0$ ($\langle \cdot, \cdot \rangle$ is the inner product in $L_2(0,T)$). Therefore $-J$ is a generator of a contraction semigroup.

After these preparations, the equality (25) implies the following result.

Theorem 2. *The operator* $\mathbb{I}^{(\mu)}$ *of distributed order integration and the integration operator* J *are connected by the relation*

$$\mathbb{I}^{(\mu)} = \mathcal{N}(J).$$

1. *Kochubei A. N.* Destributed order calculus and equations of ultraslow diffusion // J. Math. Anal. and Appl. – 2008. – **340**. – P. 252 – 281.
2. *Bakaev N. Yu., Tarasov R. P.* Semigroups and a method for stably solving the Abel equation // Sib. Math. J. – 1978. – **19**. – P. 1 – 5.
3. *Gohberg I. C., Krein M. G.* Theory and applications of Volterra operators in Hilbert space. – Providence: Amer. Math. Soc., 1970. – X + 430 p.
4. *Jacob N., Krägeloh A. M.* The Caputo derivative, Feller semigroups, and the fractional power of the first order derivative on $C_\infty(\mathbb{R}_0^+)$ // Fract. Calc. Appl. Anal. – 2002. – **5**. – P. 395 – 410.
5. *Samko S. G., Kilbas A. A., Marichev O. I.* Fractional integrals and derivatives: theory and applications. – New York: Gordon and Breach, 1993. – XXXVI + 976 p.
6. *Phillips R. S.* On the generation of semigroups of linear operators // Pacif. J. Math. – 1952. – **2**. – P. 343 – 369.
7. *Berg C., Boyadzhiev Kh., deLaubenfels R.* Generation of generators of holomorphic semigroups // J. Austral. Math. Soc. A. – 1993. – **55**. – P. 246 – 269.
8. *Schilling R. L.* Subordination in the sense of Bochner and a related functional calculus // Ibid. – 1998. – **64**. – P. 368 – 396.
9. *Hirsch F.* Intégrales de résolvantes et calcul symbolique // Ann. Inst. Fourier. – 1972. – **22**, № 4. – P. 239 – 264.
10. *Hirsch F.* Domaines d'opérateurs representés comme intégrales de résolvantes // J. Funct. Anal. – 1976. – **23**. – P. 199 – 217.
11. *Kato T.* Perturbation theory for linear operators. – Berlin: Springer, 1966. – XX + 592 p.
12. *Hille E., Phillips R. S.* Functional analysis and semigroups. – Providence: Amer. Math. Soc., 1957. – XII + 808 p.
13. *Prudnikov A. P., Brychkov Yu. A., Marichev O. I.* Integrals and series. Vol. 1: Elementary functions. – New York: Gordon and Breach, 1986. – 798 p.
14. *Olver F. W. J.* Asymptotics and special functions. – New York: Acad. Press, 1974. – XVI + 572 p.
15. *Ditkin V. A., Prudnikov A. P.* Integral transforms and operational calculus. – Oxford: Pergamon Press, 1965. – XI + 529 p.
16. *Gorbachuk V. I., Knyazyuk A. V.* Boundary values of solutions of differential operator equations // Rus. Math. Surv. – 1989. – **44**, № 3. – P. 67 – 111.

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