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QUASI-FROBENIUS RINGS AND NAKAYAMA PERMUTATIONS OF SEMI-PERFECT RINGS*

КВАЗІФРОБЕНІУСОВІ КІЛЬЦЯ ТА ПІДСТАНОВКИ НАКАЯМИ НАПІВДОСКОНАЛИХ КІЛЕЦЬ

We say that $\mathcal{A}$ is a ring with duality for simple modules, or simply a *DSM*-ring, if for each simple right (left) $\mathcal{A}$-module U, the dual module U^* is a simple left (right) $\mathcal{A}$-module. We prove that a semi-perfect ring is a *DSM*-ring if and only if it admits a Nakayama permutation. We introduce the notion of a monomial ideal of a semi-perfect ring and study the structure of the hereditary semi-perfect rings with monomial ideals. We consider perfect rings with monomial socles.

Кільце $\mathcal{A}$ називається кільцем з дуальністю для простих модулів, або *DSM*-кільцем, якщо модуль U^*, дуальний до будь-якого простого правого (лівого) $\mathcal{A}$-модуля U, є простим лівим (правим) $\mathcal{A}$-модулем. Встановлено, що напівдосконале кільце є *DSM*-кільцем тоді і тільки тоді, коли воно допускає підстановку Накаями. Введено поняття мономіального ідеалу напівдосконалого кільця та вивчено будову спадкових напівдосконалих кілець із такими ідеалами. Розглянуто досконалі кільця з мономіальними цоколями.

1. Introduction. The class of quasi-Frobenius rings, introduced by T. Nakayama in [1, 2], is one of the most interesting and intensively studied classes of artinian rings. One of the most significant results on quasi-Frobenius rings is the theorem of C. Faith and E. A. Walker (see, for example, [3]), which says that a ring A is quasi-Frobenius if and only if every projective right A-module is injective. Quasi-Frobenius rings have many interesting properties, in particular, an artinian ring A is quasi-Frobenius if and only if A is a ring with duality for simple modules (see [4], Theorem 58.6). The key concept in the classical definition of quasi-Frobenius rings, given by T. Nakayama, is a permutation of indecomposable projective modules, which is natural to call the *Nakayama permutation* (see [5 – 7]).

The starting point of this article is the Nakayama's definition of quasi-Frobenius rings (Definition 2.1) and the aim is to explore it in a somewhat "economic" way, which means that we avoid the interaction between projectivity and injectivity. Moreover, the notion of injectivity is not used at all in this work.

In Section 2 the quasi-Frobenius rings are defined and several examples are given. In Section 3 we give some basic facts about semi-perfect rings including an easy consequence of the Wedderburn – Artin Theorem, the *Lemma on annihilation of simple modules,* which is a key working tool throughout the paper. In Section 4 we introduce the notion of a *monomial ideal* of a semi-perfect ring, give a result on hereditary rings with monomial ideals and prove that the socle of a quasi-Frobenius ring is monomial (Theorem 4.2). Section 5 is dedicated to the study of socles (using the Osofsky's Lemma [8]) of perfect rings, in particular, perfect rings admitting a Nakayama permutation are considered (Theorem 5.2). In Section 6 we characterize semi-perfect rings with duality for simple modules (Theorem 6.1).

2. Quasi-Frobenius rings. Let $\mathcal{A}$ be a two-sided artinian ring and $\mathcal{R}$ be its Jacobson radical. For a (right) $\mathcal{A}$-module M we denote by M^n the direct sum of n copies of M and we set $M^0 = 0$. Then $\mathcal{A}$ can be represented as a direct sum of right

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ideals: $\mathcal{A} = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$, where $P_1, \ldots, P_s$ are pairwise non-isomorphic indecomposable right $\mathcal{A}$-modules, which are called the *principal right $\mathcal{A}$-modules.* Set $U_i = P_i / P_i \mathcal{R}$, $i = 1, \ldots, s$. It is well-known that $P_1, \ldots, P_s$ represent up to isomorphism all indecomposable projective $\mathcal{A}$-modules, while $U_1, \ldots, U_s$ form a representative set of isomorphism classes of all simple right $\mathcal{A}$-modules. Let M be a right $\mathcal{A}$-module and N be a left $\mathcal{A}$-module. We set $\operatorname{top} M = M / M\mathcal{R}$ and $\operatorname{top} N = N / \mathcal{R}N$. We denote by $\operatorname{soc} M$ (respectively $\operatorname{soc} N$) the largest semi-simple right (respectively left) submodule of M (respectively N). Since $\mathcal{A}$ is artinian, soc exists for all $\mathcal{A}$-modules. Let $1 = f_1 + \ldots + f_s$ be a decomposition of the identity element of $\mathcal{A}$ into a sum of idempotents such that $f_i \mathcal{A} = P_i^{n_i}$, $i = 1, \ldots, s$. Then $\mathcal{A} f_i = Q_i^{n_i}$, where $Q_1, \ldots, Q_s$ are the pairwise non-isomorphic indecomposable projective left $\mathcal{A}$-modules (the *principal left $\mathcal{A}$-modules*). Set $\mathcal{A}_{ij} = f_i \mathcal{A} f_j$, $i, j = 1, \ldots, s$. Then $\mathcal{A}$ has the following *canonical Peirce decomposition*:

$$\mathcal{A} = \bigoplus_{i,j=1}^{s} \mathcal{A}_{ij}. \tag{1}$$

Denote by $\mathcal{R}_i$ the radical of $\mathcal{A}_{ii}$, $i = 1, \ldots, s$. Obviously, $\mathcal{A}_{ii}$ is artinian. Since $\operatorname{Hom}(P_j^{n_j}, P_i^{n_i}) \cong \mathcal{A}_{ij}$, then $\mathcal{A}_{ij} \subset \mathcal{R}$ if $i \neq j$. The radical $\mathcal{R}$ of $\mathcal{A}$ has the following Peirce decomposition:

$$\mathcal{R} = \bigoplus_{i,j=1}^{s} f_i \mathcal{R} f_j, \tag{2}$$

where $f_i \mathcal{R} f_i = \mathcal{R}_i$ and $f_i \mathcal{R} f_j = \mathcal{A}_{ij}$, $i \neq j$, $i = 1, \ldots, s$.

We recall now the classical definition of Frobenius and quasi-Frobenius rings as given by T. Nakayama (see [2, p. 8], [3], Section 13.4).

Definition 2.1. *A two-sided artinian ring $\mathcal{A}$ is called quasi-Frobenius, if there exists a permutation ν of $\{1, 2, \ldots, s\}$ such that for each $k = 1, \ldots, s$ we have:*

(qf_1) $\operatorname{soc} P_k \cong \operatorname{top} P_{\nu(k)}$;

(qf_2) $\operatorname{soc} Q_{\nu(k)} \cong \operatorname{top} Q_k$.

A quasi-Frobenius ring $\mathcal{A}$ is called Frobenius, if $n_{\nu(i)} = n_i$ for all $i = 1, \ldots, s$.

This permutation ν is called the *Nakayama permutation* of $\mathcal{A}$. Clearly, ν is determined up to conjugation in the symmetric group on s letters, and conjugations correspond to renumberings of the principal modules $P_1, \ldots, P_s$.

We construct now some examples of quasi-Frobenius rings. Recall that a local ring O with non-zero unique maximal right ideal $\mathcal{M}$ is called a discrete valuation ring, if it has no zero divisors, the right ideals of O form the unique chain:

$$O \supset \mathcal{M} \supset \mathcal{M}^2 \supset \ldots \supset \mathcal{M}^n \supset \ldots,$$

and, moreover, this chain is also the unique chain of left ideals of $\mathcal{A}$. Then, obviously, O is noetherian, but not artinian, all powers of $\mathcal{M}$ are distinct and $\bigcap_{k=1}^{\infty} \mathcal{M}^k = 0$. Moreover, $\mathcal{M}$ is principal as a right (left) ideal.

***Example* 2.1.** Denote by $\mathcal{H}_s(O)$ the ring of all $s \times s$ matrices of the following form:

$$\mathcal{H} = \mathcal{H}_s(\mathcal{O}) = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} \\ \mathcal{M} & \mathcal{O} & \cdots & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{M} & \mathcal{M} & \cdots & \mathcal{O} \end{pmatrix}.$$

It is easily seen that the radical $\mathcal{R}$ of $\mathcal{H}_s(\mathcal{O})$ is

$$\mathcal{R} = \begin{pmatrix} \mathcal{M} & \mathcal{O} & \cdots & \mathcal{O} \\ \mathcal{M} & \mathcal{M} & \cdots & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{M} & \mathcal{M} & \cdots & \mathcal{M} \end{pmatrix}$$

and

$$\mathcal{R}^2 = \begin{pmatrix} \mathcal{M} & \mathcal{M} & \cdots & \mathcal{O} \\ \mathcal{M} & \mathcal{M} & \cdots & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{M}^2 & \mathcal{M} & \cdots & \mathcal{M} \end{pmatrix}.$$

The principal right modules of $\mathcal{H}$ are the "row-ideals" of $\mathcal{H}$ and the submodules of each of them form a chain. In particular, the submodules of the "first-row-ideal" form the following chain:

$$\begin{pmatrix} \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \supset \begin{pmatrix} \mathcal{M} & \mathcal{O} & \cdots & \mathcal{O} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \supset \ldots .$$

It is easy to see that each other row-ideal of $\mathcal{H}$ is isomorphic to a submodule of the above module. In a similar fashion, the principal left $\mathcal{H}$-modules are the column-ideals, whose submodules form corresponding chains. Thus, $\mathcal{H}$ is a serial ring in the sense of [9, p. 224]. Let $P_1, \ldots, P_s$ be the principal right modules of the quotient ring $\mathcal{A} = \mathcal{H}_s(\mathcal{O}) / \mathcal{R}^2$ and $Q_1, \ldots, Q_s$ be the principal left $\mathcal{A}$-modules numbered such that $P_i = e_{ii}\mathcal{A}$, $Q_i = \mathcal{A} e_{ii}$, $i = 1, \ldots, s$, where e_{ij} denote the elementary $s \times s$ matrix whose (i, j)'s entry is 1 and all other entries are zero. Then the submodules of every P_i, Q_i form finite chains, and a direct verification shows that

$$\operatorname{soc} P_1 \cong \operatorname{top} P_2, \quad \operatorname{soc} P_2 \cong \operatorname{top} P_3, \ldots, \operatorname{soc} P_s \cong \operatorname{top} P_1$$

and

$$\operatorname{top} Q_1 \cong \operatorname{soc} Q_2, \quad \operatorname{top} Q_2 \cong \operatorname{soc} Q_3, \ldots, \operatorname{top} Q_s \cong \operatorname{soc} Q_1.$$

Moreover, each of these modules is a one-dimensional vector space over $\mathcal{O}/\mathcal{M}$. Hence, $\mathcal{A}$ is a quasi-Frobenius ring whose Nakayama permutation is $(1, 2, \ldots, s)$.

More in general, the quotient ring $\mathcal{A} = \mathcal{H}_s(\mathcal{O}) / \mathcal{R}^m$, $m \geq 2$, is a quasi-Frobenius ring whose Nakayama permutation is $(1, 2, \ldots, s)^{m-1}$. It follows, in particular, that the Nakayama permutation of $\mathcal{A}$ is identical if and only if $m \equiv 1 (\operatorname{mod} s)$.

***Example* 2.2.** Let $\mathcal{B}_{2,s}$ be the ring of $2s \times 2s$ matrices of the form:

$$\mathcal{B}_{2,s} = \begin{pmatrix} \mathcal{H} & \mathcal{R} \\ \mathcal{R} & \mathcal{H} \end{pmatrix}.$$

It is easily seen that, the Jacobson radical of $\mathcal{B}_{2,s}$ is $\begin{pmatrix} \mathcal{R} & \mathcal{R} \\ \mathcal{R} & \mathcal{R} \end{pmatrix}$. Consider the ideal

$$J = \begin{pmatrix} \mathcal{R}^{s} & \mathcal{R}^{s+1} \\ \mathcal{R}^{s+1} & \mathcal{R}^{s} \end{pmatrix}.$$

A direct verification shows that the quotient ring $\mathcal{A} = \mathcal{B}_{2,s}/J$ is a quasi-Frobenius ring whose Nakayama permutation is

$$\nu = (1\, s+1)(2\, s+2)\ldots(s\, 2s).$$

***Remark* 2.1.** It can be verified that $\mathcal{B}_{2,s}/J$ is semi-distributive (see [10] for the definition).

***Example* 2.3.** Let G be a finite group and K be a field. The group algebra KG is a well-known example of a quasi-Frobenius ring [3]. By a result of D. S. Passman [11, p. 62] (Theorem 4.11), the Nakayama permutation of KG is identical. If A is a ring and G is a group, then it follows from results of I. G. Connell [12] that AG is quasi-Frobenius if and only if A is quasi-Frobenius and G is finite.

***Example* 2.4.** Let K be a field and S be a semigroup such that the semigroup algebra KS is quasi-Frobenius. Since KS is artinian, it follows by a result of E. Zelmanov [13] that S is finite. If S is a semi-simple semigroup, a result of J. Okniński [14, p. 196] says that KS is quasi-Frobenius if and only if S is a finite strongly p-semi-simple semigroup, where $p = \operatorname{char} K$. In particular, the semigroup algebra KS of an inverse semigroup S is quasi-Frobenius if and only if S is finite.

3. Simple modules over semi-perfect rings and Peirce decomposition. Semi-perfect rings were introduced by H. Bass in 1960. The basic facts about these rings can be found in [3, 9, 15]. In this section we denote by $\mathcal{A}$ a semi-perfect ring and denote by $\mathcal{R} = \mathcal{R}(\mathcal{A})$ its Jacobson radical. Write $\overline{\mathcal{A}} = \mathcal{A}/\mathcal{R} = M_{n_1}(\mathcal{D}_1)\times\ldots\times M_{n_s}(\mathcal{D}_s)$, where $\mathcal{D}_i$, $i = 1,\ldots,s$, are division rings. Every simple $\mathcal{A}$-module is, obviously, simple as an $\overline{\mathcal{A}}$-module. Let $\overline{1} = \bar{f}_1 + \ldots + \bar{f}_s$ be a decomposition of $\overline{1} \in \overline{\mathcal{A}}$ into a sum of central idempotents such that $\bar{f}_i\overline{\mathcal{A}} = \overline{\mathcal{A}}\,\bar{f}_i = M_{n_i}(\mathcal{D}_i)$, $i = 1,\ldots,s$, and $1 = f_1 + \ldots + f_s$ be the corresponding "lifted" decomposition, i. e. $f_i f_j = \delta_{ij} f_i$, $\bar{f}_i = f_i + \mathcal{R}$, $i = 1,\ldots,s$, where δ_{ij} is the Kronecker delta. The existence of such "lifting" follows from [15] (Chapter 3). We write $\overline{\mathcal{A}} = U_1^{n_1}\oplus\ldots\oplus U_s^{n_s}$, where $U_1,\ldots,U_s$ are the pairwise non-isomorphic simple right $\mathcal{A}$-modules. Each U_i can be identified with the set of all strings $(\alpha_1,\ldots,\alpha_{n_i})$, $\alpha_1,\ldots,\alpha_{n_i}\in\mathcal{D}_i$. Similarly, $\overline{\mathcal{A}} = V_1^{n_1} \oplus\ldots\ldots\oplus V_s^{n_s}$, where $V_1,\ldots,V_s$ are the pairwise non-isomorphic simple left $\mathcal{A}$-modules, and each V_i can be identified with the set of all columns $(\alpha_1,\ldots,\alpha_{n_i})^T$, $\alpha_1,\ldots,\alpha_{n_i}\in\mathcal{D}_i$. An idempotent $e\in\mathcal{A}$ is said to be local if $e\mathcal{A}e$ is a local ring. Observe that two principal A-modules P and P' are isomorphic if and only if $\operatorname{top} P_i \cong \operatorname{top} P_i'$.

The next two results are well-known.

Theorem 3.1 [16]. *A ring $\mathcal{A}$ is semi-perfect if and only if the identity 1 of $\mathcal{A}$ can be decomposed into a sum of pairwise orthogonal local idempotents.*

Theorem 3.2 [15]. *Let* $1 = e_1 + \ldots + e_m = h_1 + \ldots + h_n$ *be two decompositions of* $1 \in \mathcal{A}$ *into a sum of pairwise orthogonal local idempotents. Then* $m = n$ *and there exists an invertible element* $a \in \mathcal{A}$ *and a permutation* $i \to \sigma(i)$ *such that* $e_i = a h_{\sigma(i)} a^{-1}$ *for each* $i = 1, \ldots, n$.

We shall need also the following easy fact.

Lemma 3.1. *For every simple right* $\mathcal{A}$*-module* U_i *and for each* f_j *we have* $U_i f_j = \delta_{ij} U_i$, $i, j = 1, \ldots, s$. *Similarly, for every simple left* $\mathcal{A}$*-module* V_i *and for each* f_j, $f_j V_i = \delta_{ij} V_i$, $i, j = 1, \ldots, s$.

Proof. Go modulo $\mathcal{R}$ and apply the Wedderburn – Artin Theorem.

This lemma will be a useful tool in our further considerations and we shall refer to it as to *Lemma on annihilation of simple modules.* An idempotent $f \in \mathcal{A}$, which is central modulo $\mathcal{R}$, shall be called *minimal modulo* $\mathcal{R}$ if f can not be decomposed into a sum of two orthogonal idempotents, which are central modulo $\mathcal{R}$. For two idempotents e and g of $\mathcal{A}$ we shall write $e \in g$, if $g = e + e'$, where $ee' = e'e = 0$. Clearly, e' is also an idempotent in $\mathcal{A}$.

Theorem 3.3. *Let* $1 = f_1 + \ldots + f_s = g_1 + \ldots + g_t$ *be two decompositions of* $1 \in \mathcal{A}$ *into a sum of pairwise orthogonal idempotents, which are minimal central modulo* $\mathcal{R}$. *Then* $s = t$ *and there exist an invertible element* $a \in \mathcal{A}$ *and a permutation* $i \to \tau(i)$ *of* $\{1, \ldots, s\}$ *such that* $f_i = a g_{\tau(i)} a^{-1}$ *for each* $i = 1, \ldots, s$.

Proof. Applying the Wedderburn – Artin Theorem to $\overline{\mathcal{A}}$, we get immediately that $s = t$. Let $f_i = e_1^{(i)} + \ldots + e_{n_i}^{(i)}$ be a decomposition of f_i into a sum of pairwise orthogonal local idempotents. Then, obviously, $U_i e_k^{(i)} \neq 0$ for $k = 1, \ldots, n_i$. It follows from the Lemma on annihilation of simple modules that $U_i g_{\sigma(i)} = U_i$ for some $g_{\sigma(i)}$ and, moreover, $U_i g_j = 0$ if $j \neq \sigma(i)$. Renumber the idempotents $g_1, \ldots$ $\ldots, g_s$ such that $U_i g_i = U_i$, $i = 1, \ldots, s$. Take a decomposition $g_i = h_1^{(i)} + \ldots + h_{n_i}^{(i)}$ into of pairwise orthogonal local idempotents. Then we obtain two decompositions of $1 \in \mathcal{A}$, which satisfy the assumptions of Theorem 3.2. Hence, there exists a conjugating element $a \in \mathcal{A}$ which transforms one decomposition into the other, up to a permutation. It follows from our numeration of idempotents $g_1, \ldots, g_s$ that $a\left\{ h_1^{(i)}, \ldots, h_{n_i}^{(i)} \right\} a^{-1} = \left\{ e_1^{(i)}, \ldots, e_{n_i}^{(i)} \right\}$ for each $i = 1, \ldots, s$ and, consequently, $a g_i a^{-1} = f_i$, $i = 1, \ldots, s$.

Theorem 3.3 is proved.

Consider $\mathcal{A} = \oplus_{i=1}^{s} f_i \mathcal{A}$. Then clearly $f_i \mathcal{A} = P_i^{n_i}$, where P_i is an indecomposable projective $\mathcal{A}$-module whose multiplicity in the right regular module $\mathcal{A}_{\mathcal{A}}$ is n_i. Moreover, $P_i / P_i \mathcal{R} \cong U_i$, $i = 1, \ldots, s$. Similarly, $\mathcal{A}_{\mathcal{A}} = \oplus_{i=1}^{s} \mathcal{A} f_i$, where $\mathcal{A} f_i = Q_i^{n_i}$ and each Q_i is an indecomposable projective left $\mathcal{A}$-module with multiplicity n_i in the left regular module ${}_{\mathcal{A}}\mathcal{A}$. We also have $Q_i / \mathcal{R} Q_i \cong V_i$, $i = 1, \ldots, s$. Set $\mathcal{A}_{ij} = f_i \mathcal{A} f_j$. Then

$$\mathcal{A} = \bigoplus_{i,j=1}^{s} \mathcal{A}_{ij}, \quad \mathcal{R} = \bigoplus_{i,j=1}^{s} \mathcal{R}_{ij}, \tag{3}$$

where $\mathcal{R}_{ij} = f_i \mathcal{R} f_j = \mathcal{A}_{ij}$ for $i \neq j$ and $\mathcal{R}_{ii}$ is the Jacobson radical of $\mathcal{A}_{ii}$, $i = 1, \ldots, s$.

Such two-sided Peirce decompositions of $\mathcal{A}$ and $\mathcal{R}$ shall be called *canonical.* It follows from Theorem 3.3 that every other canonical Peirce decomposition of $\mathcal{A}$ can be obtained from

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \cdots & \mathcal{A}_{1s} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \cdots & \mathcal{A}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{s1} & \mathcal{A}_{s2} & \cdots & \mathcal{A}_{ss} \end{pmatrix}$$

by a simultaneous permutation of lines and columns and the substitution of all Peirce components $\mathcal{A}_{ij}$ by $a\mathcal{A}_{ij}a^{-1}$.

4. Monomial ideals. Let $1 = e_1 + \ldots + e_n$ be a decomposition of 1 into a sum of pairwise orthogonal idempotents. By an ideal we mean a two-sided ideal. For an ideal I of $\mathcal{A}$ the abelian group $e_i I e_j$, $i, j = 1, \ldots, n$, obviously lies in I, and $I = \oplus_{i,j=1}^{n} I_{ij}$ is a decomposition of I into a direct sum of abelian subgroups. Such decomposition is called the *two-sided Peirce decomposition* of I corresponding to $1 = e_1 + \ldots + e_n$. It has a natural matrix form:

$$I = \begin{pmatrix} I_{11} & I_{12} & \cdots & I_{1n} \\ I_{21} & I_{22} & \cdots & I_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ I_{n1} & I_{n2} & \cdots & I_{nn} \end{pmatrix}.$$

If $\mathcal{J} = \oplus_{i,j=1}^{n} \mathcal{J}_{ij}$ is also an ideal, then

$$I + \mathcal{J} = \begin{pmatrix} I_{11} + \mathcal{J}_{11} & I_{12} + \mathcal{J}_{12} & \cdots & I_{1n} + \mathcal{J}_{1n} \\ I_{21} + \mathcal{J}_{21} & I_{22} + \mathcal{J}_{22} & \cdots & I_{2n} + \mathcal{J}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ I_{n1} + \mathcal{J}_{n1} & I_{n2} + \mathcal{J}_{n2} & \cdots & I_{nn} + \mathcal{J}_{nn} \end{pmatrix},$$

and each Peirce component $(I\mathcal{J})_{ij}$ of the product $I\mathcal{J}$ is given by

$$(I\mathcal{J})_{ij} = \sum_{k=1}^{n} I_{ik}\mathcal{J}_{kj}, \quad i, j = 1, \ldots, n,$$

so that addition and multiplication of elements from I and $\mathcal{J}$ can be done by the addition and multiplication of corresponding matrices.

Let $\mathcal{A}$ be a semi-perfect ring and $1 = f_1 + \ldots + f_s$ be a canonical decomposition of $1 \in \mathcal{A}$ into a sum of pairwise orthogonal idempotents. Then $I = \oplus_{i,j=1}^{s} I_{ij}$ with $I_{ij} = f_i I f_j$, $i, j = 1, \ldots, s$, is called the *canonical Peirce decomposition* of I. As above, it is easily seen that one canonical Peirce decomposition of I can be obtained from another one by a simultaneous permutation of lines and columns and the substitution of each Peirce component I_{ij} by $aI_{ij}a^{-1}$.

Definition 4.1. *An ideal I of a semi-perfect ring $\mathcal{A}$ shall be called monomial if each line and each column of a canonical Peirce decomposition of I contains exactly one non-zero Peirce component.*

If I is a monomial ideal, then there exists a permutation $\nu \to \nu(i)$ of $\{1, \ldots, s\}$ such that $I_{i\nu(i)} \neq 0$. Clearly, ν is determined up to conjugation in the symmetric group on s letters. We denote this permutation by $\nu(I)$.

Lemma 4.1. *Let $\mathcal{A}$ be a semi-perfect ring. If I is a monomial ideal of $\mathcal{A}$ then each canonical Peirce component of I is an ideal in $\mathcal{A}$.*

Proof. Let $1 = f_1 + \ldots + f_s$ be a canonical decomposition of $1 \in \mathcal{A}$ into a sum of pairwise orthogonal idempotents. Write $\nu = \nu(I)$, then $I = \oplus_{i,j=1}^{s} f_i I f_{\nu(i)}$. Obviously $f_i I f_{\nu(i)} f_k \mathcal{A} f_l = 0$ if $k \neq \nu(i)$. Moreover, $f_i I f_{\nu(i)} f_{\nu(i)} \mathcal{A} f_l \subseteq f_i I f_l$ which is non-zero if and only if $l = \nu(i)$, as I is monomial. Similarly, $f_k \mathcal{A} f_l f_i I f_{\nu(i)} \neq 0$ if and only if $k = l = i$. It follows that $f_i I f_{\nu(i)}$ is an ideal in $\mathcal{A}$ for each $i = 1, \ldots, n$.

Theorem 4.1. *Let A be a right hereditary semi-perfect ring. Then the following conditions are equivalent:*

(a) *A contains a monomial ideal;*

(b) *A is isomorphic to a finite direct product of the rings $M_{n_i}(B)$ where all B_i are local hereditary domains.*

Moreover, if a ring A is semi-distributive [10], *then all rings B_i are either division rings or discrete valuation rings.*

Proof. (b) $\Rightarrow$ (a). Obvious.

(a) $\Rightarrow$ (b). Let $\mathcal{A}$ be an indecomposable ring [17, p. 73]. By [17] (Proposition 6.4.3) we obtain that $\mathcal{A}$ is a piecewise domain (see [17], § 6.4). Every semi-perfect piecewise domain has a triangular Peirce decomposition see [18] (§ 3). So $\mathcal{A}$ is a prime right hereditary semi-perfect ring. Since $\mathcal{A}$ contains a monomial ideal then $\mathcal{A} = P^n$ and $\mathcal{A} \simeq M_n(\mathrm{End}_{\mathcal{A}} P)$, where $B = \mathrm{End}_{\mathcal{A}} P$ is a local right hereditary domain.

If a ring $\mathcal{A}$ is semi-distributive, then B is a right Noetherian right hereditary local semi-distributive domain. By [19] (Theorem 3.9) B is either a division ring or a discrete valuation ring.

Lemma 4.2. *Let $\mathcal{A}$ be a semi-perfect ring. Then $\mathrm{soc}\,\mathcal{A}_{\mathcal{A}}$ coincides with the left annihilator $l(\mathcal{R})$ of $\mathcal{R} = \mathcal{R}(\mathcal{A})$, whereas $\mathrm{soc}\,{}_{\mathcal{A}}\mathcal{A}$ coincides with the right annihilator $r(\mathcal{R})$. In particular, $\mathrm{soc}\,{}_{\mathcal{A}}\mathcal{A}$ and $\mathrm{soc}\,\mathcal{A}_{\mathcal{A}}$ are two-sided ideals.*

Proof. If U is a simple right $\mathcal{A}$-module, then, obviously, $U\mathcal{R} = 0$ and, consequently, $\mathrm{soc}\,\mathcal{A}_{\mathcal{A}} \subseteq l(\mathcal{R})$. On the other hand, the equality $l(\mathcal{R})\mathcal{R} = 0$ implies that $l(\mathcal{R})$ is a semi-simple right $\mathcal{A}$-module, so it has to be contained in the right socle of $\mathcal{A}$, hence, $l(\mathcal{R}) = \mathrm{soc}\,\mathcal{A}_{\mathcal{A}}$. Similarly, $r(\mathcal{R}) = \mathrm{soc}\,{}_{\mathcal{A}}\mathcal{A}$. Lemma 4.2 is proved.

The first statement of the next theorem is well known (see [4]), however, we include a proof in order to show that the whole result is a consequence of the Lemma on annihilation of simple modules.

Theorem 4.2. *Let $\mathcal{A}$ be a quasi-Frobenius ring. Then $\mathrm{soc}\,{}_{\mathcal{A}}\mathcal{A} = \mathrm{soc}\,\mathcal{A}_{\mathcal{A}}$. Moreover, $\mathcal{Z} = \mathrm{soc}\,{}_{\mathcal{A}}\mathcal{A}$ is a monomial ideal and $\nu(\mathcal{Z})$ coincides with the Nakayama permutation $\nu(\mathcal{A})$ of $\mathcal{A}$.*

Proof. Denote by Z_l (respectively Z_r) the left (respectively right) socle of $\mathcal{A}$. It follows from the definition of quasi-Frobenius rings and from the Lemma on annihilation of simple modules that $f_i Z_l \neq 0$ for each $i = 1, \ldots, s$. Then for every local idempotent $e \in f_i$ the set $e f_i Z_l = eZ_l$ is different from 0. Therefore, the right ideal eZ_l is a non-zero submodule of the principal module P_i and, consequently, eZ_l contains $\mathrm{soc}\,P_i$, which implies that $Z_l \supseteq Z_r$. Since the Nakayama's definition of quasi-Frobenius rings is left-right symmetric, it follows that $Z_r \supseteq Z_l$, and thus, $Z_l = Z_r = \mathcal{Z}$.

It remains to show that $\mathcal{Z}$ is monomial and $\nu(\mathcal{Z}) = \nu(\mathcal{A})$. Write $\nu = \nu(\mathcal{A})$ and consider the canonical Peirce decomposition of $\mathcal{Z}$: $\mathcal{Z} = \oplus_{i,j=1}^{s} f_i \mathcal{Z} f_j$. Since $\mathcal{A}_{\mathcal{A}} = \oplus_{i=1}^{s} f_i \mathcal{A} = \oplus_{i=1}^{s} P_i^{n_i}$, we have that $\mathcal{Z} = \oplus_{i=1}^{s} \operatorname{soc} f_i \mathcal{A}$ and $f_i Z = \operatorname{soc} f_i A = \operatorname{soc} P_i^{n_i}$. It follows from Definition 2.1 that $\operatorname{soc} P_i^{n_i} \cong U_{\nu(i)}^{n_i}$, so $f_i Z \cong U_{\nu(i)}^{n_i}$, and the Lemma on annihilation of simple modules implies that $f_i Z f_j = 0$ if and only if $j \neq \nu(i)$. Hence $\mathcal{Z}$ is monomial and $\nu(\mathcal{Z})$ coincides with $\nu(\mathcal{A})$.

5. Socles of perfect rings. The following notion of *Socular Ring* was considered in [9, p. 22.10] (Chapter 22).

Definition 5.1. *A ring $\mathcal{A}$ is called left (resp. right) socular if every non-zero left (resp. right) $\mathcal{A}$-module has a non-zero socle. A ring which is right and left socular is called socular.*

Definition 5.2 [20]. *A ring $\mathcal{A}$ is called right (resp. left) perfect if every right (resp. left) $\mathcal{A}$-module has a projective cover. A ring which is right and left perfect is called perfect.*

The notion of Socular Ring is based on the following Osofsky's Lemma [8] (Lemma 9).

Lemma 5.1. *The following are equivalent*:

(i) *$\mathcal{A}$ is right perfect ($\mathcal{A}$ is left perfect)*;

(ii) *$\mathcal{A}/\mathcal{R}$ is semi-simple artinian and every cyclic left (right) $\mathcal{A}$-module has non-zero socle;*

(iii) *$\mathcal{A}/\mathcal{R}$ is semi-simple artinian and every non-zero right (left) $\mathcal{A}$-module has non-zero simple epimorphic image;*

(iv) *$\mathcal{A}/\mathcal{R}$ is semi-simple artinian and if $\{a_i \mid i = 0, 1, \ldots,\} \subseteq \mathcal{R}$ there is an n such that $a_n a_{n-1} \ldots a_0 = 0$ $(a_0 a_1 \ldots a_n = 0)$.*

The following proposition follows from Lemma 5.1 (see also, [9], Summary, 22.10).

Proposition 5.1. *For a ring $\mathcal{A}$ the following conditions are equivalent*:

(i) *$\mathcal{A}$ is socular;*

(ii) *$\mathcal{A}$ is perfect.*

In particular, if $\mathcal{A}$ is perfect, then $\operatorname{soc}_{\mathcal{A}}\mathcal{A} \neq 0$ and $\operatorname{soc}\mathcal{A}_{\mathcal{A}} \neq 0$. In [20] H. Bass gave a characterization of perfect rings, which uses the notion of T-nilpotency.

Definition 5.3. *A non-zero ideal $\mathcal{J}$ of a ring $\mathcal{A}$ is called T-nilpotent if for every sequence $a_1, a_2, \ldots, a_n \ldots$ of elements $a_i \in \mathcal{J}$ there exist positive integers k and m such that $a_k a_{k-1} \ldots a_1 = 0$ and $a_1 \ldots a_{m-1} a_m = 0$.*

Clearly, any T-nilpotent ideal is nil.

Theorem 5.1 [20]. *For a ring $\mathcal{A}$ the following conditions are equivalent*:

(i) *$\mathcal{A}$ is perfect;*

(ii) *the Jacobson radical $\mathcal{R}$ of $\mathcal{A}$ is T-nilpotent and $\mathcal{A}/\mathcal{R}$ is semi-simple.*

Now we give an example of a right noetherian serial ring $\mathcal{A}$ for which $\operatorname{soc}\mathcal{A}_{\mathcal{A}} = U_2 \oplus U_2 = U_2^2$, and $\operatorname{soc}_{\mathcal{A}}\mathcal{A} = 0$.

***Example* 5.1.** Let $\mathbf{Q}$ be the field of rational numbers, p be a prime integer, $\mathbb{Z}_p = \{ m/n \in \mathbf{Q} \mid (n, p) = 1 \}$. Set

$$\mathcal{A} = \begin{pmatrix} \mathbb{Z}_p & \mathbf{Q} \\ 0 & \mathbf{Q} \end{pmatrix} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \middle| \alpha \in \mathbb{Z}_p,\ \beta, \gamma \in \mathbf{Q} \right\}.$$

It is clear that $\mathcal{R} = \begin{pmatrix} p\mathbb{Z}_p & \mathbf{Q} \\ 0 & 0 \end{pmatrix}$ and $P_1 = (\mathbb{Z}_p, \mathbf{Q})$, $P_2 = (0, \mathbf{Q})$. The left principal $\mathcal{A}$-modules are:

$$Q_1 = \begin{pmatrix} \mathbb{Z}_p \\ 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} \mathbf{Q} \\ \mathbf{Q} \end{pmatrix}.$$

Since $\mathcal{A}$ is serial, then if the socle of a principal (left or right) module P is different from zero, then $\operatorname{soc} P$ has to be simple.

Obviously, $P_1\mathcal{R} = (p\mathbb{Z}_p, \mathbf{Q})$ and $P_1\mathcal{R} = 0$. The submodules P_1 are $P_1\mathcal{R}^j$, $j = 1, 2, \ldots$, and $(0, \mathbf{Q})$. The last module is isomorphic to U_2. Hence $\operatorname{soc}\mathcal{A}_{\mathcal{A}} = U_2 \oplus \oplus U_2 = U_2^2$.

In the left case $\mathcal{R}Q_1 = \begin{pmatrix} p\mathbb{Z}_p \\ 0 \end{pmatrix}$ and $\mathcal{R}Q_2 = \begin{pmatrix} \mathbf{Q} \\ 0 \end{pmatrix}$. It is clear that the socles of these modules are zero. Thus, $\operatorname{soc}_{\mathcal{A}}\mathcal{A} = 0$.

Let $\mathcal{A}$ be a perfect ring and $\mathcal{A} \cong P_1^{n_1} \oplus \ldots \oplus P_s^{n_s} = Q_1^{n_1} \oplus \ldots \oplus Q_s^{n_s}$ be decompositions of $\mathcal{A}$ into direct sums of right, respectively, left indecomposable ideals.

Definition 5.4. *We shall say that a perfect ring $\mathcal{A}$ admits a Nakayama permutation $\nu(\mathcal{A}) : i \to \nu(i)$ of $\{1, \ldots, s\}$ if the following conditions are verified*:

(i) $\operatorname{soc} P_k = \operatorname{top} P_{\nu(k)}$;

(ii) $\operatorname{soc} Q_{\nu(k)} = \operatorname{top} Q_k$.

Let $\mathcal{A}$ be a perfect ring, which admits a Nakayama permutation $\nu(\mathcal{A})$. By (i) the socle of every principal module is simple and, moreover, two principal modules with isomorphic socles have to be isomorphic. By (ii) the socles of the principal left modules are also simple.

Theorem 5.2 (Compare with [4], Theorem 58.12). *Let $\mathcal{A}$ be a perfect ring such that the socles of all principal right $\mathcal{A}$-modules and of all principal left $\mathcal{A}$-modules are simple. Suppose furthermore, that if the socles of two principal right $\mathcal{A}$-modules P and P' are isomorphic then $P \cong P'$. Then $\mathcal{A}$ satisfied the following conditions*:

(i) $\operatorname{soc}\mathcal{A}_{\mathcal{A}} = \operatorname{soc}_{\mathcal{A}}\mathcal{A} = \mathcal{Z}$ *and* $\mathcal{Z}$ *is a monomial ideal*;

(ii) $\mathcal{A}$ *admits a Nakayama permutation* $\nu = \nu(\mathcal{A})$ *with* $\nu(\mathcal{A}) = \nu(\mathcal{Z})$.

Proof. Let $1 = f_1 + \ldots + f_s$ be a canonical decomposition of $1 \in \mathcal{A}$ into a sum of pairwise orthogonal idempotents and $f_i\mathcal{A} = P_i^{n_i}$, $i = 1, \ldots, s$. Set $\mathcal{Z}_r = \operatorname{soc}\mathcal{A}_{\mathcal{A}}$ and $\mathcal{Z}_l = \operatorname{soc}_{\mathcal{A}}\mathcal{A}$. The equality ${}_{\mathcal{A}}\mathcal{A} = \mathcal{A}f_1 \oplus \ldots \oplus \mathcal{A}f_s$ implies that $\mathcal{Z}_l = \operatorname{soc}\mathcal{A}f_1 \oplus \ldots \ldots \oplus \operatorname{soc}\mathcal{A}f_s$ and $\mathcal{Z}_l f_i = \operatorname{soc}\mathcal{A}f_i$ for all $i = 1, \ldots, s$. Similarly, $\mathcal{Z}_r = \operatorname{soc} f_1\mathcal{A} \oplus \ldots \ldots \oplus \operatorname{soc} f_s\mathcal{A}$ and $f_j\mathcal{Z}_l = \operatorname{soc} f_j\mathcal{A}$, $j = 1, \ldots, s$. It follows from our hypothesis that $\operatorname{soc} P_1, \ldots, \operatorname{soc} P_s$ is a permutation of simple modules $U_1 = \operatorname{top} P_1, \ldots, U_s = \operatorname{top} P_s$. Denote this permutation of $\{1, \ldots, s\}$ by ν.

For a fixed $i = 1, \ldots, s$ and each local idempotent $e \in f_i$, we get by the Lemma on annihilation of simple modules that $\mathcal{Z}_r e \neq 0$. Then since $\operatorname{soc}\mathcal{A}e$ is simple, $\mathcal{Z}_r e$ must contain $\operatorname{soc}\mathcal{A}e$. Hence $\mathcal{Z}_l f_i = \operatorname{soc}\mathcal{A}f_i \subseteq \mathcal{Z}_r f_i$ for all $i = 1, \ldots, s$, which yields $\mathcal{Z}_l \subseteq$

$\subseteq \mathcal{Z}_r$. Similarly, $f_j \mathcal{Z}_r \subseteq f_j \mathcal{Z}_l$, $j = 1, \ldots, s$, which implies that $\mathcal{Z}_r \subseteq \mathcal{Z}_l$. Consequently, $\mathcal{Z}_r = \mathcal{Z}_l$.

Since $\mathcal{Z}_r \cong U_{\nu(1)}^{n_1} \oplus \ldots \oplus U_{\nu(s)}^{n_s}$, then by the Lemma on annihilation of simple modules $\mathcal{Z}_r$ has the following Peirce decomposition: $\mathcal{Z}_r = \oplus_{i=1}^{s} f_i \mathcal{Z}_r f_{\nu(i)}$. Thus $\mathcal{Z} = \mathcal{Z}_r = \mathcal{Z}_l$ is a monomial ideal and $\nu(\mathcal{Z})$ is a Nakayama permutation of $\mathcal{A}$.

6. Semi-perfect rings with duality of simple modules. The basic definitions on duality of modules can be found in [3] (Chapter 12) and [4] (§ 58).

Let M be an arbitrary right $\mathcal{A}$-module and $M^* = \mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A}_{\mathcal{A}})$. The abelian group M^* can be considered as a left $\mathcal{A}$-module if we define $(a\psi)(m) = a\psi(m)$ for $a \in \mathcal{A}$, $m \in M$, $\psi \in M^*$.

The left $\mathcal{A}$-module M^* is called dual to M.

Similarly, if N is a left $\mathcal{A}$-module then the abelian group $N^* = \mathrm{Hom}_{\mathcal{A}}(N, {}_{\mathcal{A}}\mathcal{A})$ is a right $\mathcal{A}$-module if we put $(\varphi a)(n) = \psi(n)a$ for $a \in \mathcal{A}$, $n \in N$, $\varphi \in N^*$.

Lemma 6.1. *Let U (resp. V) be a simple right (resp. left) $\mathcal{A}$-module. Then $\mathcal{A}$ has a right (resp. left) ideal isomorphic to U (resp. V) if and only if $U^* \neq 0$ (resp. $V^* \neq 0$).*

Proof. Immediately follows from the lemma of Schur.

Definition 6.1. *We say that $\mathcal{A}$ is a ring with duality for simple modules if for each simple right $\mathcal{A}$-module the dual module U^* is simple and the same holds also for simple left $\mathcal{A}$-modules.*

Proposition 6.1. *Let $\mathcal{A}$ be a semi-perfect ring with duality for simple modules, and P be a simple projective $\mathcal{A}$-module. Then $\mathcal{A} = M_{n_1}(\mathcal{D}) \times \mathcal{A}_2$ where $\mathcal{D} = \mathrm{End}\, P$. Conversely, if $\mathcal{A} = M_{n_1}(\mathcal{D})$ where $\mathcal{D}$ is a division ring, then $U^* = V$ and $V^* = U$, where U is the unique simple $\mathcal{A}$-module and V is the unique simple left $\mathcal{A}$-module.*

Proof. Let $\mathcal{R} = P_1 = e\mathcal{A}$ where e is a local idempotent of $\mathcal{A}$. Then obviously $\mathcal{A}e \subset P_1^*$. Therefore the left simple $\mathcal{A}$-module $P_1^* = V_1$ coincides with $\mathcal{A}e$. Let $\mathcal{A} = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ be a canonical decomposition of $\mathcal{A}$ into a sum of principal $\mathcal{A}$-modules.

Set $P' = P_2^{n_2} \oplus \ldots \oplus P_s^{n_s}$. Clearly $\mathrm{Hom}_{\mathcal{A}}(P', P_1^{n_1}) = 0$. Suppose, that $\mathrm{Hom}_{\mathcal{A}}(P_1^{n_1}, P') = 0$. Then there exists a canonical idempotent f_i such that $f_i a f_1 \neq 0$ and $i \neq 1$. Obviously, $f_i a f_1 \subset (f_1 \mathcal{A})^* = V_1^{n_1}$. Since $f_1 A f_1 \neq 0$ then for the simple left $\mathcal{A}$-module V_1 we have $f_1 V_1 = V_1$ and $f_i V_1 = V_1$ which contradicts the lemma on annihilation of simple modules. Hence $\mathrm{Hom}_{\mathcal{A}}(P_1^{n_1}, P') = 0$ and $\mathcal{A} = M_{n_1}(\mathrm{End}\, P_1) \times \mathcal{A}_2$. The converse is obvious. Proposition 6.1 is proved.

Let $\mathcal{A}_{\mathcal{A}} = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s}$ and ${}_{\mathcal{A}}\mathcal{A} = Q_1^{n_1} \oplus \ldots \oplus Q_s^{n_s}$ be the canonical decomposition of a semi-perfect ring $\mathcal{A}$ into a direct sum of principal right (left) modules.

Definition 6.2. *We shall say that for a semi-perfect ring $\mathcal{A}$ there exists a Nakayama permutation $\nu(\mathcal{A}) = \nu : i \to \nu(i)$ of $\{1, \ldots, s\}$ if the following conditions are verified:*

(i) $\operatorname{soc} P_k = \operatorname{top} P_{\nu(k)}$;

(ii) $\operatorname{soc} Q_{\nu(k)} = \operatorname{top} Q_k$.

Definition 6.3. *The next conditions shall be called the Nakayama's conditions for a semi-perfect ring $\mathcal{A}$:*

(α) *the socles of the principal right and left $\mathcal{A}$-module are simple*;

(β) *principal modules with isomorphic socles are isomorphic.*

We give an example of a commutative local semiprimary ring $\mathcal{A}$ whose socle is simple, such that $\mathcal{A}$ is not artinian.

***Example* 6.1.** Let $k[x_1, x_2, \ldots, x_n, \ldots]$ be the polynomial ring over a field with countable number of variables, $\mathcal{J}$ be the ideal of $k[x_1, x_2, \ldots, x_n, \ldots]$ generated by elements: $x_1^2, x_2^2, \ldots, x_n^2, \ldots$ and $x_1x_2 - x_2x_3$, $x_2x_3 - x_3x_4, \ldots, x_{n-1}x_n - x_nx_{n+1}, \ldots$. Consider the quotient ring $\mathcal{A} = k[x_1, x_2, \ldots, x_n, \ldots]/\mathcal{J}$. Denote by a_0 the image of x_1x_2 in $\mathcal{A}$. The images of $1, x_1, x_2, \ldots, x_n, \ldots$ in $\mathcal{A}$ will be denoted by the same symbols. Clearly, every element from $\mathcal{A}$ can be written as: $= \alpha \cdot 1 + \alpha_1 x_1 + \ldots + \alpha_n x_n + \alpha_0 a_0$. Observe, that $(\alpha_1 x_1 + \ldots + \alpha_n x_n)a_0 = 0$ and $a_0^2 = 0$, which implies that $r = \alpha_1 x_1 + \ldots + \alpha_n x_n + \alpha_0 x_0$ is nilpotent and $r^3 = 0$. It immediately implies that $\mathcal{R}$ is an infinite dimensional vector space over k with basic $a_0, x_1, x_2, \ldots, x_n, \ldots$ therefore the Loewy series of $\mathcal{A}$ is:

$$\mathcal{A} \supset \mathcal{R} \supset \mathcal{R}^2 \supset 0,$$

where $\mathcal{R}^2 = \operatorname{soc}\mathcal{A}$ is a simple module generated by a_0.

Thus, for $\mathcal{A}$ there exists a Nakayama permutation which maps 1 into 1 and $\mathcal{A}$ satisfy the Nakayama's conditions. A simple $\mathcal{A}$-module U can be identified with $U = \mathcal{A}/\mathcal{R}$ and $U^* = \operatorname{Hom}_{\mathcal{A}}(U, \operatorname{soc}\mathcal{A})$ is a simple $\mathcal{A}$-module.

Theorem 6.1. *Let $\mathcal{A}$ be a semi-perfect ring. Then the following are equivalent*:

(1) *$\mathcal{A}$ is a ring with duality for simple modules*;

(2) *$\mathcal{A}$ admits a Nakayama permutation $\nu(\mathcal{A})$*;

(3) *$\mathcal{A}$ satisfies the Nakayama's conditions.*

Moreover, it follows from (1), (2), (3) *that* $\operatorname{soc}(\mathcal{A}_{\mathcal{A}}) = \operatorname{soc}({}_{A}\mathcal{A}) = \mathcal{Z}$ *and* $\mathcal{Z}$ *is a monomial ideal.*

The next lemma is essential for the proof of Theorem 6.1 (see [4, p. 395]).

Lemma 6.2. *Let e be an idempotent in a ring $\mathcal{A}$, and let $\mathcal{J}$ be a two-sided ideal in $\mathcal{A}$. Then the dual of the left $\mathcal{A}$-module $\mathcal{A}e/\mathcal{J}e$ is isomorphic to the right $\mathcal{A}$-module $er(\mathcal{J})$.*

Observe that the dual of the right $\mathcal{A}$-module $e\mathcal{A}/e\mathcal{J}$ is isomorphic to the left $\mathcal{A}$-module $l(\mathcal{J})e$.

Now we are ready to prove Theorem 6.1.

Obviously, we can suppose that $\mathcal{A}$ is reduced and indecomposable. Therefore, every local idempotent is canonical.

(1) ⇒ (2). Let $\mathcal{Z}_r = \operatorname{soc}\mathcal{A}_{\mathcal{A}}$ and $\mathcal{Z}_l = \operatorname{soc}_{A}\mathcal{A}$. Since $\mathcal{A}$ is semi-perfect then $\mathcal{Z}_r = l(\mathcal{R})$ and $\mathcal{Z}_l = r(\mathcal{R})$. By Lemma 6.1 $\mathcal{Z}_r$ contains at least one copy of each simple

$\mathcal{A}$-module U_i, $i = 1, \ldots, s$, and $\mathcal{Z}_l$ contains at least one copy of each simple left $\mathcal{A}$-module V_i, $i = 1, \ldots, s$.

Therefore, by the Lemma on annihilation of simple modules, for each canonical idempotent $e_k \in \mathcal{A}$ we have $\mathcal{Z}_r e_k \neq 0$ and $e_k \mathcal{Z}_l \neq 0$.

By Lemma 6.2 the module V_i^* which is dual to $V_i = \mathcal{A} e_i / \mathcal{R} e_i$ is of the form: $V_i^* = e_i r(\mathcal{R}) = e_i \mathcal{Z}_l$. By our assumption V_i^* is simple.

Thus $e_i \mathcal{Z}_l \subset e_i \mathcal{Z}_r$ for $i = 1, \ldots, s$, from which we get $\mathcal{Z}_l \subset \mathcal{Z}_r$. By symmetry $\mathcal{Z}_r \subset \mathcal{Z}_l$ which implies $\mathcal{Z}_l = \mathcal{Z}_r = \mathcal{Z}$.

Suppose now that the right module $e_i \mathcal{Z}$ is simple and isomorphic to $U_{\nu(i)}$. Then $e_i \mathcal{Z} = e_i \mathcal{Z} e_{\nu(i)}$. Similarly $\mathcal{Z} e_{\nu(i)}$ is a simple left $\mathcal{A}$-module, and, consequently, it coincides with V_i, as $e_i \mathcal{Z} e_{\nu(i)} \neq 0$.

Therefore $\mathcal{Z} = \oplus_{i=1}^{s} e_i \mathcal{Z} e_{\nu(i)}$ is a monomial ideal with $\nu(\mathcal{Z}) = \nu(\mathcal{A})$. $\mathcal{Z}$ is determined by $V_i^* = U_{\nu(i)}$.

The equivalency of (2) and (3) is obvious.

(2) $\Rightarrow$ (1). We show that $\mathcal{Z}_r = \mathcal{Z}_l$. For each local idempotent e_i we have $e_i \mathcal{Z}_l \neq$ $\neq 0$ and $e_i \mathcal{Z}_l \supset e_i \mathcal{Z}_r$. Therefore $\mathcal{Z}_l \supset \mathcal{Z}_r$. Symmetrically $\mathcal{Z}_r \supset \mathcal{Z}_l$. Hence, $\mathcal{Z}_r =$ $= \mathcal{Z}_l = \mathcal{Z}$ and $e_i \mathcal{Z} = \mathcal{Z} e_{\nu(i)} = e_i \mathcal{Z} e_{\nu(i)}$, and by Lemma 6.2 $V_i^* = e_i \mathcal{Z} = U_{\nu(i)}$, and $U_{\nu(i)}^* = \mathcal{Z} e_{\nu(i)} = V_i$ for $i = 1, \ldots, s$.

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