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T. K. Pogány (Univ. Rijeka, Croatia)

MULTIDIMENSIONAL LAGRANGE – YEN TYPE INTERPOLATION VIA KOTEL'NIKOV – SHANNON SAMPLING FORMULAE

БАГАТОВИМІРНА ІНТЕРПОЛЯЦІЯ ТИПУ ЛАГРАНЖА – ЙЕНА ЧЕРЕЗ ВИБІРКОВІ ФОРМУЛИ КОТЕЛЬНІКОВА – ШЕННОНА

Direct finite interpolation formulae are developed for the Paley – Wiener functions spaces $L^2_{\diamond}, L^2_{[-\pi,\pi]^d}$, where $L^2_{\diamond}$ collects all twovariate entire functions which Fourier spectrum is supported by the set $\diamond = \mathrm{Cl}\{(u,v)||u|+|v|<\pi\}$, while in $L^2_{[-\pi,\pi]^d}$ the Fourier spectrum support set of its d-variate entire elements is $[-\pi,\pi]^d$. The multidimensional Kotel'nikov – Shannon sampling formula remains valid when only finitely many sampling knots are deviated from the uniform spacing. By this interpolation procedure we realize truncating sampling sum to its irregularly sampled part. Upper bounds of truncation error are derived in both cases.

According to the Sun – Zhou extension of the Kadets's $\dfrac{1}{4}$-theorem, the magnitudes of deviations are limited coordinatewise to 1/4. To avoid this inconvenience, we introduce weighted Kotel'nikov – Shannon sampling sums. For $L^2_{[-\pi,\pi]^d}$ Lagrange-type direct finite interpolation formulae are given. Finally, convergence rate questions are discussed.

Прямі скінченні інтерполяційні формули отримано для функціональних просторів $L^2_{\diamond}, L^2_{[-\pi,\pi]^d}$, де $L^2_{\diamond}$ містить усі цілі функції з двома змінними, для яких носієм спектра Фур'є є множина $\diamond = \mathrm{Cl}\{(u,v)||u|+|v|<\pi\}$, а в $L^2_{[-\pi,\pi]^d}$ носієм цілих елементів спектра Фур'є з d змінними є множина $[-\pi,\pi]^d$. Багатовимірна вибіркова формула Котельнікова – Шеннона залишається справедливою у випадку, коли тільки скінченна кількість вибіркових вузлів відхиляється від рівномірного розташування. За допомогою цієї інтерполяційної процедури ми обрізаємо вибіркову суму до її нерегулярної вибіркової частини. Верхні межі похибки відсікання отримано для обох випадків.

Відповідно до розширення Сун – Жу для $\dfrac{1}{4}$-теореми Кадеца значення відхилень покоординатно обмежені значенням 1/4. Щоб уникнути цієї незручності, введено зважені вибіркові суми Котельнікова – Шеннона. Для простору $L^2_{[-\pi,\pi]^d}$ наведено прямі скінченні інтерполяційні формули типу Лагранжа. Розглянуто питання про швидкість збіжності.

1. Introduction with historical background. The Kotel'nikov – Shannon (sampling) reconstruction formula is devoted to bandlimited to w function coming from some function space $\mathbf{S}$, say, and uses the infinite linear combination of uniformly sampled (digitalized) values of the considered signal function with the well-known sampling (reconstruction) function

$$\mathrm{sinc}(t) := \frac{\sin(t)}{t}\chi_{\mathbb{R}\setminus\{0\}}(t) + \delta_{t0},$$

where $\chi_A(t)$ is the indicator function of the event $t \in A$, while $\delta_{\mu\nu}$ denotes the Kronecker's delta. Precisely that means

$$f(t) = \sum_{n\in\mathbb{Z}} f\left(\frac{\pi}{w}n\right)\mathrm{sinc}(wt - n\pi). \tag{1}$$

The latter results precises functions spaces, and convergence results in (1). Anyway, the history of the formula (1) is widely known, we refer e. g. to [1, 2] and [3] for the interested reader.

The uniform setting in measuring the input analogue signal function is not a reasonable idealization, since the timing of the digitalization is rarely exactly uniform. This statement led Yen to give a generalization of (1) in few directions. The most interesting for us from these is the *migration of a finite number of uniform sample points* ((**M**) in the sequel), speaking in Yen's terminology [4] (Theorem I). By him, some of time points are disturbed from the uniform spacing $\pi k/w$, $k \in \mathbb{Z}$, but only for the k's from set $\{1, \ldots, N\} \subset \mathbb{N}$. Otherwords, he prescribes that t_p are the new irregularly placed sample points migrated to N new positions from $\pi n_p/w$, that wt_p/π, $p = \overline{1, N}$, is not an integer and the new sample is τ_m, $m \in \mathbb{Z}$. This results with

$$f(t) = \sum_{n\in\mathbb{Z}} f(\tau_n)\Psi_n^Y(t), \tag{2}$$

where the sampling function is

$$\Psi_n^Y(t) = (w\pi)^N \operatorname{sinc}(wt - n\pi) \prod_{q=1}^{N} \frac{(t-t_q)(n-n_q)}{(wt-n_q\pi)(n\pi - wt_q)}\, \chi_{\{\tau_m = \frac{\pi}{w}n \neq \frac{\pi}{w}n_q\}} +$$

$$+ \frac{\sin(wt)}{\sin(wt_p)} \prod_{p\neq q=1}^{N} \frac{(t-t_q)}{(t_p - t_q)} \prod_{q=1}^{N} \frac{(wt_p - n_q\pi)}{(wt - n_q\pi)}\, \chi_{\{\tau_m = t_p\}}, \tag{3}$$

where χ_A denotes the indicator function of the event A. Yen does not precise convergence rates of the sampling series, the kind of the convergence and the function space **S**. Following Yen's idea Bond & Cahn [5] and Bond, Cahn & Hancock [6] consider the so-called *zero-crossing* in the signal reconstruction procedure supposing (**M**) is satisfied. His approach is outside of our framework.

Considering the same problem (**M**) for the so-called L^2_w-functions (such that are in $L^2(\mathbb{R})$ and their Fourier spectrum is bandlimited to w simultaneously), Flornes, Lyubarskii and Seip examine

$$f(t) = \sum_{|n-t|\le L} f(\lambda_n)\Psi_n^F(t) + \sum_{|n-t|>L} f\left(\frac{\pi}{w}n\right)\Psi_n^F(t) \tag{4}$$

where for

$$\Lambda_{h,M} = \{\lambda_n\} := \left\{\lambda_n = \frac{\pi}{w}n + h_n : |n-t| \le L\right\} \cup \left\{\lambda_n = \frac{\pi}{w}n : |n-t| > L\right\},$$

and $|h_n| \le M < \dfrac{\pi}{4w}$ (Kadets's theorem!), it is

$$\Psi_n^F(t) = \frac{G(\Lambda_t, t)}{G'(\Lambda_t, \lambda_n)(t - \lambda_n)}. \tag{5}$$

Here the auxiliary function G we get from the infinite product representation of the sine by migrating $2L+1$ its zeros from the uniform (regular) spacing to get Λ_t, i. e.

$$G(\Lambda_t, t) = (t - \lambda_0) \prod_{|n-t|\le L}{}' \left(1 - \frac{t}{\lambda_n}\right) \lim_{R\to\infty} \prod_{L<|n-t|<R}{}' \left(1 - \frac{wt}{n\pi}\right),$$

compare especially [7] (§C-2) and [8].

By rewriting the sampling function $\Psi_n^Y(t)$ putting $N = 2L+1$, translating the migrated knots $t_p, p = \overline{1, 2L+1}$, into

$$\left\{\lambda_n = \frac{\pi}{w} n + h_n : |n - t| \leq L\right\},$$

we get exactly $\Psi_n^F(t)$, therefore (4) is the actually the Yen's formula (1).

Further, Flornes et al. modified (4) in the Valiron manner (they said Boas – Bernstein formula) with a weight function

$$\omega_n(t) = \operatorname{sinc}^l(\delta(t - \lambda_n)/l), \quad \delta \in (0, \pi), \ l \in \mathbb{N}. \tag{6}$$

They derive the sampling formula

$$f(t) = \sum_{|n-t| \leq L} f(\lambda_n) \omega_n(t) \Psi_n^F(t) + \sum_{|n-t| > L} f\left(\frac{\pi}{w} n\right) \omega_n(t) \Psi_n^F(t) \tag{7}$$

and develop truncation error estimates, and evaluate L, M, l for some already given reconstruction error level. Obviously (2), (4) and (7) are Lagrange-type formulae.

The next author revisited (**M**) was Houdré, who is assumes not only that either finitely many sampling knots are missing and/or finitely many sampling knots migrate from the uniform distribution. His results for the class of weakly stationary stochastic processes can be interpreted as statistical *missing data* results. These allow interpolation, using finite data, of sets containing mixtures of uniform and nonuniform sample points with gaps, satisfying the uniform density $d > 0$ condition (we write (**H**) for this approach). Obviously (**M**) $\subseteq$ (**H**). The set of sampling knots is

$$\{t_k\}_{k \in \mathbb{Z}} = \left\{\frac{k}{d}\right\}_{\mathbb{Z} \setminus \{k_1, \ldots, k_r\}} \cup \{s_1, \ldots, s_l\}.$$

Therefore, when $\{X(t), \ t \in \mathbb{R}\}$ is weakly stationary stochastic process bandlimited to $w < \pi d$, we have

$$f(t) = \sum_{n \in \mathbb{Z}} f(\tau_n) \Psi_n^H(t),$$

in the mean square sense uniformly on compact sybsets of $\mathbb{R}$. Here

$$\Psi_n^H(t) =$$

$$= \frac{w}{\pi d} \Bigg\{ \operatorname{sinc} w(t_k/d - t) + \frac{w}{\pi d} \sum_{p=1}^{r} \sum_{q=1}^{r} V_{q,p} \operatorname{sinc} w(k_p/d - t) \operatorname{sinc} w(t_k - k_q)/d -$$

$$- \frac{w}{\pi d} \sum_{p=1}^{l} \sum_{q=1}^{l} V_{q,p} \operatorname{sinc} w(s_p/d - t) \operatorname{sinc} w(t_k - s_q)/d \Bigg\},$$

and $V_{q,p}$ is the (q, p)-entry of the inverse of the matrix:

$$V = \left(I - \frac{w}{\pi} \operatorname{sinc} w(f_p - f_q)\right)_{p,q = 1, \ldots, r+l},$$

with $t \in \mathbb{R}$ and $f_p = n_p$ (s_p) when $p = \overline{1, r}$ $(p = \overline{r+1, l})$ compare [9] and [10] (Theorem 3.7).

The main goal of the paper such that precises and generalizes the results of [11] is to give direct finite optimal size interpolation formulae of Lagrange – Yen type for the functions spaces L_B^2, $B = \diamond$, $[-\pi, \pi]^d$, truncating the sampling sums to the irregularly sampled signal functions, having on mind the already known relative approximation error. The simple truncation error upper bounds are the main results in numerical implementations of the derived results, since they not contain infinite products, iterative procedures and unknown function values. Convergence rates are obtained in all considered cases, when the interpolation sum size grows to the infinity.

2. Lagrange – Yen interpolation in $L_\diamond^2$. *2.1. Introduction with preparation.* Let us consider the closed, connected twodimensional region $B \subset \mathbb{R}^2$ such that tessellates the same plane. (That means, the union of all integer translates of B covers $\mathbb{R}^2$.) The bivariate Fourier transform couple $f, f^\wedge$ is given as usual:

$$f^\wedge(u,v) = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} e^{-i(ux+vy)} f(x,y)\,dxdy,$$

$$f(x,y) = \frac{1}{2\pi} \int\limits_{\mathrm{Cl}(B)} e^{i(xu+yv)} f^\wedge(u,v)\,dudv.$$

The set of all square integrable on $\mathbb{R}^2$ twovariate functions f having the Fourier spectrum of f supported by closure $\mathrm{Cl}(B)$ is oftenly called Paley – Wiener space L_B^2. The most common space of this kind is $L^2_{[-\pi,\pi]^2}$ and the Whittaker – Kotel'nikov – Shannon theorem for such that f is well-known standard result:

$$f(x,y) = \sum_{(m,n)\in\mathbb{Z}^2} f(m,n)\,\mathrm{sinc}\,\pi(x-m)\,\mathrm{sinc}\,\pi(y-n), \qquad x,y \in \mathbb{R}^2,$$

where the convergence is absolute and uniform on $\mathbb{R}^2$ [3].

In this paper we consider the functions space $L^2_{\mathrm{Cl}(\{(\lambda,\mu)|\ |\lambda|+|\mu|<\pi\})}$, where for the sake of brevity we will write $\diamond$ for the closure $\mathrm{Cl}(\{(\lambda,\mu)|\ |\lambda| + |\mu| < \pi\})$. By obvious reasons we can arrange just finite size samples for f reading on the lattice $\mathbb{Z}^2$, and these sampling nodes (m,n) are moved from the uniform spacing into $\zeta_{mn} = (x_m, y_n) = (m + h_m, n + g_n)$, where $|h_m| \le M, |g_n| \le N$ (by the Sun – Zhou extension of the Kadets theorem the best possible bounds are $M, N < 1/4$, see [12]), assume that this situation appears when $|x - m| \le L,\ |y - n| \le L$, compare the approaches in [7, 8]. (For additional informations on sampling bandlimited homogeneous random fields consult e.g. [13].)

We have to point out that $L_\diamond^2$ is Hilbert space with the scalar product

$$\langle f, g\rangle := \int\limits_{\mathbb{R}^2} f(x,y)\overline{g(x,y)}\,dxdy \equiv \int\limits_{\diamond} f^\wedge(u,v)\overline{g^\wedge(u,v)}\,dudv \qquad \forall f,g \in L_\diamond^2,$$

which implies the norm $\|f\|_{2,\diamond} = \sqrt{\langle f, f\rangle}$ which is oftenly calling *signal energy*. Our main goal in this chapter is to develop a direct Lagrange – Yen type interpolation formula $S_L(f; x, y)$ for $L_\diamond^2$-functions when the interpolation, relative error level ε and the parameters M, N are already known. The size of parameter L can be directly computed by this method fixing the values of the arguments x, y.

*2.2. **Main results.*** According to the notations introduced in the previous section denote in the sequel

$$\Lambda_{h,M} := \{\lambda_n\} = (\mathbb{Z} \setminus \{n \mid |t-n| \le L\}) \cup \{\lambda_n = n + h_n \mid |t-n| \le L\}, \qquad |h_n| \le M,$$

for fixed $t \in \mathbb{R}$, and some positive integer L. Using the well known infinite product representation of the sine one defines the sampling function:

$$G_L(t) = (t - h_0)\operatorname{sinc}(\pi t) \prod_{|t-k| \le L} \left(1 - \frac{h_k}{t-k}\right) \frac{k}{\lambda_k}, \qquad |h_k| \le M,$$

called *window canonical product* in [7, 8].

For $f \in L^2_\diamond$ and $\mathbf{\Lambda}_2 := \Lambda_{h,M} \times \Lambda_{g,N}$ we write $\|f|\mathbf{\Lambda}_2\| = \sum_{(\lambda_n,\mu_m)\in\mathbf{\Lambda}_2} |f(\lambda_n, \mu_m)|^2$. Then $\mathbf{\Lambda}_2$ is said to be a *set of sampling* if there exists absolute constants C_1, C_2 such that

$$C_1\|f\|_\diamond \le \|f|\mathbf{\Lambda}_2\| \le C_2\|f\|_\diamond.$$

A set $\mathbf{\Lambda}_2$ is said to be a *set of uniqueness* if $\forall f \in L^2_\diamond$ that vanishes on $\mathbf{\Lambda}_2$ vanishes identically. The classical Plancherel–Pólya theorem states that the upper sampling inequality it holds for *uniformly discrete sets* in L^2_π [14, p. 97][1]. With the condition $M < 1/2$ the set $\Lambda_{h,M}$ becomes uniformly discrete, so $\mathbf{\Lambda}_2$ is uniformly discrete as well, so it is a set of uniqueness. It is well-known that the Lagrange interpolation formula remains valid when only a finite number of sampling nodes deviate from the integers, therefore the following reconstruction formula holds for the function $f \in L^2_\diamond$ reading on the lattice $\mathbf{\Lambda}_2 \equiv \Lambda_{h,M} \times \Lambda_{g,N}$:

$$f(x,y) = \underbrace{\sum_{|x-m|\le L} \sum_{|y-n|\le L} \frac{f(x_m + y_n, x_m - y_n) G_L\left(\frac{x+y}{2}\right) G_L\left(\frac{x-y}{2}\right)}{G'_L(x_m) G'_L(y_n) \left(\frac{x+y}{2} - x_m\right)\left(\frac{x-y}{2} - y_n\right)}}_{S_L(f;x,y)} +$$

$$+ \underbrace{\sum_{|x-m|> L} \sum_{|y-n|\le L} \frac{f(m + y_n, m - y_n) G_L\left(\frac{x+y}{2}\right) G_L\left(\frac{x-y}{2}\right)}{G'_L(m) G'_L(y_n) \left(\frac{x+y}{2} - m\right)\left(\frac{x-y}{2} - y_n\right)}}_{R_L^{(1)}(f;x,y)} +$$

$$+ \underbrace{\sum_{|x-m|\le L} \sum_{|y-n|> L} \frac{f(x_m + n, x_m - n) G_L\left(\frac{x+y}{2}\right) G_L\left(\frac{x-y}{2}\right)}{G'_L(x_m) G'_L(n) \left(\frac{x+y}{2} - x_m\right)\left(\frac{x-y}{2} - n\right)}}_{R_L^{(2)}(f;x,y)} +$$

[1] The increasing sequence of reals $\Lambda = \{\lambda_n\}$ is uniformly discrete if $|\lambda_n - \lambda_m| \ge \varepsilon > 0, \; n \ne m$.

$$+\underbrace{\sum_{|x-m|>L}\sum_{|y-n|>L}\frac{f(m+n,m-n)G_L\left(\frac{x+y}{2}\right)G_L\left(\frac{x-y}{2}\right)}{G'_L(m)G'_L(n)\left(\frac{x+y}{2}-m\right)\left(\frac{x-y}{2}-n\right)}}_{R_L^{(3)}(f;x,y)}. \tag{8}$$

The right-side series converges absolutely and uniformly on $\mathbb{R}^2$. (The case of 2D bandlimited to $\diamond$ homogeneous random fields with $M,N=0$ is discussed in [15].) For convenience the deviation bounds M,N have to be precised in (12); we are taking such M,N that $M+N<1/2$; this condition is used according to the Sun–Zhou result [12].

In the approximation $f\approx S_L(f;x,y)$ we have to be under the prescribed error level ε, even the already precised values of parameters M,N are considered. The truncation error is the remainder $T_L(f;x,y)=|f(x,y)-S_L(f;x,y)|$ of the expansion (8). Then there remains the problem of choosing L which minimizes an estimate of $T_L(f;x,y)$ under the constraint

$$T_L(f;x,y)\le\varepsilon\|f\|_{2,\diamond}.$$

In the sequel we need estimates for the truncation error in our approximation. In order to choose optimal L we evaluate the Lagrange-type sampling function

$$\psi_n(\Lambda_2,t)\equiv\psi_n(t):=\frac{G_L(t)}{G'_L(\lambda_n)(t-\lambda_n)},\qquad \lambda_n\in\Lambda_{\alpha,M}.$$

Lemma 1. *For all k satisfying $|t-k|>L$, t fixed and $M<1/2$, $L\ge3$ we have*

$$|\psi_k(t)|=\left|\frac{G_L(t)}{G'_L(k)(t-k)}\right|\le A_M\frac{L^{2M-1}(2L+1)^{M+1}}{|t-k|}, \tag{9}$$

where

$$A_M=\frac{3(4e^{2+2M/3})^M(1+2M)^{1-2M}(2M+1)}{4\pi(1-M)^{1+M}(3-M)}. \tag{10}$$

Moreover for all k given by $|t-k|\le L$ and for all $M<1/3$, $L\ge3$ it holds

$$|\psi_k^*(t)|=\left|\frac{G_L(t)}{G'_L(\lambda_k)(t-\lambda_k)}\right|\le B_M\frac{L^{2M}(2L+1)^{3M}}{|t-\lambda_k|}, \tag{11}$$

where

$$B_M=\frac{22(4e^2\sqrt[9]{e^2})^M(1+2M)^{1-2M}}{(1-3M)^{1+3M}(11-12M)}. \tag{12}$$

Proof. Following the procedure used by Flornes in [7, p. 59–60], we clearly get (9) with the constant A_M which modestly differs from hers.

Othersides, since $\sin\pi u\ge 2u$ for all $u\in[0,1/2]$ we deduce

$$|\psi_k^*(t)|=\left|\frac{h_k}{t-\lambda_k}\,\frac{\sin\pi t}{(-1)^k\sin\pi h_k}\prod_{|j-t|\le L,\ j\ne k}\frac{\lambda_j-t}{\lambda_j-\lambda_k}\,\frac{j-\lambda_k}{j-t}\right|\le$$

$$\leq \frac{2}{|t-\lambda_k|} \prod_{|j-t|\leq L,\ j\neq k} \frac{1+\dfrac{M}{|j-t|}}{1-\dfrac{3M}{|j-k|+M}} \leq \frac{2}{|t-\lambda_k|} \prod_{|j-t|\leq L,\ j\neq k} \frac{1+\dfrac{M}{|j-t|}}{1-\dfrac{3M}{|j-k|}} =$$

$$= \frac{2}{|t-\lambda_k|} \exp\left\{ \sum_{|j-t|\leq L,\ j\neq k} \ln\left(1+\frac{M}{|j-t|}\right) - \ln\left(1-\frac{3M}{|j-k|}\right)\right\}.$$

Now estimating the sums with integrals repeating the evaulation procedure by [7] we clearly derive (11) with the constant B_M which is depending just from M.

The lemma is proved.

Theorem 1. *Let* $(x,y)\in\mathbb{R}^2$, $M,N<1/3$ *and let* $L\geq 3$ *be positive integer. Then for all* $f\in L^2_\diamond$ *and* $\min\{x,y\}\geq L+1/2$, *in the approximation procedure*

$$f(x,y)\approx S_L(f;x,y)=$$

$$=\sum_{|x-m|\leq L}\sum_{|y-n|\leq L} \frac{f(x_m+y_n,x_m-y_n)G_L\left(\dfrac{x+y}{2}\right)G_L\left(\dfrac{x-y}{2}\right)}{G'_L(x_m)G'_L(y_n)\left(\dfrac{x+y}{2}-x_m\right)\left(\dfrac{x-y}{2}-y_n\right)}$$

the following truncation error upper bound appears

$$T_L(f;x,y)\leq 2\sqrt{6}\left(A_MB_N\frac{L^{2M+2N-1}(1+2L)^{M+3N+1}}{\sqrt{L-1}}+\right. \tag{13}$$

$$+A_NB_M\frac{L^{2M+2N-1}(1+2L)^{3M+N+1}}{\sqrt{L-1}}+ \tag{14}$$

$$\left.+A_MA_N\frac{L^{2M+2N-2}(1+2L)^{M+N+2}}{\sqrt{6}(L-1)}\right)\|f\|_\diamond. \tag{15}$$

Proof. At first it is easy to see that

$$T_L(f;x,y)=|f(x,y)-S_L(f;x,y)|\leq\sum_{j=1}^{3}|R_L^{(j)}(f;(x,y)|,$$

where $R_L^{(j)}(f;x,y)$, $j=1,2,3$, are defined by (8). Concentrate at first to $R_L^{(1)}(f;x,y)$. Applying twice the Cauchy–Buniakowsky–Schwarz inequality, having on mind that $f^\wedge$ is square integrable on $\diamond$, by the estimates (9), (11) it follows that

$$R_L^{(1)}(f;x,y)=\left|\sum_{|x-m|>L}\sum_{|y-n|\leq L}\frac{f(m+y_n,m-y_n)G_L\left(\dfrac{x+y}{2}\right)G_L\left(\dfrac{x-y}{2}\right)}{G'_L(m)G'_L(y_n)\left(\dfrac{x+y}{2}-m\right)\left(\dfrac{x-y}{2}-y_n\right)}\right|\leq$$

$$\leq\sum_{|x-m|>L}\sum_{|y-n|\leq L}\left|\frac{f(m+y_n,m-y_n)G_L\left(\dfrac{x+y}{2}\right)G_L\left(\dfrac{x-y}{2}\right)}{G'_L(m)G'_L(y_n)\left(\dfrac{x+y}{2}-m\right)\left(\dfrac{x-y}{2}-y_n\right)}\right|\leq$$

$$\leq \sqrt{\sum_{|x-m|>L} \sum_{|y-n|\leq L} |f(m+y_n, m-y_n)|^2} \times$$

$$\times \sqrt{\sum_{|x-m|>L} \left|\psi_m\left(\frac{x+y}{2}\right)\right|^2 \sum_{|y-n|\leq L} \left|\psi_n^*\left(\frac{x-y}{2}\right)\right|^2} \leq$$

$$\leq \|f\|_{2,\diamond} A_M B_N L^{2M+2N-1}(1+2L)^{M+3N+1} \times$$

$$\times \sqrt{\sum_{|x-m|>L} \frac{1}{\left|\frac{x+y}{2}-m\right|^2}} \sqrt{\sum_{|y-m|\leq L} \frac{1}{\left|\frac{x-y}{2}-y_n\right|^2}} \leq$$

$$\leq 2\sqrt{6} A_M B_N \frac{L^{2M+2N-1}(1+2L)^{M+3N+1}}{\sqrt{L-1}} \|f\|_\diamond.$$

In similar manner we collect the estimates of $R^{(j)}(f;x,y)$, $j = 2,3$, getting (14) and (15). The desired upper bound is just the sum of estimated remainders. Finally, it remains the estimation of the sum

$$\sum_{|x-m|>L} \frac{1}{|x-m|^2} \leq 2 \int\limits_{L-1+x}^{\infty} \frac{dt}{(t-x)^2} = \frac{2}{L-1}. \tag{16}$$

Similarly, for all m satisfying the reasonable bounds $1/2 \leq |x-m| \leq L$ and all $N < 1/3$, it is

$$\sum_{|y-n|\leq L} \frac{1}{|y-y_n|^2} \leq 12. \tag{17}$$

Indeed, we clearly see that

$$\sum_{|y-n|\leq L} \frac{1}{|y-y_n|^2} \leq \sum_{|y-n|\leq L} \frac{1}{(|y-n|-|y_n-n|)^2} \leq \sum_{|y-n|\leq L} \frac{1}{(|y-n|-N)^2} \leq$$

$$\leq \int\limits_{\frac{1}{2}\leq|y-n|\leq L}^{\infty} \frac{dt}{(t-N)^2} = 2\left(\frac{2}{1-2N} - \frac{1}{L-N}\right) \leq \frac{4}{1-2N} \leq 12.$$

This ends the proof of the theorem.

***Example* 1.** Assume that the deviation parameters M, N belong to the simplex

$$\mathrm{Simp}_2 := \{(u,v)|\ u,v \geq 0, 6u+10v < 1,\ 10u+6v < 1,\ 3(u+v) < 1\} \subset [0,1/4)^2.$$

Then no oversampling arises, because the Kadets – Sun – Zhou condition, $M, N < 1/4$ is satisfied.

On the other side for such one (M,N) it holds uniformly on bounded (x,y)-regions from $\mathbb{R}^2$ that

$$\lim_{L\to\infty} S_L(f;x,y) = f(x,y) \quad \forall f \in L^2_\diamond.$$

2.3. Sample size parameter discussion. The Theorem 1 gives us the following mathematical model.

***Model* 1.** For approximation

$$f(x,y) \approx S_L(f;x,y) =$$

$$= \sum_{|x-m|\le L} \sum_{|y-n|\le L} \frac{f(x_m+y_n, x_m-y_n) G_L\left(\frac{x+y}{2}\right) G_L\left(\frac{x-y}{2}\right)}{G'_L(x_m)G'_L(y_n)\left(\frac{x+y}{2}-x_m\right)\left(\frac{x-y}{2}-y_n\right)},$$

find the minimal L with $\min\{x,y\} \ge L+1/2$, such that

$$2\sqrt{6}\frac{L^{2M+2N-1}(1+2L)^{M+N+2}}{\sqrt{L-1}}\Bigg(A_M B_N (1+2L)^{2N-1} +$$

$$+A_N B_M (1+2L)^{2M-1} + \frac{A_M A_N}{L\sqrt{6(L-1)}}\Bigg) \le \epsilon,$$

where $A(\cdot), B(\cdot)$ are defined by (10), (12) respectively.

To find the optimal L we avoid the approximative computation of infinite product in $\operatorname{sinc}\pi t$ and thanks to Yen's method we avoid the trap of the unknown values of $f \in L^2_\diamond$ at nodes which contain some indices $m,n:\ |x-m|>L,\ |y-n|>L$. The norm of the considered signal function is incorporated in the left-hand term in the last relation such that coincides with the relative error of the truncation in $f \approx S_L$.

On the other side we cannot be really satisfied with the developed method because of simple calculation showes us that the approximation procedure $f \approx S_L$ converges with the growing L just for $(M,N) \in \mathrm{Simp}_2$, see the example in the previous chapter. Now it is evident that there is a "huge" space in applications, since Simp_2 is far from exhausting the Kadets region $[0,1/4)^2$. In the goal to escape from this inconveniences we will introduce a weight-function method such that is developed for L^2_π (see the Introduction chapter and [7, 8]). There are weight-functions used, involving new parameters, since by the Kadets's $\frac{1}{4}$-theorem no oversapling occurs just for $4M<1$, i. e.

$$T_L(f;t) = |f(t) - S_L(f;t)| \le C\|f\|_2 L^{2M-\frac{1}{2}} \quad \forall f \in L^2_\pi,$$

compare [16, 17]. If the weight-function ensures the convergence of approximant $S_L(f)$ to f altough M is from $[0,1/2]$, then the whole real axis will be convered by possible measuring times (sampling nodes). The bill for this effort will be applyed with oversampling.

Finally, we point out that no generalizations are possible to functions spaces with higher dimensional support sets on traces of the presented method, because the tessellation of $\mathbb{R}^2$ by $\diamond$ cannot be followed. For example consider $\mathbb{R}^3$. The octahedron $\mathrm{Cl}(\{(u,v,w)|\ |u|+|v|+|w|<\pi\})$ does not tessellate $\mathbb{R}^3$. Altough them we can cover the basic tessellation cell B always with a suitably large cube $[-w,w]^3$, say, but the use

Kotel'nikov – Shannon sampling formulae with respect of $[-w, w]^3 \supset B$ automathically means oversampled sampling sums.

3. Weighted Lagrange – Yen interpolation in $L^2_{[-\pi,\pi]^d}$. ***3.1. Introductionary remarks.*** Denote

$$\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d, \quad \mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}^d, \quad \mathbf{M} = (M_1, \ldots, M_d) \geq 0$$

and let $\mathbf{L} = (L_1, \ldots, L_d)$ be a d-tuple of positive integers up to 3; endly we will write $D := \{1, \ldots, d\} \subset \mathbb{N}$. The entire product of two d-tuples $\mathbf{a}, \mathbf{b}$, say, we write $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j=1}^d a_j b_j$. Let us denote $L^2_{[-\pi,\pi]^d}$ the Paley – Wiener type functions space of entire d-variate functions f which Fourier spectrum is supported by $[-\pi, \pi]^d$, $d \geq 2$, i. e.

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} \int\limits_{([-\pi,\pi]^d)} e^{i\langle \mathbf{x}, \mathbf{u} \rangle} f^\wedge(\mathbf{u})\, d\mathbf{u}, \qquad f^\wedge(\mathbf{u}) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^d} e^{-i\langle \mathbf{u}, \mathbf{x} \rangle} f(\mathbf{x})\, d\mathbf{x}.$$

The elements of $L^2_{[-\pi,\pi]^d}$ we oftenly call *signals* bandlimited to $[-\pi, \pi]^d$. We have to mention that $L^2_{[-\pi,\pi]^d}$ is Hilbert space equipped with scalar product

$$\langle f, g \rangle := \int\limits_{\mathbb{R}^d} f(\mathbf{x}) \overline{g(\mathbf{y})}\, d\mathbf{x} d\mathbf{y},$$

so the depending norm of f is ${}_d\|f\|_2 = \sqrt{\langle f, f \rangle}$.

The Kotel'nikov – Shannon theorem with regular/uniform sample nodes for such signal f is standard result:

$$f(\mathbf{x}) = \sum_{\mathbf{m} \in \mathbb{Z}^d} f(\mathbf{m}) \prod_{j=1}^d \operatorname{sinc} \pi(x_j - m_j), \qquad \mathbf{x} \in \mathbb{R}^d,$$

where the convergence is absolute and uniform on $\mathbb{R}^d$ [3].

For numerical implementations only finite size samples of F can be measured reading from the lattice $\mathbb{Z}^d$ around fixed value of the arguments $\mathbf{x}$ that $|x_j - m_j| \leq L_j$, $j = \overline{1, d}$. The sampling nodes $\mathbf{m}$, where f will be measured, could deviate from the uniform spacing into $\lambda_{\mathbf{m}} = (\lambda^1_{m_1}, \ldots, \lambda^d_{m_d}) = \mathbf{m} + \mathbf{h} = (m_1 + h^1_{m_1}, \ldots, m_d + h^d_{m_d})$, where $|h^j_{l_j}| \leq M_j$, $j = \overline{1, d}$, $l_j \in \mathbb{Z}$. The set of sampling we denote now with $\Lambda_d = \Lambda_{h^1, M_1} \times \ldots \times \Lambda_{h^d, M_d}$.

3.2. Main results. In this chapter our main goal is to develop a direct weighted Lagrange – Yen type interpolation formula for $L^2_{[-\pi,\pi]^d}$-functions when the interpolation relative error level ε and the parameter vector $\mathbf{M}$ is already known. The interpolation sum size vector $\mathbf{L}$ can be directly computed by the proposed method. Let us recall the notations:

$$\psi_{m_k}(x_k) = \frac{G_{L_k}(x_k)}{G'_{L_k}(m_k)(x_k - m_k)}, \qquad \psi^*_{m_j}(x_j) = \frac{G_{L_j}(x_j)}{G'_{L_j}(\lambda^j_{m_j})(x_j - \lambda^j_{m_j})},$$

where $k: \ |x_k - m_k| > L_k$ and $j: \ |x_j - m_j| \leq L_j$, $j, k = \overline{1, d}$.

Theorem 2. *For every $f \in L^2_{[-\pi,\pi]^d}$ we have*

$$f(\mathbf{x}) = \underbrace{\sum_{j=1}^{d} \sum_{|x_j - m_j| \le L_j} f(\lambda_{\mathbf{m}}) \prod_{l=1}^{d} \psi^*_{m_l}(x_l)}_{S_{\mathbf{L}}(f;\mathbf{x})} +$$

$$+ \sum_{j=1}^{d} \sum_{|x_j - m_j| > L_j} f(\mathbf{m}) \prod_{l=1}^{d} \psi_{m_l}(x_l) +$$

$$+ \sum_{|x_1 - m_1| \le L_1} \sum_{p=2}^{d} \sum_{|x_p - m_p| > L_p} f(\lambda^1_{m_1}, m_2, \ldots, m_d) \psi^*_{m_1}(x_1) \prod_{l=2}^{d} \psi_{m_l}(x_l) +$$

$$+ \sum_{|x_2 - m_2| \le L_2} \sum_{\substack{p=1 \\ p \ne 2}}^{d} \sum_{|x_p - m_p| > L_p} f(m_1, \lambda^2_{m_2}, m_3, \ldots, m_d) \psi^*_{m_2}(x_2) \prod_{\substack{l=1 \\ l \ne 2}}^{d} \psi_{m_l}(x_l) + \ldots$$

$$\ldots + \sum_{|x_d - m_d| \le L_d} \sum_{p=1}^{d-1} \sum_{|x_p - m_p| > L_p} f(m_1, \ldots, m_{d-1}, \lambda^d_{m_d}) \psi^*_{m_d}(x_d) \prod_{l=1}^{d-1} \psi_{m_l}(x_l) + \ldots$$

$$\ldots + \sum_{|x_d - m_d| > L_d} \sum_{p=1}^{d-1} \sum_{|x_p - m_p| \le L_p} f(\lambda^1_{m_1}, \ldots, \lambda^{d-1}_{m_{d-1}}, m_d) \psi_{m_d}(x_d) \prod_{l=1}^{d-1} \psi^*_{m_l}(x_l). \tag{18}$$

Here the right-side series converges absolutely and uniformly on $\mathbb{R}^d$.

Proof. As for all fixed real t the integer lattice $\mathbb{Z}$ (or its constant multiple!) is set of uniqueness for L^2_π, the same is valid when only finite number of its elements deviates from the uniform (integer) spacing, compare e.g. [7]. The Kotel'nikov–Shannon restoration formula (18) holds reading on the lattice $\Lambda_d = \Lambda_{h^1, M_1} \times \ldots \times \Lambda_{h^d, M_d}$.

Indeed, the right-hand series in (18) converges to certain function $g \in L^2_{[-\pi,\pi]^d}$. It is not hard to see that $f(\lambda_{\mathbf{m}}) = g(\lambda_{\mathbf{m}})$ where $\lambda_{\mathbf{m}} \in \Lambda_d$. Hence, the function

$$\nu(\mathbf{x}) = (f(\mathbf{x}) - g(\mathbf{x})) \prod_{l=1}^{d} \prod_{m_l : h^l_{m_l} \ne 0} \frac{x_l - m_l}{x_l - \lambda^l_{m_l}}$$

is in $L^2_{[-\pi,\pi]^d}$ too and vanishes on $\mathbb{Z}^d$. Therefore $\nu(\mathbf{x}) = 0$, i.e. $f = g$.

The theorem is proved.

Since the next results are connected to error estimates in the approximation $f \approx S_{\mathbf{L}}(f;\mathbf{x})$, we denote with

$$T_{\mathbf{L}}(f;\mathbf{x}) = |f(\mathbf{x}) - S_{\mathbf{L}}(f;\mathbf{x})|$$

the truncation error upper bound.

Theorem 3. *Let* $0 \le M_j < 1/3$, $j = \overline{1,d}$, *and* $\mathbf{x} \in \mathbb{R}^d$ *such that* $\min(x_j - L_j) \ge 1/2$. *Then for all* $f \in L^2_{[-\pi,\pi]^d}$, *we have*

$$T_{\mathbf{L}}(f;\mathbf{x}) \le \left(\sqrt{2^d} \sum_{s=1}^{d} \sqrt{6^s} \sum_{1 \le j_1 < \ldots < j_s \le d} \prod_{r=1}^{s} B(M_{j_r}) L_{j_r}^{2M_{j_r}} (1 + 2L_{j_r})^{3M_{j_r}} \times \right.$$

$$\left. \times \prod_{k \in D \setminus \{j_1, \ldots, j_s\}} A(M_k) \frac{L_k^{2M_k - 1}(1 + 2L_k)^{M_k + 1}}{\sqrt{(L_k - 1)}} \right) {}_d\|f\|_2. \tag{19}$$

Proof. At first denote

$$R^{(j_1,\ldots,j_s)}_{L_{j_1},\ldots,L_{j_s}}(f;\mathbf{x}) =$$

$$= \underbrace{\prod_{r=1}^{s} \sum_{|x_{j_r} - m_{j_r}| \le L_{j_r}} \prod_{k \in D \setminus \{j_1,\ldots,j_s\}} \sum_{|x_k - m_k| > L_k}}_{\Sigma_4} f(\lambda^{j_s}_{j_1}) \psi^*_{m_{j_r}}(x_{j_r}) \psi_{m_k}(x_k),$$

the addend indexed by the multiindex $(j_1, \ldots, j_s)$ addend in the remainder of the approximation $f(\mathbf{x}) \approx S_{\mathbf{L}}(f : \mathbf{x})$, using the abbreviated notation

$$f(\lambda^{j_s}_{j_1}) := f(m_1, \ldots, \lambda^{j_1}_{m_{j_1}}, \ldots, \lambda^{j_s}_{m_{j_s}}, \ldots, m_d).$$

Now, by the Cauchy – Bunyakowsky – Schwarz inequality one gets

$$R^{(j_1,\ldots,j_s)}_{L_{j_1},\ldots,L_{j_s}}(f;\mathbf{x}) \le \sqrt{\textstyle\sum_4 |f(\lambda^{j_s}_{j_1})|^2} \sqrt{\textstyle\sum_4 |\psi^*_{m_{j_r}}(x_{j_r})|^2 |\psi_{m_k}(x_k)|^2} \equiv H.$$

As $\sqrt{\sum_4 |f(\lambda^{j_s}_{j_1})|^2} \le {}_d\|f\|_2$ and $\sum_4 |\psi^*_{m_{j_r}}(x_{j_r})|^2 |\psi_{m_k}(x_k)|^2$ is splitting into product of two multiple sums, by Lemma 1 we deduce

$$H \le {}_d\|f\|_2 \prod_{k \in D \setminus \{j_1,\ldots,j_s\}} \sum_{|x_k - m_k| > L_k} A(M_k) L_k^{2M_k - 1} (1 + 2L_k)^{m_k + 1} \times$$

$$\times \sqrt{\sum_{|x_k - m_k| > L_k} \frac{1}{|x_k - m_k|^2}} \prod_{r=1}^{s} \sum_{|x_{j_r} - m_{j_r}| \le L_{j_r}} B(M_{j_r}) L_{j_r}^{2M_{j_r}} (1 + 2L_{j_r})^{3M_{j_r}} \times$$

$$\times \sqrt{\sum_{|x_{j_r} - m_{j_r}| \le L_{j_r}} \frac{1}{|x_{j_r} - m_{j_r}|^2}} \le$$

$$\le \left(\sqrt{2^d 6^s} \prod_{k \in D \setminus \{j_1,\ldots,j_s\}} \sum_{|x_k - m_k| > L_k} A(M_k) \frac{L_k^{2M_k - 1}(1 + 2L_k)^{m_k + 1}}{\sqrt{L_k - 1}} \times \right.$$

$$\left. \times \prod_{r=1}^{s} \sum_{|x_{j_r} - m_{j_r}| \le L_{j_r}} B(M_{j_r}) L_{j_r}^{2M_{j_r}} (1 + 2L_{j_r})^{3M_{j_r}} \right) {}_d\|f\|_2,$$

where the sum evaluations (16) and (17) are used. Finally, by

$$T_{\mathbf{L}}(f;\mathbf{x}) \le \sum_{s=1}^{d} \sum_{1\le j_1<\ldots<j_s\le d} |R^{(j_1,\ldots,j_s)}_{L_{j_1},\ldots,L_{j_s}}(f;\mathbf{x})|$$

we deduce the asserted truncation error upper bound (19).

The theorem is proved.

In all numerical implementations it is more convenient to put $L_j = L$, $j=\overline{1,d}$, i. e. the rectangle of sampling nodes we replace with a sampling cube, $\mathbf{L} = (L,\ldots,L)_{1\times d}$.

***Corollary* 1.** *When* $\mathbf{M} = (M_1,\ldots,M_d)$ *belongs to the convex d-dimensional region* $\mathrm{Simp}_d = \{\mathbf{u}|\ (C)\cap \mathbf{u}\ge 0\}$, *where*

$$6\sum_{j=1}^{d} u_j < d,$$

$$5\sum_{r=1}^{s} u_{j_r} + 3\sum_{k\in D\setminus\{j_1,\ldots,j_s\}} u_k < \frac{1}{2}(d-s), \qquad (C)$$

as $1\le j_1<\ldots<j_s\le d$, $s=\overline{1,d-1}$, *then*

$$\lim_{L\to\infty} S_{\mathbf{L}}(f;\mathbf{x}) = f(\mathbf{x})$$

uniformly for all bounded $\mathbf{x}$*-domains from* $\mathbb{R}^d$ *and in sampling reconstruction formula* (18) *no oversampling occurs.*

Proof. Put $L_j = L$, $j=\overline{1,d}$, in (19) and $C_{M,s}$ for suitable absolute constants, we get

$$T_{\mathbf{L}}(f;\mathbf{x}) \le$$

$$\le \sum_{s=1}^{d} \sum_{1\le j_1<\ldots<j_s\le d} C_{M,s} \frac{L^{2\sum_k M_k - d + s + 2\sum_{r=1}^{s} M_{j_r}} (1+2L)^{\sum_k M_k + d - s + 3\sum_{l=1}^{s} M_{j_l}}}{\sqrt{(L-1)^{d-s}}} =$$

$$= \mathcal{O}\left(L^{\max_{1\le s\le d}\{5\sum_{r=1}^{s} M_{j_r} + 3\sum_{k\in D\setminus\{j_1,\ldots,j_s\}} M_k + \frac{s}{2}\} - \frac{d}{2}}\right).$$

Therefore by (C) we deduce the assertion of the Corollary. As $\mathrm{Simp}_d \subseteq [0,1/4)^d$, no oversampling occurs in our case.

***Model* 2.** In the approximation

$$f(\mathbf{x}) \approx S_{\mathbf{L}}(f;\mathbf{x}) = \sum_{j=1}^{d} \sum_{|x_j - m_j|\le L_j} f(\lambda_{\mathbf{m}}) \prod_{l=1}^{d} \psi^{*}_{m_l}(x_l),$$

for already known $\varepsilon > 0$ find the minimal L with $\min_{1\le j\le d}(x_j - L_j) \ge 1/2$, such that

$$\sqrt{2^d}\sum_{s=1}^{d}\sqrt{6^s} \sum_{1\le j_1<\ldots<j_s\le d} \prod_{l=1}^{s} B(M_{j_l}) L_{j_l}^{2M_{j_l}} (1+2L_{j_l})^{3M_{j_l}} \times$$

$$\times \prod_{k\in D\setminus\{j_1,\dots,j_s\}} A(M_k)\frac{L_k^{2M_k-1}(1+2L_k)^{M_k+1}}{\sqrt{L_k-1}} \le \varepsilon$$

where $A(\cdot), B(\cdot)$ are defined by (10), (12) respectively.

4. Weighted interpolation in $L^2_{[-\pi,\pi]^d}$. The efficiency of the approximation procedure

$$f(\mathbf{x}) \approx S_{\mathbf{L}}(f;\mathbf{x}) = \sum_{j=1}^{d}\sum_{|x_j-m_j|\le L_j} f(\lambda_{\mathbf{m}})\prod_{p=1}^{d}\psi^*_{m_p}(x_p)$$

is not quietly satisfactory, because for positive parameter vector $\mathbf{M}$ the interpolation formula $S_{\mathbf{L}}(f;\mathbf{x})$ is to long and for convergence purposes we have only very limited choice for choosing suitable $\mathbf{M}$ that the truncation error upper bound vanishes when the minimal coordinatewise sampling size parameter $L^* = \min_{1\le j\le d} L_j$ runs to the infinity.

Therefore from now on we assume that the signal function f belongs to Paley–Wiener type functions space $L^2_{\mathbf{T}_d}$, $\mathbf{T}_d := [-\tau_1,\tau_1]\times\ldots\times[-\tau_d,\tau_d]$, where $\delta = (\delta_1,\dots,\delta_p) = (\pi-\tau_1,\dots,\pi-\tau_d) > 0$. So, we follow the onedimensional approach of exposed in [7] and one introduces the weight-function

$$w_{a,l}(t) = (\mathrm{sinc}(at/l))^l.$$

Of course $L^2_{\mathbf{T}_d} \subset L^2_{[-\pi,\pi]^d}$. At this point we have to remark that the type of the entire function $w_{a,l}(t)$ is equal to a.

Theorem 4. *Let f be in $L^2_{\mathbf{T}_d}$. Then it holds uniformly for $\mathbf{x}\in\mathbb{R}^d$ the following restoration formula:*

$$f(\mathbf{x}) = \underbrace{\sum_{j=1}^{d}\sum_{|x_j-m_j|\le L_j} f(\lambda_{\mathbf{m}})\prod_{k=1}^{d}\psi^*_{m_k}(x_k)w_{\delta_k,l_k}(x_k-\lambda^k_{m_k})}_{\sigma_{\mathbf{L}}(f;\mathbf{x})} +$$

$$+\sum_{j=1}^{d}\sum_{|x_j-m_j|>L_j} f(\mathbf{m})\prod_{k=1}^{d}\psi_{m_k}(x_k)w_{\delta_k,l_k}(x_k-m_k) +$$

$$+\sum_{|x_1-m_1|\le L_1}\sum_{p=2}^{d}\sum_{|x_p-m_p|>L_p} f(\lambda^1_{m_1},m_2,\dots,m_d)\psi^*_{m_1}(x_1)w_{\delta_1,l_1}(x_1-\lambda^1_{m_1})\times$$

$$\times\prod_{k=2}^{d}\psi_{m_k}(x_k)w_{\delta_k,l_k}(x_k-m_k) +$$

$$+\sum_{|x_2-m_2|\le L_2}\sum_{\substack{p=1\\ p\ne 2}}^{d}\sum_{|x_p-m_p|>L_p} f(m_1,\lambda^2_{m_2},m_3,\dots,m_d)\psi^*_{m_2}(x_2)w_{\delta_2,l_2}(x_2-\lambda^2_{m_2})\times$$

$$\times\prod_{2\ne k=1}^{d}\psi_{m_k}(x_k)w_{\delta_k,l_k}(x_k-m_k) + \ldots$$

$$\ldots + \sum_{|x_d-m_d|\le L_d} \sum_{p=1}^{d-1} \sum_{|x_p-m_p|>L_p} f(m_1,\ldots,m_{d-1},\lambda^d_{m_d})\psi^*_{m_d}(x_d)w_{\delta_d,l_d}(x_d-\lambda^d_{m_d}) \times$$

$$\times \prod_{k=1}^{d-1} \psi_{m_k}(x_k)w_{\delta_k,l_k}(x_k-m_k) + \ldots$$

$$\ldots + \sum_{|x_d-m_d|>L_d} \sum_{p=1}^{d-1} \sum_{|x_p-m_p|\le L_p} f(\lambda^1_{m_1},\ldots,\lambda^{d-1}_{m_{d-1}},m_d)\psi_{m_d}(x_d)w_{\delta_d,l_d}(x_d-m_d) \times$$

$$\times \prod_{k=1}^{d-1} \psi^*_{m_k}(x_k)w_{\delta_k,l_k}(x_k-\lambda^k_{m_k}). \tag{20}$$

Proof. As $f \in L^2_{\mathbf{T}_d}$, we apply (18) to the function

$$f(\mathbf{x})\prod_{p=1}^{d} w_{\delta_p,l_p}(t_p-x_p) \in L^2_{[-\pi,\pi]^d}$$

specifying $t_s = x_s$, $s = \overline{1,d}$. All limit processes and the convergence are valid because Λ_d is the set of sampling in $L^2_{[-\pi,\pi]^d}$. Therefore we derive (20) by the Valiron–Boas–Bernstein method.

The theorem is proved.

In the next step we introduce the truncation error, i. e. the remainder

$$\tau_{\mathbf{L}}(f;\mathbf{x}) = |f(\mathbf{x}) - \sigma_{\mathbf{L}}(f;\mathbf{x})|$$

in the expansion (20), and we give an estimate of this quantity.

Theorem 5. *Let f be band-limited to $\mathbf{T}_d$ Paley–Wiener type function, i. e.*

$$f \in L^2_{\mathbf{T}_d}, \quad \mathbf{T}_d := [-\tau_1,\tau_1]\times\ldots\times[-\tau_d,\tau_d],$$

where $\delta = (\delta_1,\ldots,\delta_p) = (\pi-\tau_1,\ldots,\pi-\tau_d) > 0$, $M_j < 1/3$, $j = \overline{1,d}$, and for all positive integers $\mathbf{l} = (l_1,\ldots,l_d)$ and $\mathbf{x}\in\mathbb{R}^d : \min_{1\le j\le d}(x_j - L_j) \ge 1/2$, we have

$$\tau_{\mathbf{L}}(f;\mathbf{x}) \le \left(2^{d/2}\left[\prod_{j=1}^{d}\frac{(l_j/\delta_j)^{l_j}}{\sqrt{2l_j+1}}\right]\sum_{s=1}^{d}\sum_{1\le j_1<\ldots<j_s\le d} 6^{s/2+\sum_{r=1}^{s} l_{j_r}} \times\right.$$

$$\times \prod_{k\in D\setminus\{j_1,\ldots,j_s\}}^{*} A(M_k)\frac{L_k^{2M_k-1}(1+2L_k)^{M_k+1}}{\sqrt{(L_k-1)^{2(l_k+1)}}} \times$$

$$\left.\times \prod_{p=1}^{s} B(M_{j_p})L_{j_p}^{2M_{j_p}}(1+2L_{j_p})^{3M_{j_p}}\right)_d \|f\|_2. \tag{21}$$

Proof. It is not hard to see that

$$|w_{\delta,l}(t)| = |\mathrm{sinc}^l(\delta t/l)| \le l^l(\delta|t|)^{-l}.$$

So following the steps of the proof of Theorem 2 it only remains to apply the upper bounds for the following sums:

$$\sum_{|x_{j_p}-m_{j_p}|\le L_{j_p}} \frac{1}{|x_{j_p}-\lambda^{j_p}_{m_{j_p}}|^{2(l_p+1)}} \le \frac{36^{l_p+1}}{3(2l_p+1)},$$

$$\sum_{|x_k-m_k|>L_k} \frac{1}{|x_k-m_k|^{2(l_k+1)}} \le \frac{2}{(2l_k+1)(L_k-1)^{2l_k+1}}.$$

(We derive these upper bounds following the procedure in getting the evaluations (16), (17).) Now, obvious transformations gives us (21).

The theorem is proved.

Similarly as in the previous subsection, we are interested now in the convergence question discussion in the approximation $f \approx \sigma_{\mathbf{L}}$, when we specify $L_j \equiv L \to \infty$.

Corollary 2. *When* $\mathbf{M} = (M_1, \ldots, M_d)$ *belongs to the convex d-dimensional region* $\mathrm{Simp}_d = \{\mathbf{u}|\ (E) \cap \mathbf{u} \ge 0\}$, *where*

$$6\sum_{j=1}^d u_j < d + \sum_{j=1}^d l_j,$$

$$5\sum_{r=1}^s u_{j_r} + 3\sum_{k\in D\setminus\{j_1,\ldots,j_s\}} u_k < \frac{1}{2}(d-s) + \sum_{k\in D\setminus\{j_1,\ldots,j_s\}} l_k, \qquad (E)$$

and $1 \le j_1 < \ldots < j_s \le d,\ s = \overline{1, d-1}$, *then*

$$\lim_{L\to\infty} \sigma_L(f;\mathbf{x}) = f(\mathbf{x}), \qquad (22)$$

uniformly in any bunded $\mathbf{x}$*-region in* $\mathbb{R}^d$ *without oversampling.*

Proof. Specifying $L_j = L$, $j = \overline{1, n}$, in the truncation error upper bound (21), by the same procedure as in the proof of the Corollary 2, we deduce

$$\tau_{\mathbf{L}}(f;\mathbf{x}) = \mathcal{O}\left(L^{\max_{1\le s\le d}\{5\sum_{r=1}^s M_{j_s}+3\sum_{k\in D\setminus\{j_1,\ldots,j_s\}}(M_k-l_k)+\frac{s}{2}\}-\frac{d}{2}}\right).$$

Therefore as $L \to \infty$ we get (22). Finally, since Simp_d is proper subset of the Kadets–Sun–Zhou region $[0, 1/4)^d$, no oversampling occurs in our case.

Model 3. To optimize $\mathbf{L}$ in the approximation

$$f(\mathbf{x}) \approx \sigma_{\mathbf{L}}(f;\mathbf{x}) = \sum_{j=1}^d \sum_{|x_j-m_j|\le L_j} f(\lambda_{\mathbf{m}}) \prod_{k=1}^d \psi^*_{m_k}(x_k) w_{\delta_k,l_k}(x_k - \lambda^k_{m_k}),$$

for already known $\varepsilon > 0$ solve with respect to $\mathbf{L}$ with $\min_{1\le j\le d}(x_j - L_j) \ge 1/2$, such that

$$2^{d/2}\left[\prod_{j=1}^d \frac{(l_j/\delta_j)^{l_j}}{\sqrt{2l_j+1}}\right] \sum_{s=1}^d \sum_{1\le j_1<\ldots<j_s\le d} 6^{s/2+\sum_{r=1}^s l_{j_r}} \times$$

$$\times \prod_{k \in D \setminus \{j_1, \ldots, j_s\}} A(M_k) \frac{L_k^{2M_k - 1}(1 + 2L_k)^{M_k + 1}}{\sqrt{(L_k - 1)^{2l_k + 1}}} \times$$

$$\times \prod_{r=1}^{s} B(M_{j_r}) L_{j_r}^{2M_{j_r}} (1 + 2L_{j_r})^{3M_{j_r}} < \epsilon,$$

where $A(\cdot), B(\cdot)$ are defined by (10), (12) respectively.

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