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GENERALIZED RELAXED ELASTIC LINE ON AN ORIENTED SURFACE

УЗАГАЛЬНЕНА РЕЛАКСОВАНА ПРУЖНА ЛІНІЯ НА ОРІЄНТОВАНІЙ ПОВЕРХНІ

We study the relaxed elastic line in a more general case on an oriented surface. In particular, we obtain a differential equation with three boundary conditions for the generalized relaxed elastic line. Then we analyze the results in a plane, on a sphere, on a cylinder, and on the geodesics of these surfaces.

Вивчається релаксована пружна лінія у більш загальному випадку на орієнтованій поверхні. Зокрема, отримано диференціальне рівняння з трьома граничними умовами для узагальненої релаксованої пружної лінії. Отримані результати проаналізовано на площині, сфері, циліндрі та на геодезичних цих поверхонь.

1. Introduction. A relaxed elastic line of length ℓ on a connected oriented surface in three-dimensional Euclidean space E^3 as defined by G. S. Manning in [2] and characterized in [3], minimizes the total square curvature, $\int_0^\ell \kappa^2(s)ds$, in the family of all arcs of length ℓ having the same initial point and initial direction. In [2], he finds that whether or not the solutions are geodesic curves of the surface depends on the boundary conditions and on the surface. Because physical motivation for study of the problem on surface may be found in the nucleosome core partical of DNA molecule. In [3], H. K. Nickerson and G. S. Manning consider the relaxed elastic line model on an oriented surface and they derive an intrinsic equation with two boundary conditions for a relaxed elastic line on this surface. They give several illustrations and apply this formulation to give important results about relaxed elastic lines on various surface. They find the geodesics in a plane and on a sphere are relaxed elastic lines, but the geodesics on the other surface are not.

In this paper, our purpose is to study extremal for the variational problem of minimizing the functional

$$\mathcal{F}(\alpha) = \int\limits_0^\ell (\kappa^2 + \lambda_2\tau + \lambda_1)ds$$

within the family of all arcs of length ℓ on a connected oriented surface in three-dimensional Euclidean space E^3 having the same initial point and initial direction. The functional consist of the addition of twisting energy to bending energy. Then the functional is a generalizing of Manning's relaxed elastic line functional. Therefore, we call as "generalized relaxed elastic line" the curve which is extremal of this functional. We obtain a differential equation with three boundary conditions for generalized relaxed elastic line on a connected oriented surface in three-dimensional Euclidean space E^3. Then, we apply the results to analyze three important situations: in a plane and its geodesic, on a sphere and its geodesic and on a cylinder and its geodesics. These examples pave the way for comparing the relaxed elastic line and the generalized relaxed elastic line.

2. Intrinsic equations for a generalized relaxed elastic line on an oriented surface. Let S be a connected oriented surface in the three-dimensional Euclidean space E^3 and let $\alpha\colon I \subset \mathbb{R} \to S$ be an arc parametrized by arc length s, $0 \le s \le \ell$, with curvature $\kappa(s)$ and torsion $\tau(s)$. The arc α is called as a generalized relaxed elastic line if it is an extremal for variational problem of minimizing value of the functional $\mathcal{F}$,

$$\mathcal{F}(\alpha) = \int_0^\ell (\kappa^2 + \lambda_2\tau + \lambda_1)ds, \tag{2.1}$$

which stands in the family of all arcs of length ℓ on S having the same initial point and initial direction as α. If $\lambda_2 = \lambda_1 = 0$ in the functional $\mathcal{F}$, then the arc which is extremal of the functional is a relaxed elastic line (see [2] and [3]).

Assume that the coordinate functions of S be of class C^5 and that the equations of α, as functions of s, be of class C^5 in these coordinates.

At a point $\alpha(s)$ of α, let $T(s) = \alpha'(s)$ denote the unit tangent vector to α and let n denote the unit normal vector field of S. Then the Darboux frame T, Q, n along α on S is the orthonormal frame defined by

$$T(s) = \alpha'(s), \qquad Q(s) = n(s) \times T(s), \qquad n(s) = n\big(\alpha(s)\big).$$

So, the derivative equations of Darboux frame is

$$\begin{pmatrix} T' \\ Q' \\ n' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ Q \\ n \end{pmatrix}, \tag{2.2}$$

where κ_g, κ_n and τ_g are the geodesic curvature, the normal curvature and the geodesic torsion of α, respectively [1].

The square curvature κ^2 and the torsion τ of α on S is given by

$$\kappa^2 = \kappa_g^2 + \kappa_n^2 \tag{2.3}$$

and

$$\tau = \tau_g + \frac{\kappa_n'\kappa_g - \kappa_g'\kappa_n}{\kappa_g^2 + \kappa_n^2}, \tag{2.4}$$

respectively.

Let

$$x\colon D \subset \mathbb{R}^2 \to \mathbb{R}^3,$$

$$(u,v) \to x(u,v) = \big(x\,(u,v)\,, y\,(u,v)\,, z\,(u,v)\,\big)$$

be a coordinate patch of S. The partial velocities of x are given by $x_u = \dfrac{\partial x}{\partial u}$, $x_v = \dfrac{\partial x}{\partial v}$. Then an arc α is expressed as

$$\alpha(s) = x\big(u(s), v(s)\big), \quad 0 \le s \le \ell,$$

with

$$T(s) = \alpha'(s) = \frac{du}{ds}x_u + \frac{dv}{ds}x_v$$

and

$$Q(s) = p(s)x_u + q(s)x_v$$

for suitable scalar functions $p(s)$ and $q(s)$.

Now we need to define a variational field for constructing a family of the curves of length ℓ, which have the same initial point and initial direction. In order to obtain variational arcs of length ℓ, we extend α to an arc $\alpha^*(s)$ defined for $0 \leq s \leq \ell$, with $\ell^* > \ell$ but sufficiently close to ℓ so that α^* lies in the coordinate patch. Let $\mu(s)$, $0 \leq s \leq \ell^*$, be a scalar function of class C^3, not vanishing identically. Then it can be defined as

$$\eta(s) = \mu(s)p^*(s), \qquad \zeta(s) = \mu(s)q^*(s)$$

and so we can write

$$\mu(s)Q(s) = \eta(s)x_u + \zeta(s)x_v \tag{2.5}$$

along α. We also suppose that μ has only the restrictions

$$\mu(0) = 0, \qquad \mu'(0) = 0. \tag{2.6}$$

No further restrictions may be placed on μ. By this way we define

$$\beta\,(\sigma; t) = x\big(u(\sigma), v\,(\sigma)\,\big) + t\big(\eta(\sigma), \zeta(\sigma)\big), \tag{2.7}$$

for $0 \leq \sigma \leq \ell^*$. For $|t| < \varepsilon_1$ (where $\varepsilon_1 > 0$ depends upon the choice of α^* and μ), the point $\beta\,(\sigma; t)$ lies in the coordinate patch. Because of μ has the restrictions (2.6), $\beta(\sigma; t)$ gives an arc which is the same initial point and initial direction at fixed t. For $t = 0$, $\beta\,(\sigma; t)$ is the same as α^* and σ is arc length. For $t \neq 0$, the parameter σ has not arc length in general.

For fixed t, $|t| < \varepsilon_1$, let $L^*(t)$ denote the length of the arc $\beta\,(\sigma; t)\,$, $0 \leq \sigma \leq \ell^*$. Then

$$L^*(t) = \int\limits_0^{\ell^*} \sqrt{\left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle} d\sigma \tag{2.8}$$

with

$$L^*(0) = \ell^* > \ell. \tag{2.9}$$

It is clear in (2.7) and (2.8) that $L^*(t)$ is continuous (even differentiable) in t. In particular, it follows from (2.9) that

$$L^*(t) > \frac{\ell + \ell^*}{2} > \ell \qquad \text{for} \qquad |t| < \varepsilon \tag{2.10}$$

for a suitable ε satisfying $0 < \varepsilon \leq \varepsilon_1$. Because of (2.10) we can restrict $\beta(\sigma; t)$, $0 \leq |t| < \varepsilon$, to an arc of length ℓ by restricting the parameter σ to an interval of $0 \leq \sigma \leq \lambda(t) \leq \ell^*$ by requiring

$$\int\limits_0^{\lambda(t)} \sqrt{\left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle} d\sigma = \ell.$$

Note that $\lambda(0) = \ell$. The function $\lambda(t)$ need not be determined explicitly but we shall need.

Lemma 1 [3]**.**

$$\left.\frac{d\lambda}{dt}\right|_{t=0}=\int\limits_0^{\ell}\mu\kappa_g ds. \tag{2.11}$$

Now, we will calculate some derivatives of $\beta(\sigma;t)$. The partial derivative of (2.7) with respect to parameter σ is

$$\left.\frac{\partial\beta}{\partial\sigma}\right|_{t=0}=T,\quad 0\le s\le\ell. \tag{2.12}$$

By taking (2.2) into consideration, we calculate the partial derivative of (2.12) with respect to σ as

$$\left.\frac{\partial^2\beta}{\partial\sigma^2}\right|_{t=0}=\kappa_g Q+\kappa_n n \tag{2.13}$$

and

$$\left.\frac{\partial^3\beta}{\partial\sigma^3}\right|_{t=0}=-\left(\kappa_g^2+\kappa_n^2\right)T+\left(\kappa_n'+\kappa_g\tau_g\right)n+\left(\kappa_g'+\kappa_n\tau_g\right)Q. \tag{2.14}$$

Also we get

$$\left.\frac{\partial\beta}{\partial t}\right|_{t=0}=\mu Q \tag{2.15}$$

from (2.5). Further differentiation of (2.15) with respect to σ gives

$$\left.\frac{\partial^2\beta}{\partial t\partial\sigma}\right|_{t=0}=\left.\frac{\partial^2\beta}{\partial\sigma\partial t}\right|_{t=0}=-\mu\kappa_g T+\mu\tau_g n+\mu' Q \tag{2.16}$$

by using (2.2), and

$$\left.\frac{\partial^3\beta}{\partial t\partial\sigma^2}\right|_{t=0}=\left(-2\mu'\kappa_g-\mu\kappa_g'-\mu\kappa_n\tau_g\right)T+\left(2\mu'\tau_g+\mu\tau_g'-\mu\kappa_g\kappa_n\right)n+\left(\mu''-\mu\kappa_g^2-\mu\tau_g^2\right)Q. \tag{2.17}$$

Finally we get

$$\begin{gathered}\left.\frac{\partial^4\beta}{\partial t\partial\sigma^3}\right|_{t=0}=\Big(-3\mu''\kappa_g+3\mu'\kappa_g'+\mu\kappa_g''+\mu\kappa_n'\tau_g+2\mu\kappa_n\tau_g'+\\ +3\mu'\kappa_n\tau_g-\mu\kappa_n^2\kappa_g-\mu\kappa_g^3-\mu\kappa_g\tau_g^2\Big)T-\\ -\Big(3\mu'\kappa_g\kappa_n+2\mu\kappa_g'\kappa_n+\mu\kappa_g^2\tau_g-3\mu''\tau_g-3\mu'\tau_g'-\\ -\mu\tau_g''+\mu\kappa_g\kappa_n'+\mu\tau_g^3+\mu\kappa_n^2\tau_g\Big)n-\\ -\Big(3\mu'\kappa_g^2+3\mu'\tau_g^2+3\mu\kappa_g\kappa_g'+3\mu\tau_g\tau_g'-\mu'''\Big)Q.\end{gathered} \tag{2.18}$$

Now let $\mathcal{F}(t)$ denote the generalized relaxed elastic line functional of arc $\beta(\sigma;t)$, $0\leq\sigma\leq\lambda(t)$, $|t|<\varepsilon$. Since σ is not generally arc length for $t\neq 0$, the functional (2.1) calculates as

$$\mathcal{F}(t)=\int\limits_0^{\lambda(t)}\left\{\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{-3/2}\left\langle\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial^2\beta}{\partial\sigma^2}\right\rangle-\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{-5/2}\left\langle\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial\beta}{\partial\sigma}\right\rangle^{2}\right\}d\sigma+$$

$$+\lambda_2\int\limits_0^{\lambda(t)}\frac{\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{-2}\left\langle\frac{\partial\beta}{\partial\sigma}\times\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial^3\beta}{\partial\sigma^3}\right\rangle}{\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{-1}\left\langle\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial^2\beta}{\partial\sigma^2}\right\rangle-\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{-2}\left\langle\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial\beta}{\partial\sigma}\right\rangle^{2}}d\sigma+$$

$$+\lambda_1\int\limits_0^{\lambda(t)}\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{-1/2}d\sigma.$$

A necessary condition that α be an extremal is that $\left.\dfrac{d\mathcal{F}}{dt}\right|_{t=0}=0$ for arbitrary μ satisfying (2.6). In calculating $\dfrac{d\mathcal{F}}{dt}$, we give explicitly only those terms which do not vanish for $t=0$. The omitted terms are those with a factor $\left\langle\dfrac{\partial^2\beta}{\partial\sigma^2},\dfrac{\partial\beta}{\partial\sigma}\right\rangle$, which vanishes at $t=0$ since $\langle T,T'\rangle=0$. Thus, we get

$$\frac{d\mathcal{F}}{dt}=\frac{d\lambda}{dt}\left\{\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{-3/2}\left\langle\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial^2\beta}{\partial\sigma^2}\right\rangle\right\}_{\sigma=\lambda(t)}-$$

$$-3\int\limits_0^{\lambda(t)}\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{-5/2}\left\langle\frac{\partial^2\beta}{\partial t\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle\left\langle\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial^2\beta}{\partial\sigma^2}\right\rangle d\sigma+$$

$$+2\int\limits_0^{\lambda(t)}\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{-3/2}\left\langle\frac{\partial^3\beta}{\partial t\partial\sigma^2},\frac{\partial^2\beta}{\partial\sigma^2}\right\rangle d\sigma+$$

$$+\lambda_2\frac{d\lambda}{dt}\left\{\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{-1}\left\langle\frac{\partial\beta}{\partial\sigma}\times\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial^3\beta}{\partial\sigma^3}\right\rangle\left\langle\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial^2\beta}{\partial\sigma^2}\right\rangle^{-1}\right\}_{\sigma=\lambda(t)}-$$

$$-2\lambda_2\int\limits_0^{\lambda(t)}\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{-2}\left\langle\frac{\partial^2\beta}{\partial t\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle\left\langle\frac{\partial\beta}{\partial\sigma}\times\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial^3\beta}{\partial\sigma^3}\right\rangle\left\langle\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial^2\beta}{\partial\sigma^2}\right\rangle^{-1}d\sigma+$$

$$+\lambda_2\int\limits_0^{\lambda(t)}\left\langle\frac{\partial\beta}{\partial\sigma},\frac{\partial\beta}{\partial\sigma}\right\rangle^{-1}\left\langle\frac{\partial^2\beta}{\partial t\partial\sigma}\times\frac{\partial^2\beta}{\partial\sigma^2}+\frac{\partial\beta}{\partial\sigma}\times\frac{\partial^3\beta}{\partial t\partial\sigma^2},\frac{\partial^3\beta}{\partial\sigma^3}\right\rangle\left\langle\frac{\partial^2\beta}{\partial\sigma^2},\frac{\partial^2\beta}{\partial\sigma^2}\right\rangle^{-1}d\sigma+$$

$$+\lambda_2 \int\limits_0^{\lambda(t)} \left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle^{-1} \left\langle \frac{\partial\beta}{\partial\sigma} \times \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^4\beta}{\partial t\partial\sigma^3} \right\rangle \left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2} \right\rangle^{-1} d\sigma-$$

$$-2\lambda_2 \int\limits_0^{\lambda(t)} \left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle^{-1} \left\langle \frac{\partial\beta}{\partial\sigma} \times \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^3\beta}{\partial\sigma^3} \right\rangle \left\langle \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2} \right\rangle^{-2} \left\langle \frac{\partial^3\beta}{\partial t\partial\sigma^2}, \frac{\partial^2\beta}{\partial\sigma^2} \right\rangle d\sigma+$$

$$+\lambda_1 \frac{d\lambda}{dt} \left\{ \left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle^{-1/2} \right\}_{\sigma=\lambda(t)} - \lambda_1 \int\limits_0^{\lambda(t)} \left\langle \frac{\partial\beta}{\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle^{-3/2} \left\langle \frac{\partial^2\beta}{\partial t\partial\sigma}, \frac{\partial\beta}{\partial\sigma} \right\rangle d\sigma + \ldots .$$

To make it easy, we will use left-hand sides of the equations (2.3) and (2.4). But, while we are working on an oriented surface, we have to use the geodesic curvature, the normal curvature and the geodesic torsion. So, we will use right-hand sides of the equations (2.3) and (2.4) on the surface.

Then, by using (2.11), (2.3), (2.4), (2.12), (2.16), (2.13), (2.17), (2.14) and (2.18), we obtain

$$\left.\frac{d\mathcal{F}}{dt}\right|_{t=0} = \int\limits_0^{\ell} \mu(\kappa_g\kappa^2(\ell) + 3\kappa_g\kappa^2 - 2\kappa_g^3 - 2\kappa_g\tau_g^2 + 2\kappa_n\tau_g' - 2\kappa_g\kappa_n^2)ds+$$

$$+\lambda_2 \int\limits_0^{\ell} \mu(\kappa_g\tau(\ell) + 2\kappa_g\tau + \tau_g\kappa_g - 3\kappa_g^2\kappa_n'\kappa^{-2} - 3\kappa_g^3\tau_g\kappa^{-2})ds+$$

$$+\lambda_2 \int\limits_0^{\ell} \mu\left(-\kappa_n'\tau_g^2\kappa^{-2} - 2\kappa_g\tau_g^3\kappa^{-2} - \kappa_g'\tau_g'\kappa^{-2} + 4\kappa_n\tau_g\tau_g'\kappa^{-2}\right) ds+$$

$$+\lambda_2 \int\limits_0^{\ell} \mu\left(3\kappa_n\kappa_g\kappa_g'\kappa^{-2} - 3\kappa_g\kappa_n^2\tau_g\kappa^{-2} + \kappa_g\tau_g''\kappa^{-2} + 2\kappa_g^3\tau\kappa^{-2}\right) ds+$$

$$+\lambda_2 \int\limits_0^{\ell} \mu\left(2\kappa_g\tau_g^2\tau\kappa^{-2} - 2\kappa_n\tau_g'\tau\kappa^{-2} + 2\kappa_n^2\kappa_g\tau\kappa^{-2}\right) ds + 2\lambda_1 \int\limits_0^{\ell} \mu\kappa_g ds+$$

$$+4 \int\limits_0^{\ell} \mu'\kappa_n\tau_g ds + \lambda_2 \int\limits_0^{\ell} \mu'\left(-\kappa_n - 2\kappa_g'\tau_g\kappa^{-2} + 5\tau_g^2\kappa_n\kappa^{-2}\right) ds+$$

$$+\lambda_2 \int\limits_0^{\ell} \mu'\left(3\kappa_g\tau_g'\kappa^{-2} - 4\tau_g\kappa_n\tau\kappa^{-2}\right) ds + 2 \int\limits_0^{\ell} \mu''\kappa_g ds+$$

$$+\lambda_2\int\limits_0^\ell \mu''\left(\kappa_n'\kappa^{-2}+4\kappa_g\tau_g\kappa^{-2}-2\kappa_g\tau\kappa^{-2}\right)ds-\lambda_2\int\limits_0^\ell \mu'''\kappa_n\kappa^{-2}ds.$$

However, with integration by parts and (2.6), we can write

$$\tfrac{d\mathcal{F}}{dt}\big|_{t=0}=\int\limits_0^\ell \mu\big(\kappa_g\kappa^2(\ell)+3\kappa_g\kappa^2-2\kappa_g^3-2\kappa_g\tau_g^2-2\kappa_n\tau_g'-2\kappa_g\kappa_n^2-4\kappa_n'\tau_g+2\kappa_g''+$$

$$+\lambda_2(\kappa_g\tau(\ell)+2\kappa_g\tau+\kappa_g\tau_g-3\kappa_g^2\kappa_n'\kappa^{-2}+\kappa_n'-3\kappa_g^3\tau_g\kappa^{-2}-6\kappa_n'\tau_g^2\kappa^{-2}-$$

$$-2\kappa_g\tau_g^3\kappa^{-2}-6\kappa_g'\tau_g'\kappa^{-2}-6\kappa_n\tau_g\tau_g'\kappa^{-2}+3\kappa_g\kappa_g'\kappa_n\kappa^{-2}-3\kappa_g\kappa_n^2\tau_g\kappa^{-2}+$$

$$+2\kappa_g\tau_g''\kappa^{-2}-12\kappa_g'\tau_g\kappa^{-3}\kappa'+2\kappa_g\tau_g^2\tau\kappa^{-2}+2\kappa_n\tau_g'\tau\kappa^{-2}+2\kappa_n^2\kappa_g\tau\kappa^{-2}+$$

$$+6\kappa_g''\tau_g\kappa^{-2}+2\kappa_g^3\tau\kappa^{-2}+10\tau_g^2\kappa_n\kappa^{-3}\kappa'-2\kappa_g\tau_g'\kappa^{-3}\kappa'+4\tau_g\kappa_n\tau'\kappa^{-2}+$$

$$+2\kappa_n'''\kappa^{-2}-8\tau_g\kappa_n\tau\kappa^{-3}\kappa'+4\kappa_n'\tau_g\tau\kappa^{-2}-10\kappa_n''\kappa^{-3}\kappa'+24\kappa_n'\kappa^{-4}\kappa'^2-$$

$$-8\kappa_n'\kappa^{-3}\kappa''-2\kappa_g\tau''\kappa^{-2}+8\kappa_g\tau'\kappa^{-3}\kappa'-4\kappa_g'\tau'\kappa^{-2}-12\kappa_g\tau\kappa^{-4}\kappa'^2+4\kappa_g\tau\kappa^{-3}\kappa''+$$

$$+8\kappa_g'\tau\kappa^{-3}\kappa'-2\kappa_g''\tau\kappa^{-2}-24\kappa_n\kappa^{-5}\kappa'^3+18\kappa_n\kappa^{-4}\kappa'\kappa''-2\kappa_n\kappa^{-3}\kappa''')+2\lambda_1\kappa_g\big)ds+$$

$$+\mu(\ell)(4\kappa_n(\ell)\tau_g(\ell)-2\kappa_g'(\ell)+\lambda_2(-\kappa_n(\ell)-$$

$$-6\kappa_g'(\ell)\tau_g(\ell)\kappa^{-2}(\ell)+5\tau_g^2(\ell)\kappa_n(\ell)\kappa^{-2}(\ell)-\kappa_g\left(\ell\right)\tau_g'(\ell)\kappa^{-2}\left(\ell\right)-$$

$$-4\tau_g(\ell)\kappa_n(\ell)\tau\left(\ell\right)\kappa^{-2}(\ell)-2\kappa_n''(\ell)\kappa^{-2}(\ell)+6\kappa_n'(\ell)\kappa^{-3}(\ell)\kappa'\left(\ell\right)+$$

$$+2\kappa_g(\ell)\tau'(\ell)\kappa^{-2}(\ell)-4\kappa_g(\ell)\tau\left(\ell\right)\kappa^{-3}(\ell)\kappa'(\ell)+2\kappa_g'(\ell)\tau(\ell)\kappa^{-2}(\ell)-$$

$$-6\kappa_n(\ell)\kappa^{-4}(\ell)\kappa'^2(\ell)+2\kappa_n(\ell)\kappa^{-3}(\ell)\kappa''(\ell)))+\mu'(\ell)\big(2\kappa_g(\ell)+\lambda_2\big(2\kappa_n'(\ell)\kappa^{-2}\left(\ell\right)+$$

$$+4\kappa_g(\ell)\tau_g(\ell)\kappa^{-2}(\ell)-2\kappa_g(\ell)\tau\left(\ell\right)\kappa^{-2}(\ell)-2\kappa_n(\ell)\kappa^{-3}(\ell)\kappa'(\ell)\big)\big)-$$

$$-\lambda_2\kappa_n(\ell)\kappa^{-2}(\ell)\mu''(\ell)+\lambda_2\kappa_n(0)\kappa^{-2}(0)\mu''(0).$$

In order that $\left.\dfrac{d\mathcal{F}}{dt}\right|_{t=0}=0$ for all choices of the function $\mu(s)$ to satisfying (2.6), with arbitrary values of $\mu(\ell)$ and $\mu'(\ell)$, the given arc α must satisfy three boundary conditions

$$4\kappa_n(\ell)\tau_g(\ell)-2\kappa_g'(\ell)+\lambda_2(-\kappa_n\left(\ell\right)+\left(\kappa_g^2(\ell)+\kappa_n^2\left(\ell\right)\right)^{-1}(-6\kappa_g'(\ell)\tau_g(\ell)+$$

$$+5\tau_g^2(\ell)\kappa_n(\ell)+\kappa_g(\ell)\tau_g'(\ell)-4\tau_g^2(\ell)\kappa_n(\ell)-4\tau_g\left(\ell\right)\kappa_n(\ell)\kappa_n'\left(\ell\right)\kappa_g(\ell)(\kappa_g^2(\ell)+$$

$$+\kappa_n^2(\ell))^{-1}+4\tau_g(\ell)\kappa_n^2(\ell)\kappa_g'(\ell)\left(\kappa_g^2(\ell)+\kappa_n^2\left(\ell\right)\right)^{-1}-2\kappa_n''(\ell)+2\kappa_g(\ell)(\kappa_n''\left(\ell\right)\kappa_g(\ell)-$$

$$-\kappa_n(\ell)\kappa_g''(\ell))(\kappa_g^2(\ell)-\kappa_n(\ell)\kappa_g''(\ell))(\kappa_g^2(\ell)+\kappa_n^2(\ell))^{-1}-4\kappa_g(\ell)(\kappa_n'(\ell)\kappa_g(\ell)-$$

$$-\kappa_g'(\ell)\kappa_n(\ell))(2\kappa_g(\ell)\kappa_g'(\ell)+2\kappa_n(\ell)\kappa_n'(\ell))\left(\kappa_g^2(\ell)+\kappa_n^2(\ell)\right)^{-2}+2\kappa_g'(\ell)\tau_g(\ell)+$$

$$+2\kappa_g'(\ell)\kappa_n'(\ell)\kappa_g(\ell)\left(\kappa_g^2(\ell)+\kappa_n^2(\ell)\right)^{-1}-2\left(\kappa_g'(\ell)\right)^2\kappa_n(\ell)\left(\kappa_g^2(\ell)+\kappa_n^2(\ell)\right)^{-1}))+$$

$$+\left(\kappa_n(\ell)\kappa_n'(\ell)+\kappa_g(\ell)\kappa_g'(\ell)\right)(\kappa_n^2(\ell)+\kappa_g^2(\ell))^{-2}(6\kappa_n'(\ell)-4\kappa_g^2(\ell)\kappa_n'(\ell)(\kappa_g^2(\ell)+$$

$$+\kappa_n^2(\ell))^{-1}+4\kappa_g(\ell)\kappa_g'(\ell)\kappa_n(\ell)\left(\kappa_g^2(\ell)+\kappa_n^2(\ell)\right)^{-1})-6\kappa_n(\ell)(\kappa_n^2(\ell)+$$

$$+\kappa_g^2(\ell))^{-3}\left(\kappa_n(\ell)\kappa_n'(\ell)+\kappa_g(\ell)\kappa_g'(\ell)\right)^2+(2\kappa_n(\ell)\left(\kappa_n'\right)^2(\ell)+2\kappa_n^2(\ell)\kappa_n''(\ell)+$$

$$+2\kappa_n(\ell)\left(\kappa_g'\right)^2(\ell)+2\kappa_n(\ell)\kappa_g(\ell)\kappa_g''(\ell))\left(\kappa_n^2(\ell)+\kappa_g^2(\ell)\right)^{-2}-$$

$$-\left(2\kappa_n^2(\ell)\kappa_n'(\ell)+2\kappa_n(\ell)\kappa_g(\ell)\kappa_g'(\ell)\right)\left(\kappa_n^2(\ell)+\kappa_g^2(\ell)\right)^{-3})=0, \tag{2.19}$$

$$2\kappa_g(\ell)+\lambda_2\Big(\left(\kappa_g^2(\ell)+\kappa_n^2(\ell)\right)^{-1}(2\kappa_n'(\ell)+4\kappa_g(\ell)\tau_g(\ell)-$$

$$-2\kappa_g(\ell)\tau_g(\ell))+\left(\kappa_g^2(\ell)+\kappa_n^2(\ell)\right)^{-2}\left(-2\kappa_g^2(\ell)\kappa_n'(\ell)-\kappa_g'(\ell)\kappa_n(\ell)-2\kappa_n(\ell)\right)\Big)=0, \tag{2.20}$$

$$-\lambda_2\kappa_n(\ell)\left(\kappa_g^2(\ell)+\kappa_n^2(\ell)\right)^{-1}\mu''(\ell)+\lambda_2\kappa_n(0)\left(\kappa_g^2(0)+\kappa_n^2(0)\right)^{-1}\mu''(0)=0 \tag{2.21}$$

and the differential equation

$$\kappa_g\left(\kappa_g^2(\ell)+\kappa_n^2(\ell)\right)+3\kappa_g^3+3\kappa_g\kappa_n^2-2\kappa_g^3-2\kappa_g\tau_g^2-2\kappa_n\tau_g'-2\kappa_g\kappa_n^2-4\kappa_n'\tau_g+2\kappa_g''+$$

$$+\lambda_2(\kappa_g\tau_g(\ell)+\kappa_g\kappa_n'(\ell)\kappa_g(\ell)\left(\kappa_g^2(\ell)+\kappa_n^2(\ell)\right)^{-1}-\kappa_g\kappa_g'(\ell)\kappa_n(\ell)\left(\kappa_g^2(\ell)+\kappa_n^2(\ell)\right)^{-1}+$$

$$+2\kappa_g\tau_g+\tau_g\kappa_g+\left(\kappa_g^2+\kappa_n^2\right)^{-1}(\kappa_g\kappa_n\kappa_g'-\kappa_g^2\kappa_n'-\kappa_g^3\tau_g-6\kappa_n'\tau_g^2-10\kappa_g'\tau_g'-\kappa_g\kappa_n^2\tau_g+$$

$$+\left(\kappa_g^2+\kappa_n^2\right)^{-1}(2\kappa_g^4\kappa_n'-2\kappa_g^3\kappa_g'\kappa_n+2\kappa_g^2\tau_g^2\kappa_n'-2\kappa_g\tau_g^2\kappa_g'\kappa_n+2\kappa_n\tau_g'\kappa_n'\kappa_g-2\kappa_n^2\tau_g'\kappa_g'+$$

$$+2\kappa_n^2\kappa_g^2\kappa_n'-2\kappa_n^3\kappa_g\kappa_g'+4\tau_g\kappa_n\left(\kappa_n''\kappa_g-\kappa_g''\kappa_n\right)-4\kappa_g'\left(\kappa_n''\kappa_g-\kappa_g''\kappa_n\right)-2\kappa_g''\kappa_n'\kappa_g+$$

$$+2\kappa_g''\kappa_g'\kappa_n)+4\kappa_g''\tau_g+4\kappa_n'\tau_g^2-8\tau_g\kappa_n\left(\kappa_n'\kappa_g-\kappa_g'\kappa_n\right)\left(\kappa_g\kappa_g'+\kappa_n\kappa_n'\right)\left(\kappa_g^2+\kappa_n^2\right)^{-2}+$$

$$+(4\left(\kappa_n'\right)^2\tau_g\kappa_g-4\kappa_n'\tau_g\kappa_g'\kappa_n-2\kappa_g\left(\kappa_n'''\kappa_g+\kappa_n''\kappa_g'-\kappa_g'''\kappa_n-\kappa_g''\kappa_n'\right))\left(\kappa_g^2+\kappa_n^2\right)^{-1}+$$

$$+2\kappa_n'''+\left(\kappa_g^2+\kappa_n^2\right)^{-2}(8\kappa_g(\kappa_n''\kappa_g-\kappa_g''\kappa_n)(\kappa_g\kappa_g'+\kappa_n\kappa_n')+4\kappa_g\left(\kappa_n'\kappa_g-\kappa_g'\kappa_n\right)(\kappa_g''\kappa_g+$$

$$+\left(\kappa_g'\right)^2+\kappa_n''\kappa_n+\left(\kappa_n'\right)^2)+8\kappa_g'\left(\kappa_n'\kappa_g-\kappa_g'\kappa_n\right)\left(\kappa_g\kappa_g'+\kappa_n\kappa_n'\right))-8\kappa_g(\kappa_n'\kappa_g-$$

$$-\kappa_g'\kappa_n)(\kappa_g\kappa_g'+\kappa_n\kappa_n')^2\left(\kappa_g^2+\kappa_n^2\right)^{-3})+\kappa_n'+\Big(\kappa_n\kappa_n'+\kappa_g\kappa_g'\left(\kappa_n^2+\kappa_g^2\right)^{-2}\Big)(-12\kappa_g'\tau_g+$$

$$+10\tau_g^2\kappa_n - 2\kappa_g\tau_g' - 8\tau_g^2\kappa_n + \left(\kappa_g^2+\kappa_n^2\right)^{-1}\left(-8\tau_g\kappa_n\kappa_n'\kappa_g + 8\tau_g\kappa_n^2\kappa_g'\right) - 10\kappa_n'' +$$

$$+8\kappa_g\tau_g' + 8\kappa_g\left(\kappa_n''\kappa_g - \kappa_g''\kappa_n\right)\left(\kappa_g^2+\kappa_n^2\right)^{-1} - 16\kappa_g\left(\kappa_n'\kappa_g - \kappa_g'\kappa_n\right)\left(\kappa_g\kappa_g' +\right.$$

$$\left.+\kappa_n\kappa_n'\right)\left(\kappa_g^2+\kappa_n^2\right)^{-2} + 8\kappa_g'\tau_g + 8\kappa_g'\kappa_n'\kappa_g\left(\kappa_g^2+\kappa_n^2\right)^{-1} - 8\kappa_g'\kappa_g'\kappa_n\left(\kappa_g^2+\kappa_n^2\right)^{-1}) +$$

$$+24\kappa_n'\left(\kappa_n\kappa_n' + \kappa_g\kappa_g'\right)^2\left(\kappa_g^2+\kappa_n^2\right)^{-3} - 8\kappa_n'(\left(\kappa_n'\right)^2 + \kappa_n\kappa_n'' + \left(\kappa_g'\right)^2 + \kappa_g\kappa_g'')\left(\kappa_g^2+\kappa_n^2\right)^{-2} +$$

$$+8\kappa_n'\left(\kappa_n\kappa_n' + \kappa_g\kappa_g'\right)^2\left(\kappa_g^2+\kappa_n^2\right)^{-3} - 12\kappa_g\tau_g\left(\kappa_n\kappa_n' + \kappa_g\kappa_g'\right)^2\left(\kappa_g^2+\kappa_n^2\right)^{-3} -$$

$$-12\kappa_g^2\kappa_n'\left(\kappa_n\kappa_n' + \kappa_g\kappa_g'\right)^2\left(\kappa_g^2+\kappa_n^2\right)^{-4} + 12\kappa_g\kappa_g'\kappa_n\left(\kappa_n\kappa_n' + \kappa_g\kappa_g'\right)^2\left(\kappa_g^2+\kappa_n^2\right)^{-4} +$$

$$+4\kappa_g\tau_g\left(\left(\kappa_n'\right)^2 + \kappa_n\kappa_n'' + \left(\kappa_g'\right)^2 + \kappa_g\kappa_g''\right)\left(\kappa_g^2+\kappa_n^2\right)^{-2} + 4\kappa_g^2\kappa_n'(\left(\kappa_n'\right)^2 + \kappa_n\kappa_n'' +$$

$$+\left(\kappa_g'\right)^2 + \kappa_g\kappa_g'')\left(\kappa_g^2+\kappa_n^2\right)^{-3} - 4\kappa_g\kappa_g'\kappa_n\left(\left(\kappa_n'\right)^2 + \kappa_n\kappa_n'' + \left(\kappa_g'\right)^2 + \kappa_g\kappa_g''\right)\left(\kappa_g^2+\kappa_n^2\right)^{-3} -$$

$$-4\kappa_g\tau_g\left(\kappa_n\kappa_n' + \kappa_g\kappa_g'\right)^2\left(\kappa_g^2+\kappa_n^2\right)^{-3} - 4\kappa_g^2\kappa_n'\left(\kappa_n\kappa_n' + \kappa_g\kappa_g'\right)^2\left(\kappa_g^2+\kappa_n^2\right)^{-4} +$$

$$+4\kappa_g\kappa_g'\kappa_n\left(\kappa_n\kappa_n' + \kappa_g\kappa_g'\right)^2\left(\kappa_g^2+\kappa_n^2\right)^{-4} - 24\kappa_n\left(\kappa_n\kappa_n' + \kappa_g\kappa_g'\right)^3\left(\kappa_g^2+\kappa_n^2\right)^{-4} +$$

$$+18\kappa_n\left(\left(\kappa_n'\right)^2 + \kappa_n\kappa_n'' + \left(\kappa_g'\right)^2 + \kappa_g\kappa_g''\right)\left(\kappa_n\kappa_n' + \kappa_g\kappa_g'\right)\left(\kappa_g^2+\kappa_n^2\right)^{-3} - 18\kappa_n(\kappa_n\kappa_n' +$$

$$+\kappa_g\kappa_g')^3\left(\kappa_g^2+\kappa_n^2\right)^{-4} - 2\kappa_n(3\kappa_n'\kappa_n'' + \kappa_n\kappa_n''' + 3\kappa_g'\kappa_g'' + \kappa_g\kappa_g''')\left(\kappa_g^2+\kappa_n^2\right)^{-2} +$$

$$+6\kappa_n(\left(\kappa_n'\right)^2 + \kappa_n\kappa_n'' + \left(\kappa_g'\right)^2 + \kappa_g\kappa_g'')\left(\kappa_n\kappa_n' + \kappa_g\kappa_g'\right)\left(\kappa_g^2+\kappa_n^2\right)^{-3} -$$

$$-6\kappa_n\left(\kappa_n\kappa_n' + \kappa_g\kappa_g'\right)\left(\kappa_g^2+\kappa_n^2\right)^{-4}) + 2\lambda_1\kappa_g = 0. \tag{2.22}$$

Then we can give the following theorem.

Theorem 1. *The intrinsic equations for a generalized relaxed elastic line on a connected oriented surface in three-dimensional Euclidean space E^3 are given by the differential equation* (2.22) *together with the boundary conditions* (2.19), (2.20) *and* (2.21) *at the free end. Here κ_g, κ_n and τ_g are the functions giving the geodesic curvature, the normal curvature and the geodesic torsion as functions of arc length along the line.*

It is clear that the solutions give us the relaxed elastic line in a special case $\lambda_2 = \lambda_1 = 0$. The following expression is a natural conclusion of this theorem.

***Corollary* 1.** *The critical point of the functional*

$$\mathcal{F}(\alpha) = \int\limits_0^\ell \kappa^2 ds$$

is an arc which satisfies two boundary conditions

$$\kappa_g(\ell) = 0,$$

$$\kappa_g'(\ell) = 2\kappa_n(\ell)\tau_g(\ell)$$

and differential equation

$$2\kappa_g'' - 4\kappa_n'\tau_g - 2\kappa_n\tau_g' + \kappa_g^3 + \kappa_g\kappa_n^2 - 2\kappa_g\tau_g^2 + \kappa_g\kappa_n^2(\ell) = 0,$$

for arbitrary value of the function $\mu(\ell)$ *satisfying* (2.6). *Then, this differential equation with two boundary conditions is the relaxed elastic line derived by Nickerson and Manning* [3].

3. Applications. ***3.1. Generalized relaxed elastic line in a plane.*** The critical point of the functional (2.1) is a relaxed elastic line since plane curves have identically zero torsion. Then we can give the following corollary.

***Corollary* 2.** *Geodesic of a plane is a generalized relaxed elastic line.*

3.2. Generalized relaxed elastic line on a sphere. The geodesic torsion τ_g vanishes for all curves on a sphere of radius R and normal curvature $\kappa_n = -\dfrac{1}{R}$. Then (2.22) reduces to

$$2\kappa_g'' + \kappa_g^3 + \left(\frac{2}{R^2} + \kappa_g^2(\ell)\right)\kappa_g + \lambda_2\Big(\kappa_g\kappa_g'(\ell)\left(R^2\kappa_g^2(\ell)+1\right)^{-1} +$$

$$+ (2-3R)\,\kappa_g\kappa_g'\left(R^2\kappa_g^2+1\right)^{-1} + 2R^2\kappa_g^3\kappa_g'\left(R^2\kappa_g^2+1\right)^{-2} +$$

$$+2\kappa_g\kappa_g'\left(R^2\kappa_g^2+1\right)^{-2} - 2R^2(1+R)\,\kappa_g\kappa_g'''\left(R^2\kappa_g^2+1\right)^{-2} -$$

$$-6R^2(1+R)\,\kappa_g'\kappa_g''\left(R^2\kappa_g^2+1\right)^{-2} - 18R^5\kappa_g\kappa_g'\left((\kappa_g')^2+\kappa_g\kappa_g''\right)\left(R^2\kappa_g^2+1\right)^{-2} +$$

$$+2R^4(4-R)\,\kappa_g^2\kappa_g'\kappa_g''\left(R^2\kappa_g^2+1\right)^{-3} +$$

$$+2R^4\left(8+3R+9R^3\kappa_g^2\right)\kappa_g(\kappa_g')^3\left(R^2\kappa_g^2+1\right)^{-3} +$$

$$+2R^6(9R-8)\,\kappa_g^3(\kappa_g')^3\left(R^2\kappa_g^2+1\right)^{-4}\Big) + 2\lambda_1\kappa_g = 0. \tag{3.1}$$

The boundary conditions (2.19), (2.20) and (2.21), which reduces to

$$-2\kappa_g'(\ell) + \lambda_2\Bigg(\frac{1}{R} + 2R^2(1-R)\left(\kappa_g(\ell)\kappa_g''(\ell) + \kappa_g'^2(\ell)\right)\left(R^2\kappa_g^2(\ell)+1\right)^{-2} +$$

$$+8R^4(R-1)\kappa_g^2(\ell)(\kappa_g')^2(\ell)\left(R^2\kappa_g^2(\ell)+1\right)^{-3}\Bigg) = 0, \tag{3.2}$$

$$2\kappa_g(\ell) + \lambda_2 2R^2(R-1)\,\kappa_g(\ell)\kappa_g'(\ell)\left(R^2\kappa_g^2(\ell)+1\right)^{-2} = 0, \tag{3.3}$$

and

$$\frac{\lambda_2 R\mu''(\ell)}{R^2\kappa_g^2(\ell)+1} - \frac{\lambda_2 R\mu''(0)}{R^2\kappa_g^2(0)+1} = 0. \tag{3.4}$$

Then, a generalized relaxed elastic line on the sphere is given by the differential equation (3.1) with three boundary conditions (3.2), (3.3) and (3.4). We know that geodesic on a sphere is a relaxed elastic line, but the following expression gives us an important case.

***Corollary* 3.** *The geodesic of the sphere is a generalized relaxed elastic line in case of* $\lambda_2 = 0$, *but it is not a generalized relaxed elastic line in case of* $\lambda_2 \neq 0$.

3.3. Generalized relaxed elastic line on a cylinder. Let the cylinder be parametrized by

$$x(u,v) = \left(R\cos\frac{u}{R}, R\sin\frac{u}{R}, v\right),$$

where R is radius of the circle. Then for an arbitrary arc α on the cylinder

$$\kappa_g = \frac{d\theta}{ds}, \qquad \kappa_n = -\frac{1}{R}\cos^2\theta \qquad \text{and} \qquad \tau_g = \frac{1}{R}\cos\theta\sin\theta,$$

where $\theta = \theta(s)$ is the angle between the u-coordinate curve through $\alpha(s)$ and the arc α. The geodesics on the cylinder are characterized by $\theta = \text{constant}$ and satisfy the generalized relaxed elastic line differential equation (2.22) only if $\theta = 0$, $\theta = \pm\pi$ and $\theta = \pm\frac{\pi}{2}$. But the boundary conditions (2.19) and (2.21) (the boundary condition (2.20) is already zero)

$$-\frac{4}{R^2}\cos^3\theta(\ell)\sin\theta\,(\ell) + \frac{\lambda_2}{R}\cos 2\theta(\ell) = 0$$

and

$$\frac{\lambda_2 R}{\cos^2\theta(\ell)}\left(\mu''(\ell) - \mu''(0)\right) = 0.$$

Then, we clearly see the following corollary.

***Corollary* 4.** *If* $\theta = 0$, $\theta = \pm\pi$ *or* $\theta = \pm\frac{\pi}{2}$, *the geodesics of the cylinder are generalized relaxed elastic lines in case of* $\lambda_2 = 0$, *but they are not a generalized relaxed elastic lines in case of* $\lambda_2 \neq 0$.

Conclusion. In this work we generalize the notion of "relaxed elastic line" by using the definition given in [2] and [3], and define the notion "generalized relaxed elastic line". Then we obtain the formulation to determine a generalized relaxed elastic line on an oriented surface. We apply this formulation to give results about generalized relaxed elastic line on various surfaces. We show that the geodesic of a plane is always a generalized relaxed elastic line. The geodesic of a sphere and geodesics of a cylinder are generalized relaxed elastic line only special cases, but they are not a generalized relaxed elastic lines in case of $\lambda_2 \neq 0$.

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