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ON THE DEGREE OF HOLOMORPHIC MAPPINGS FROM AN ANNULUS INTO AN ANNULUS

ПРО СТЕПЕНІ ГОЛОМОРФНИХ ВІДОБРАЖЕНЬ ІЗ КІЛЬЦЯ У КІЛЬЦЕ

We establish an estimate for the degree of the holomorphic mapping $f: K_1 \to K_2$ (here, K_1 and K_2 are doubly-connected domains) in terms of the modulus of a family of curves in K_2. This estimate generalizes a certain result obtained by Kobayashi.

Встановлено оцінку степеня голоморфного відображення $f: K_1 \to K_2$ (K_1, K_2 — двозв'язні області) в термінах модуля сімей кривих у K_2, яка узагальнює один результат Кобаясі.

Introduction. Let $K_i = \{ z \in C: 0 < r_i < |z| < R_i \}$, $i = 1, 2$, be two standard annuli. Let $f: K_1 \to K_2$ be a holomorphic mapping and let $f_*: \pi_1(K_1) \to \pi_1(K_2)$ be the induced homomorphism of the fundamental groups of K_1 and K_2. Let α_i be the generator of $\pi_1(K_i) \cong Z$, $i = 1, 2$. It is well known that the degree of f, denoted by $\deg f$, is defined by

$$f_*(\alpha_1) = (\deg f)\alpha_2.$$

In [1, p. 14] (Theorem 6.1), Kobayashi claimed that

$$|\deg f| \le \frac{\log(R_2/r_2)}{\log(R_1/r_1)}.$$

The aim of this note is to extend the above-mentioned theorem to the case of holomorphic maps between two arbitrary annuli in the complex plane.

1. Preliminaries. *1.1.* A continuous mapping $\gamma: [a, b] \to C$ $(a < b)$ is called a flat curve if it is nonconstant on every segment $[\alpha, \beta] \subset [a, b]$, where $\alpha < \beta$. A curve $\gamma: [a, b] \to C$ is called a closed curve if $\gamma(a) = \gamma(b)$.

A curve $\gamma: [a, b] \to C$ is said to be a Jordan curve if the restricted mapping $\gamma_{(a,b)}: (a, b) \to \gamma((a, b))$ is homeomorphic.

1.2. Let γ be a rectifiable curve in C and ρ be a Borel function (i. e., ρ is Borel measurable). Then there exists the integral of ρ along γ, i. e.,

$$\int_\gamma \rho\, ds < \infty.$$

Let Γ be a family of rectifiable curves in a domain $\Omega \subset C$. We say that a Borel function ρ is Γ-admissible iff

$$\int_\gamma \rho\, ds \ge 1 \quad \text{for every} \quad \gamma \in \Gamma.$$

The modulus of Γ is defined by

$$M(\Gamma) = \inf \int\int_\Omega \rho^2\, dx\, dy.$$

Here, infimum is taken over all Γ-admissible Borel functions ρ. It is well-known that $M(\Gamma)$ is a biconformal invariance (see [2, p. 50]).

1.3. A domain $\Omega \subset C$ is called an annulus iff Ω is homeomorphic to a standard annulus $K = \{ x \in C : 0 < r < |z| < R \}$ and its boundary components are two closed Jordan curves.

The modulus $M(\Omega)$ of the annulus Ω is the modulus of the family of closed rectifiable Jordan curves in Ω which separate the boundary components of Ω.

1.4. Let Ω_1, Ω_2 be annuli and $f : \Omega_1 \to \Omega_2$ be a holomorphic mapping. Then the degree of f is defined in the same way as in Introduction.

2. The main theorem.

2.1. **Theorem.** *Let Ω_1, Ω_2 be annuli in the complex plane. Assume that γ_1, γ_2 are the boundary components of Ω_1 such that $d(\gamma_1, \gamma_2) = \delta < \infty$ and the length L of γ_1 is finite.*

Assume that $f : \Omega_1 \to \Omega_2$ is a holomorphic mapping. Then the following assertions hold:

$$|\deg f| \le \frac{2\pi M(\Omega_2)}{\log(1 + 2\pi\delta / L)} \quad \text{if} \quad \gamma_2 \subset \operatorname{ext} \gamma_1$$

and

$$|\deg f| \le \frac{-2\pi M(\Omega_2)}{\log(1 - 2\pi\delta / L)} \quad \text{if} \quad \gamma_2 \subset \operatorname{int} \gamma_1 .$$

To prove that main theorem, we need the following lemmas:

2.2. **Lemma** (see [3], Theorem 2). *Let Ω be an annulus. Assume that γ_1, γ_2 are the boundary components of Ω such that $\gamma_2 \neq \varnothing$ and γ_1 is a closed Jordan curve of the finite length L.*

Then the following inequalities hold:

$$M(\Omega) \ge \frac{1}{2\pi} \log\left(1 + \frac{2\pi d(\gamma_1, \gamma_2)}{L}\right) \quad \text{if} \quad \gamma_2 \subset \operatorname{ext} \gamma_1$$

and

$$M(\Omega) \ge -\frac{1}{2\pi} \log\left(1 - \frac{2\pi d(\gamma_1, \gamma_2)}{L}\right) \quad \text{if} \quad \gamma_2 \subset \operatorname{int} \gamma_1 ,$$

where $d(\gamma_1, \gamma_2)$ is the distance between γ_1 and γ_2.

2.3. **Lemma.** *Let $\Omega = \{ z \in C : 0 < r < R \}$ be a standard annulus. Then*

$$M(\Omega) = \frac{1}{2\pi} \log\left(\frac{R}{r}\right).$$

Proof. By Lemma 2.2, we have

$$M(\Omega) \ge \frac{1}{2\pi} \log\left(\frac{R}{r}\right).$$

We must prove the converse inequality. Consider the function $\rho(z) = 1/2\pi|z|$.

Let Γ be a family of closed Jordan curves which separate the boundary components of Ω. Then ρ is Γ-admissible. Indeed, for every $\gamma \in \Gamma$, we have

$$\int_\gamma \frac{|d\theta|}{2\pi} \ge 1. \tag{1}$$

Assume that γ is an arbitrary curve of Γ. Put $z = \gamma(t) = r(t) e^{i\theta(t)}$. Then

$$\gamma(t) = r(t) e^{i\theta(t)} + i r(t) \theta'(t) e^{i\theta(t)}.$$

Therefore,

$$|\gamma'(t)|^2 = |r'(t)|^2 + |r(t)|^2|\theta'(t)|^2 \Rightarrow$$

$$\Rightarrow \frac{|\gamma'(t)|^2}{|r(t)|^2} \geq |\theta'(t)|^2 \Rightarrow \frac{ds(z)}{|z|} \geq |d\theta| \Rightarrow$$

$$\Rightarrow \frac{ds(z)}{2\pi|z|} \geq \frac{|d\theta|}{2\pi} \Rightarrow \rho(z)\,ds(z) \geq \frac{|d\theta|}{2\pi}.$$

Thus,

$$\int_\gamma \rho(z)ds(z) \geq \int \frac{|d\theta|}{2\pi} \geq 1$$

by (1). Hence, ρ is Γ-admissible.

On the other hand, we have

$$\iint_\Omega \rho^2 dx\,dy = \iint_\Omega \frac{dx\,dy}{4\pi^2(x^2+y^2)} = \frac{1}{4\pi^2}\int_r^R dr \int_0^{2\pi} d\theta = \frac{1}{2\pi}\log\left(\frac{R}{r}\right).$$

This implies that

$$M(\Omega) \leq \frac{1}{2\pi}\log\left(\frac{R}{r}\right).$$

Thus,

$$M(\Omega) = \frac{1}{2\pi}\log\left(\frac{R}{r}\right)$$

and Lemma 2.3 is proved.

The proof of the main theorem. Let f be a holomorphic mapping from Ω_1 into Ω_2. It is well known that there exist standard annuli $K_i = \{z \in C: 0 < r_i < |z| < R_i\}$, $i = 1, 2$, such that Ω_1, Ω_2 are conformally equivalent to K_1 and K_2, respectively (see [2, p. 53]). Assume that $\varphi_1 : \Omega_1 \to K_1$ and $\varphi_2 : \Omega_2 \to K_2$ are biconformal mappings (i. e., φ_i and φ_i^{-1} are conformal, $i = 1, 2$).

Consider the holomorphic mapping $F = \varphi_2 \circ f \circ \varphi_i^{-1} : K_1 \to K_2$. By Kobayashi's Theorem [1, p. 14] and by Lemma 2.3, we have

$$|\deg F| \leq \frac{M(K_2)}{M(K_1)}. \tag{2}$$

Since the degree of a holomorphic mapping is invariant under biconformal maps, we have

$$|\deg f| = |\deg \varphi_2^{-1} \circ F \circ \varphi_1| = |\deg F|. \tag{3}$$

On the other hand, since modulus of a domain is also invariant under biconformal maps, we have

$$M(\Omega_1) = M(K_1), \quad M(\Omega_2) = M(K_2). \tag{4}$$

Combining (2) – (4), we obtain

$$|\deg F| \leq \frac{M(\Omega_2)}{M(\Omega_1)}.$$

By Lemma 2.2,

$$M(\Omega_1) \geq \frac{1}{2\pi}\log\left(1+\frac{2\pi\delta}{L}\right) \text{ if } \gamma_2 \subset \operatorname{ext}\gamma_1$$

and

$$M(\Omega) \geq -\frac{1}{2\pi}\log\left(1-\frac{2\pi\delta}{L}\right) \text{ if } \gamma_2 \subset \operatorname{int}\gamma_1,$$

where $\delta = d(\gamma_1, \gamma_2)$, L is the length of γ_1. Thus,

$$|\deg f| \leq \frac{2\pi M(\Omega_2)}{\log(1+2\pi\delta/L)} \text{ if } \gamma_2 \subset \operatorname{ext}\gamma_1$$

and

$$|\deg f| \leq \frac{-2\pi M(\Omega_2)}{\log(1-2\pi\delta/L)} \text{ if } \gamma_2 \subset \operatorname{int}\gamma_1.$$

The main theorem is proved.

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