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BOGOLIUBOV AVERAGING AND PROCEDURES OF NORMALIZATION IN NONLINEAR MECHANICS. IV*

УСЕРЕДНЕННЯ ЗА БОГОЛЮБОВИМ ТА ПРОЦЕДУРИ НОРМАЛІЗАЦІЇ У НЕЛІНІЙНІЙ МЕХАНІЦІ. IV

In this paper we aim to apply the theory developed in [1 – 3] to some classes of problems. Linear systems in zero approximation are considered. The question of preserving integral manifolds under perturbations is investigated. Unlike nonlinear systems, the linear one has centralized systems which are always decomposable. Also, limitations connected with impossibility to diagonalize the coefficient matrix in zero approximations are removed. In conclusion the method of local asymptotic decomposition is applied to some mechanical problems.

Теорія, розвинена в роботах [1 – 3], застосовується до деяких класів проблем. Розглянуто лінійні в нульовому наближенні системи. Досліджено питання збереження інтегральних многовидів під дією збурень. На відміну від нелінійних систем лінійні мають централізовані системи, які завжди можуть бути декомпозовані. При цьому зн'ято обмеження, які пов'язані з недіагональністю системи в нульовому наближенні. На завершення метод локальної асимптотичної декомпозиції застосовано до деяких задач механіки.

1. Linear systems with constant coefficients and a small parameter. 1. 1. Realization of the algorithm. Consider the system of linear differential equations

$$\dot{x}' = \mathcal{A}x' + \varepsilon\tilde{\mathcal{A}}x', \tag{1}$$

where $x' = \operatorname{col}\|x_1', \ldots, x_n'\|$; $\mathcal{A}$, $\tilde{\mathcal{A}}$ are constant matrices of dimension $n \times n$. The linear differential operator

$$U_0' = U' + \varepsilon\tilde{U}',$$

where

$$U' = \hat{\hat{x}}_{m_1}' \mathcal{F}\partial', \quad \tilde{U}' = \hat{\hat{x}}_{m_1}' \tilde{\mathcal{A}}^T\partial', \quad \mathcal{F} = \mathcal{A}^T,$$

$$\hat{\hat{x}}_{m_1}' = \|x_1', \ldots, x_n'\|, \quad \partial' = \operatorname{col}\|\partial/\partial x_1', \ldots, \partial/\partial x_n'\|,$$

corresponds to (1).

Assume that the matrix $\mathcal{A}$ of the zero approximation system is diagonable. The case of the general structure of matrix $\mathcal{A}$ will be considered in section 2. According to the asymptotic decomposition algorithm, we can make a change of variables in system (1)

$$x_k' = \exp(\varepsilon S)x_k, \quad x_k = \exp(-\varepsilon S')x_k', \quad k = \overline{1,n},$$

where $S = S_1 + \varepsilon S_2 + \ldots$.

The operator S_i has linear coefficients and is determined by the square matrix Γ_i:

$$S_i = \hat{\hat{x}}_{m_1}\Gamma_i\partial.$$

As shown in [2], we should solve the operator equation

$$[\mathbf{U}, S_\nu] = F_\nu, \quad \nu = 1, 2, \ldots . \tag{2}$$

The right-hand side F_ν of the equations is obtained by calculating the Poisson bracket

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of $\boldsymbol{U}$, $\tilde{\boldsymbol{U}}$, $\boldsymbol{S}_1, \ldots, \boldsymbol{S}_{\nu-1}$. Therefore, the coefficients of this operator are also linear functions.

According to the general theory, equation (2) is reduced to solving the sequence of independent matrix equations

$$\mathcal{F}_1 \Gamma_\nu - \Gamma_\nu \mathcal{F} = Q_\nu, \quad \nu = 1, 2, \ldots, \tag{3}$$

where $\mathcal{F}_1 \equiv \mathcal{F} \equiv \mathcal{A}^T$, and Q_ν is the matrix of the operator $\boldsymbol{F}_\nu$.

The matrix equation (3) in the linear space $\hat{R}^{(n,n)}$ is substituted by the equations

$$G_{\mathcal{F}} \hat{\Gamma}_\nu = \hat{Q}_\nu, \quad \nu = 1, 2, \ldots,$$

where

$$G_{\mathcal{F}} = \mathcal{F}_1 \otimes \mathcal{E}_n - \mathcal{E}_n \mathcal{F}^T.$$

Let $\hat{Q}_{\nu N}$ be a projection onto the kernel of the matrix $G_{\mathcal{F}}$. The matrix $Q_{\nu N}$ corresponds to this vector in the linear space $R^{(n,n)}$. The vector $\hat{\Gamma}_\nu$ is obtained as a solution of the nonhomogeneous equations

$$G_{\mathcal{F}} \hat{\Gamma}_\nu = \hat{Q}_\nu - \hat{Q}_{\nu N}.$$

We can recover the matrix Γ_ν from the vector $\hat{\Gamma}_\nu$. The operator of the transformation

$$\boldsymbol{S}_\nu = \hat{\hat{x}}_{m_1} \Gamma_\nu \partial$$

can be recovered from the this matrix.

In the linear systems under consideration, we can show two properties that simplify solution of the problem. They are the unchanged order of equations (3) and the solvability of the homogenerous equation $[\mathcal{F}, \Gamma_\nu] = 0$ for any ν.

The operator

$$\boldsymbol{N}_\nu = \hat{\hat{x}}_{m_1} Q_{\nu N} \partial,$$

which commutes with the operator $\boldsymbol{U}$, corresponds to the matrix $Q_{\nu N}$.

After necessary transformations and by using the expressions for $\boldsymbol{N}_\nu$, we can represent the operator $\boldsymbol{U}_0$ in the form

$$\boldsymbol{U}_0 = \boldsymbol{U} + \varepsilon \boldsymbol{N}_1 + \varepsilon^2 \boldsymbol{N}_2 + \ldots .$$

The centralized system

$$\dot{x}_i = (\boldsymbol{U} + \varepsilon \boldsymbol{N}_1 + \varepsilon^2 \boldsymbol{N}_2 + \ldots) x_i, \quad i = \overline{1, n}, \tag{4}$$

can be easily recovered the operator $\boldsymbol{U}_0$.

Denote $\mathcal{M}_{\nu N} = Q_{\nu N}^T$. Then equation (4) can be rewritten in the usual matrix from

$$\frac{dx}{dt} = (\mathcal{A} + \varepsilon \mathcal{M}_{1N} + \varepsilon^2 \mathcal{M}_{2N} + \ldots) x, \tag{5}$$

where

$$[\mathcal{A}, \mathcal{M}_{\nu N}] \equiv 0.$$

1. 2. The centralized system is always decomposable. By virtue of commutativity of the matrices $\mathcal{A}$ and $\mathcal{M}_{\nu N}$, the centralized system (5) is always decomposable. This simplifies its integration in comparison with the initial system (1).

Let us formulate the principal result on decomposition.

Theorem 1. *Let* λ_j *be characteristic numbers of multiplicities* r_j, $j = 1, \ldots, m$, *of the matrix* $\mathcal{A}$; *then centralized system*

$$\frac{dx}{dt} = (\mathcal{A} + \varepsilon \mathcal{M}_{1N} + \varepsilon^2 \mathcal{M}_{2N} + \ldots)x$$

is transformed into the block diagonal form

$$\frac{dz}{dt} = \begin{Vmatrix} \lambda_1 \mathcal{E}_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 \mathcal{E}_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_m \mathcal{E}_2 \end{Vmatrix} z + \sum_{\nu=1}^{m} \varepsilon^{\nu} \begin{Vmatrix} \tilde{\mathcal{M}}_{\nu 1} & 0 & \ldots & 0 \\ 0 & \tilde{\mathcal{M}}_{\nu 2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \tilde{\mathcal{M}}_{\nu m} \end{Vmatrix} z, \qquad (6)$$

where the matrices $\tilde{\mathcal{M}}_{\nu 1}$, $\tilde{\mathcal{M}}_{\nu 2}$, $\ldots$, $\tilde{\mathcal{M}}_{\nu m}$, $\nu = 1, 2, \ldots$, *located on the block diagonal, have the dimensions* $r_1 \times r_1$, $r_2 \times r_2$, $\ldots$, $r_m \times r_m$, *respectively. The vector* $z = \mathrm{col}\,\| z_1, \ldots, z_m \|$ *of new variables is connected with the old variables by the matrix of reducing transformation* $\mathcal{L}$, $\det \mathcal{L} \neq 0$, $x = \mathcal{L}z$. *The matrix* $\mathcal{L}$ *is composed of the basis vectors the root subspaces* $P(\lambda_i)$ *of the matrix* $\mathcal{A}$.

Before proving the formulated theorem, we make some remarks. The solution of system (6), in comparison with the solution of the initial perturbed system, is simplified due to decomposition of the subsystem into m subsystems of smaller dimension. To reduce the centralized system (5) to block diagonal form (6), we use only the information about the characteristic numbers of matrix $\mathcal{A}$ of the zero approximation system, since while calculating the matrices $\mathcal{M}_{1N}$, $\mathcal{M}_{2N}$, $\ldots$, we do not need the above-mentioned characteristic numbers.

***Proof of Theorem* 1.** The identity $\mathcal{A}\mathcal{M}_{\nu N} - \mathcal{M}_{\nu N}\mathcal{A} \equiv 0$, $\nu = 1, 2, \ldots$, implies $(\mathcal{A} - \lambda_i \mathcal{E})\mathcal{M}_{\nu N} - \mathcal{M}_{\nu N}(\mathcal{A} - \lambda_i \mathcal{E}) \equiv 0$. If $\eta = \mathrm{col}\,\| \eta_1, \ldots, \eta_n \|$ is the vector of the root subspace $P(\lambda_i)$ defined by the equation $(\mathcal{A} - \lambda_i \mathcal{E})\eta$, then the relation is valid:

$$(\mathcal{A} - \lambda_i \mathcal{E})\mathcal{M}_{\nu N}\eta = 0, \quad i = \overline{1,m}.$$

Therefore, $\mathcal{M}_{\nu N}$ maps the vectors of $P(\lambda_i)$ into themselves and the subspace $P(\lambda_i)$ is invariant with respect to the matrices $\mathcal{M}_{\nu N}$. n linearly independent columns of the projection matrix $P_1, \ldots, P_n$ can be chosen to be the columns of the matrix of the transformation $\mathcal{L}$.

***Corollary* 1.** *Assume that matrix* $\mathcal{A}$ *has a zero root of multiplicity, for example,* r_1. *Then in the centralized system* (6), *the first* r_1 *coordinates are proportional to the parameter* ε, *i. e. they are slow variables. Then the centralized system* (6) *has* r_1 *slow and* $n - r_1$ *fast variables.*

In practical calculations, it can be difficult to apply theorem 1. We do not need eigenvalues to obtain the matrices $\mathcal{M}_{1N}, \ldots, \mathcal{M}_{\nu N}, \ldots$ of the centralized system. While reducing the centralized system (5) to block diagonal form, the eigenvalues are assumed to be known according to Theorem 1. The following statement can be sometimes useful.

Theorem 2. *Let the matrix* $\mathcal{A}$ *acting in the space* R^n *possess an invariant subspace* $L_1 \subset R^n$ *of dimension* k_1. *Then in the space* R^n, *we can find a subspace* $L_2 \subsetneq R^n$, $L_1 \subseteq L_2$, *of dimension* k_1 $(k_1 \geq k_2)$, *which is invariant* *with respect to the matrices* $\mathcal{A}$ *and* $\mathcal{M}_{\nu N}$, $\nu = 1, 2, \ldots$: $\mathcal{A}L_2 = L_2$, $\mathcal{M}_{\nu N}L_2 = L_2$. *The centralized system* (5) *can be reduced to a block diagonal form with square blocks of dimensions* $k_2 \times k_2$ *and* $(n-k_2)\times(n-k_2)$ *an the principal diagonal*:

$$\frac{dz}{dt} = \sum_{\nu=0}^{\infty} \varepsilon^{\nu} \begin{Vmatrix} q_{11}^{\nu} & \ldots & q_{1k_1}^{\nu} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ q_{k_2 1}^{\nu} & \ldots & q_{k_2 k_2}^{\nu} & 0 & \ldots & 0 \\ q_{k_2+1,1}^{\nu} & \ldots & q_{k_2+1,k_2}^{\nu} & q_{k_2+1,k_2+1}^{\nu} & \ldots & q_{k_2+1,n}^{\nu} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ q_{n1}^{\nu} & \ldots & q_{nk_2}^{\nu} & q_{n,k_2+1}^{\nu} & \ldots & q_{nn}^{\nu} \end{Vmatrix} z. \tag{7}$$

Proof. By virtue of diagonability of the matrix $\mathcal{A}$, a basis in R^n can be composed of n eigenvectors of the matrix $\mathcal{A}$. Denote this basis $\xi_1, \ldots, \xi_n$. If $\eta_1, \ldots, \eta_{k_1}$ is a basis of the invariant subspace L_1, then it can be linearly expressed in terms of k_1 vectors $\xi_1, \ldots, \xi_n$. For example, $\xi_1, \ldots, \xi_{k_1}$. Assume that the vector ξ_j from $\xi_1, \ldots, \xi_{k_1}$ belongs to the root subspaces $P(\lambda_i)$ corresponding to the root λ_j of the multiplicity r_j. Then the identity $(\mathcal{A}-\lambda_i\mathcal{E})\mathcal{M}_{\nu N} = \mathcal{M}_{\nu N}(\mathcal{A}-\lambda_i\mathcal{E})$ implies $(\mathcal{A}-\lambda_i\mathcal{E})\mathcal{M}_{\nu N}\xi_j = 0$. Therefore, any vector ξ_j can be transformed into a vector of the root subspace $P(\lambda_i)$. Let us give an algorithm for calculation. Supplement the system of the vectors $\eta_1, \ldots, \eta_{k_1}$ with linearly independent vectors of $\mathcal{M}_{\nu N}\eta_1, \ldots \ldots, \mathcal{M}_{\nu N}\eta_{k_1}$, $\nu = 1, 2, \ldots$. Let them be $\eta_{k_1+1}, \ldots, \eta_{k_2}$. Calculate the vectors $\mathcal{M}_{\nu N}\eta_{k_1+1}, \ldots, \mathcal{M}_{\nu N}\eta_{k_2}$ and supplement the set of vectors $\eta_1, \ldots, \eta_{k_2}$, $\eta_{k_1+1}, \ldots \ldots, \eta_{k_2}$ with linearly independent ones. It is clear that in each step, the new vectors, linearly independent of the previous ones, should be obtained or the calculation process stops. After a finite number of steps, we obtain a maximal number of linearly independent vectors $\eta_1, \ldots, \eta_{k_1}, \eta_{k_1+1}, \ldots, \eta_{k_2}$. Denote the linear subspace by spanned these vectors by L_2. The dimension of the space in a limit is equal to the sum of dimensions of the root subspaces $P(\lambda_j)$, $j = \overline{1, k_1}$, which involves the vectors $\xi_1, \ldots, \xi_{k_1}$.

Let the dimension k_2 of subspace L_2 be less than n. We choose the basis of L_2 as the first k_2 columns of the transformation matrix $\mathcal{L}$. The rest of the $n - k_2$ columns supplement the basis of L_2 to a complete basis of R^n. Upon a change of variables $x = \mathcal{L}z$, the matrices $\mathcal{A}$ and $\mathcal{M}_{\nu N}$ are transformed to block diagonal form (7).

Introduce the notations

$$Q_{\nu 1} = \begin{Vmatrix} q_{11}^{\nu} & \ldots & q_{1k_2}^{\nu} \\ \ldots & \ldots & \ldots \\ q_{k_2 1}^{\nu} & \ldots & q_{k_2 k_2}^{\nu} \end{Vmatrix}, \quad Q_{\nu 2} = \begin{Vmatrix} q_{k_2+1,k_2+1}^{\nu} & \ldots & q_{k_2,n}^{\nu} \\ \ldots & \ldots & \ldots \\ q_{n,k_2+1}^{\nu} & \ldots & q_{nn}^{\nu} \end{Vmatrix},$$

$$Q_{\nu 21} = \begin{Vmatrix} q_{k_2 1}^{\nu} & \ldots & q_{k_2+1,k_2}^{\nu} \\ \ldots & \ldots & \ldots \\ q_{n1}^{\nu} & \ldots & q_{nk_2}^{\nu} \end{Vmatrix}, \quad Q_{\nu} = \begin{Vmatrix} Q_{\nu 1} & 0 \\ Q_{\nu 21} & Q_{\nu 2} \end{Vmatrix}, \quad \nu = 0, 1, 2, \ldots .$$

Corollary 2. *Let the matrices Q_{01} and Q_{02} have no common characteristic numbers. System (7) can be transformed to a block diagonal form.*

To prove this, consider the matrix equation

$$\begin{Vmatrix} \mathcal{E}_{k_2} & 0 \\ \chi & \mathcal{E}_{n-k_2} \end{Vmatrix} \begin{Vmatrix} Q_{01} & 0 \\ Q_{021} & Q_{02} \end{Vmatrix} = \begin{Vmatrix} Q_{01} & 0 \\ 0 & Q_{02} \end{Vmatrix} \begin{Vmatrix} \mathcal{E}_{k_2} & 0 \\ \chi & \mathcal{E}_{n-k_2} \end{Vmatrix},$$

where $\mathcal{E}_{k_2}$ and $\mathcal{E}_{n-k_2}$ are unit matrices of dimensions k_2 and $n-k_2$, and χ is an unknown matrix of dimensions $k_2 \times (n-k_2)$. After multiplication, we obtain the equation for χ $Q_{02}\chi - \chi Q_{01} = Q_{021}$. This equation can be solved for any form of the right-hand side.

Introduce the matrix

$$\mathcal{P} = \begin{Vmatrix} \mathcal{E}_{k_2} & 0 \\ -\chi & \mathcal{E}_{n-k_2} \end{Vmatrix}.$$

It is evident that the matrix $\mathcal{A}$ is reduced to a block diagonal form by the matrix $\mathcal{LP}$.

$$\mathcal{P}^{-1}\mathcal{L}^{-1}\mathcal{A}\mathcal{L}\mathcal{P} = \operatorname{diag}\| Q_{01}, Q_{02} \|.$$

Then, all matrices $Q_{\nu N}$, $\nu = 1, 2, \ldots$, are reduced to a block diagonal form by the matrix $\mathcal{LP}$. If the converse is assumed, then the conditions of commutativity of the matrices $\mathcal{P}^{-1}\mathcal{L}^{-1}\mathcal{A}\mathcal{L}\mathcal{P}$ and Q_{ν}, $\nu = 1, 2, \ldots$.

$$\begin{Vmatrix} Q_{01} & 0 \\ 0 & Q_{02} \end{Vmatrix} \begin{Vmatrix} Q_{\nu 1} & 0 \\ Q_{\nu 21} & Q_{\nu 2} \end{Vmatrix} = \begin{Vmatrix} Q_{\nu 1} & 0 \\ Q_{\nu 21} & Q_{\nu 2} \end{Vmatrix} \begin{Vmatrix} Q_{01} & 0 \\ 0 & Q_{02} \end{Vmatrix}$$

imply the identity $Q_{02}Q_{\nu 21} - Q_{\nu 21}Q_{01} \equiv 0$. Under the assumptions that the matrices Q_{01} and Q_{02} have no common characteristic numbers, this identity implies $Q_{\nu 21} \equiv 0$.

The following statement is a criterion for the absence of the common roots of matrices Q_{01} and Q_{02}:

(a) the matrix equation $Q_{10}\chi - \chi Q_{02} \equiv 0$ has only zero solutions;

(b) the matrix determinant $G = Q_{10} \otimes \mathcal{E}_{n-k} - \mathcal{E}_{k_2} \otimes Q_{20}^{T}$ is not equal to zero.

The matrices $\mathcal{M}_{\nu N}$ involved in the centralized system often can be successfully used while constructing invariant subspaces of the matrix $\mathcal{A}$ and decomposing the centralized system.

Theorem 3. *Let the matrix $\mathcal{M}_{\nu N}$ have the zero root of multiplicity k. The subspace $L \subset R^n$, defined by the solutions of the equation $\mathcal{A}\mathcal{M}_{\nu N}\eta = 0$, $\eta \in R^n$, is invariant with respect to the matrix $\mathcal{A}$.*

Proof. The matrix commutativity condition $\mathcal{A}\mathcal{M}_{\nu N} - \mathcal{M}_{\nu N}\mathcal{A} \equiv 0$ implies that $\mathcal{A}\mathcal{M}_{\nu N}\eta = 0$. With the help of the described theorem, we have obtained a subspace L, which is invariant under the matrix $\mathcal{A}$. The centralized system can be decomposed by Theorem 2.

1. 3. The case of decomposability of the zero approximation. Consider the perturbed linear system when the zero approximation

$$\frac{dx'}{dt} = \mathcal{A}x' \tag{8}$$

is an algebraically reducible system. We can assume that (8) has been already reduced to a block diagonal form. Consider the most general case of the structure of the matrix $\mathcal{A}$. Suppose that $\mathcal{A}$ has the block diagonal form

$$\mathcal{A} = \begin{Vmatrix} \mathcal{A}_{11} & 0 & \dots & 0 \\ 0 & \mathcal{A}_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathcal{A}_{mm} \end{Vmatrix}. \tag{9}$$

The blocks $\mathcal{A}_{11}, \mathcal{A}_{22}, \dots, \mathcal{A}_{mm}$ of dimensions $r_1 \times r_1, r_2 \times r_2, \dots, r_m \times r_m$, have a block structure too

$$\mathcal{A} = \begin{Vmatrix} \mathcal{A}_i & 0 & \dots & 0 \\ 0 & \mathcal{A}_i & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathcal{A}_i \end{Vmatrix}, \quad i = \overline{1,m}. \tag{10}$$

The blocks $\mathcal{A}_i$, $i = \overline{1,m}$, have dimension $k_i \times k_i$. The multiplicity of the block $\mathcal{A}_i$ in the matrix $\mathcal{A}_{ii}$ is determined as

$$\mu_i = r_i / k_i, \quad i = \overline{1,m}. \tag{11}$$

Assume, that the system of zero approximation (8) with the matrix $\mathcal{A}$, given by formulae (9) and (10), is under a small perturbation of matrix $\mathcal{M}$ of general structure. Represent the perturbed system under consideration as follows:

$$\frac{dx'}{dt} = \begin{array}{|c|c|c|c|} \hline \mathcal{A}_{11} & 0 & \dots & 0 \\ \hline 0 & \mathcal{A}_{22} & \dots & 0 \\ \hline \dots & \dots & \dots & \dots \\ \hline 0 & 0 & \dots & \mathcal{A}_{mm} \\ \hline \end{array}\, x' + \varepsilon \begin{array}{|c|} \hline \quad \mathcal{M} \quad \\ \hline \end{array}\, x', \tag{12}$$

where $x' = \operatorname{col} \| x_1', \dots, x_n' \|$. The matrix $\mathcal{M}$ of dimension $n \times n$ has an arbitrary structure.

The perturbation in the right-hand side of (12) does not allow to reduce this system to a block diagonal form like the system of zero approximation (8). We prove the theorem allowing the possibility of decomposing the perturbed system (12) to block diagonal form.

Theorem 4. *If the blocks* $\mathcal{A}_i$, $i = \overline{1,m}$, *in matrix* (10) *of the system of zero approximation have no common characteristic numbers, then the centralized system corresponding to perturbed system* (12) *has the block diagonal structure*

$$\frac{dx}{dt} = \left\{ \begin{Vmatrix} \mathcal{A}_{11} & 0 & \dots & 0 \\ 0 & \mathcal{A}_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathcal{A}_{mm} \end{Vmatrix} + \sum_{\nu=1}^{\infty} \varepsilon^{\nu} \begin{Vmatrix} \mathcal{M}_{11}^{(\nu)} & 0 & \dots & 0 \\ 0 & \mathcal{M}_{22}^{(\nu)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathcal{M}_{mm}^{(\nu)} \end{Vmatrix} \right\}, \tag{13}$$

where the block matrices $\mathcal{M}_{ii}^{(\nu)}$ *of dimensions* $r_i \times r_i$, $i = \overline{1,m}$,

$$\mathcal{M}_{ii}^{(\nu)} = \begin{Vmatrix} \mathcal{T}_{11}^{(\nu i)} & \mathcal{T}_{12}^{(\nu i)} & \dots & \mathcal{T}_{1\mu_i}^{(\nu i)} \\ \mathcal{T}_{21}^{(\nu i)} & \mathcal{T}_{22}^{(\nu i)} & \dots & \mathcal{T}_{2\mu_i}^{(\nu i)} \\ \dots & \dots & \dots & \dots \\ \mathcal{T}_{\mu_i 1}^{(\nu i)} & \mathcal{T}_{\mu_i 2}^{(\nu i)} & \dots & \mathcal{T}_{\mu_i \mu_i}^{(\nu i)} \end{Vmatrix} \tag{14}$$

are composed of square matrices $\mathcal{T}^{(\nu i)}_{j'j''}$, $j', j'' = \overline{1, \mu_i}$, *of dimensions* $k_i \times k_i$. *The matrices* $\mathcal{T}^{(\nu i)}_{j'j''}$ *involved in matrix* (14), *commutate with the matrix* $\mathcal{A}_i$:

$$\mathcal{A}_i \mathcal{T}^{(\nu i)}_{j'j''} = \mathcal{T}^{(\nu j)}_{j'j''} \mathcal{A}_i, \quad j', j'' = \overline{1, \mu_i}, \quad i = \overline{1, m}.$$

Before proving Theorem 4, we make the following remark. Integration of the centralized system (13) is much simpler than that of the initial perturbed system (12) since this system and the system of zero approximation is decomposable into m independently integrated subsystems of dimension $r_1, \ldots, r_m$. Let us prove two auxiliary statement which will be used while proving Theorem 4. They are of special interest too. These statements show that owing to the decomposability of the system of zero approximation, all calculations can be much simplified.

Lemma 1. *Determination of the elements of the algebra of the centralizer* $\mathcal{B}_0^{(1)}$ *from the equation of the order* m

$$G_{\mathcal{F}} \hat{\chi} = 0, \tag{15}$$

where $G_{\mathcal{F}} = \mathcal{F}_1 \otimes \mathcal{E} - \mathcal{E} \otimes \mathcal{F}^T$, *is reduced to the solution of* m *independent algebraic equations of order* k_i^2

$$G_{\mathcal{F}_{1i}} \hat{\chi}^{(i)} = 0, \quad i = \overline{1, m}, \tag{16}$$

where $G_{\mathcal{F}_{1i}} = \mathcal{F}_{1i} \otimes \mathcal{E} - \mathcal{E} \otimes \mathcal{F}_i^T$.

Proof. Represent (15) as the equivalent matrix equation

$$\mathcal{F}\chi - \chi \mathcal{F} = 0. \tag{17}$$

since matrix $\mathcal{F}$ has the block diagonal form (9), matrix χ will be determined in a block matrix, too

$$\chi = \begin{Vmatrix} \chi_{11} & \chi_{12} & \cdots & \chi_{1m} \\ \chi_{21} & \chi_{22} & \cdots & \chi_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \chi_{m1} & \chi_{m2} & \cdots & \chi_{mm} \end{Vmatrix},$$

where the element χ_{ij} has the dimension $r_i \times r_j$, $i, j = \overline{1, m}$. Substituting the values of the matrices $\mathcal{A}$ and χ into equation (17), we obtain the system of m^2 independent matrix equations

$$\mathcal{A}_{ii} \chi_{ij} - \chi_{ij} \mathcal{A}_{ii} = 0, \quad i, j = \overline{1, m}. \tag{18}$$

If we assume that the matrices $\mathcal{A}_i$ and $\mathcal{A}_j$, for $i \neq j$, have no common characteristic numbers, matrices $\mathcal{A}_{ii}$ and $\mathcal{A}_{jj}$ will also have no common characteristic numbers.

Therefore, when $i \neq j$, the trivial matrix $\chi_{ij} \equiv 0$ is a unique solution of the homogeneous system (18). We have only form m equations

$$\mathcal{A}_{11} \chi_{11} - \chi_{11} \mathcal{A}_{11} = 0, \ldots, \mathcal{A}_{mm} \chi_{mm} - \chi_{mm} \mathcal{A}_{mm} = 0. \tag{19}$$

Due to the block diagonal form of the matrices $\mathcal{A}_{ii}$, $i = \overline{1, m}$, further simplification of (19) is possible. Since the matrix $\mathcal{A}_{ii}$ has μ_i (see (11)) blocks $\mathcal{A}_i$ on the principal diagonal, χ_{ii} can be determined as the block matrix

$$\chi_{ii} = \begin{Vmatrix} \chi_{11}^{(i)} & \chi_{12}^{(i)} & \dots & \chi_{1\mu_i}^{(i)} \\ \chi_{21}^{(i)} & \chi_{22}^{(i)} & \dots & \chi_{2\mu_i}^{(i)} \\ \dots & \dots & \dots & \dots \\ \chi_{\mu_i 1}^{(i)} & \chi_{\mu_i 2}^{(i)} & \dots & \chi_{\mu_i \mu_i}^{(i)} \end{Vmatrix},$$

where $\chi_{j'j''}^{(i)}$, $j', j'' = \overline{1, \mu_i}$ has the dimension $k_i \times k_i$, $i = \overline{1, m}$.

Upon substituting the values $\mathcal{A}_{ii}$, χ_{ii} into equations (19) with the index i, we obtain the system of μ_i^2 independent matrix equations

$$\mathcal{A}_i \chi_{j'j''}^{(i)} = \chi_{j'j''}^{(i)} \mathcal{A}_i, \quad j', j'' = \overline{1, \mu_i}.$$

Solution of these equations is reduced to determination of the general solution of one equation,

$$\mathcal{A}_i \chi^{(i)} = \chi^{(i)} \mathcal{A}_i, \tag{20}$$

where $\chi^{(i)}$ is a square matrix of order k_i. Passing from equations (20) to the equivalent system (16), we obtain the necessary statement.

***Corollary* 3.** *The dimension k of the algebra of the centralizer $\mathcal{B}_0^{(1)}$ is determined by the formula*

$$k = \sum_{i=1}^{m} \mu_i^2 g_i,$$

where g_i is the number of linearly independent solutions of (20).

Let the linearly independent square matrices

$$\chi_1^{(i)}, \dots, \chi_{q_i}^{(i)}, \quad i = \overline{1, m},$$

of dimension $r_i \times r_i$, where $q_i = \mu_i^2 g_i$, be a solution of (19). Denote a matrix of dimension $r_i \times r_i$ composed only of zero elements by $0^{(i)}$. The basis of $\mathcal{B}_0^{(1)}$ can be represented as

$$Z_j^{(i)} = \operatorname{diag} \left\| 0^{(1)}, \dots, 0^{(i-1)}, \chi_j^{(i)}, 0^{(i+1)}, \dots, 0^{(m)} \right\|, \quad j = \overline{1, q_i}, \quad i = \overline{1, m}.$$

The block diagonal structure of the matrix $\mathcal{A}$ of zero approximation also simplifies the problem of finding the projection $\operatorname{pr} \boldsymbol{F}_\nu$ of the right parts of the operator equation (2). This takes place because the appropriate algebraic system splits into subsystems of a smaller dimension.

***Proof of Theorem* 4.** The centralized system corresponding to perturbed system (12) can be written in the form

$$\frac{dx}{dt} = (\mathcal{A} + \varepsilon \mathcal{M}_{1N} + \varepsilon^2 \mathcal{M}_{2N} + \dots) x,$$

where the matrices $\mathcal{M}_{1N}, \mathcal{M}_{2N}, \dots$ commutate with $\mathcal{A}$.

The set of $\mathcal{M}_{1N}, \dots, \mathcal{M}_{\nu N}, \dots$ is not empty in the general case. Moreover, matrices $\mathcal{M}_{ii}^{(\nu)}$ have a structure which has been described in Lemma 1.

Let us prove two more statements on the structure of solution of the centralized system (13). Introduce the vectors

$$x^{(1)} = \left\| x_{r_1 1}, \dots, x_{r_1 r_1} \right\|, \dots, x^{(m)} = \left\| x_{1 r_m}, \dots, x_{r_m r_m} \right\|.$$

The vector of variables x is expressed by these vectors in the following way: $x = \operatorname{col}\|x^{(1)}, \ldots, x^{(m)}\|$.

Integration of (13) can be substituted by integration of m independent subsystems $i = \overline{1,m}$,

$$\frac{dx^{(i)}}{dt} = \left\{ \begin{Vmatrix} \mathcal{A}_i & 0 & \ldots & 0 \\ 0 & \mathcal{A}_i & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \mathcal{A}_i \end{Vmatrix} + \sum_{\nu=1}^{\infty} \varepsilon^{\nu} \begin{Vmatrix} \mathcal{T}_{11}^{(\nu i)} & \mathcal{T}_{12}^{(\nu i)} & \ldots & \mathcal{T}_{1\mu_i}^{(\nu i)} \\ \mathcal{T}_{21}^{(\nu i)} & \mathcal{T}_{22}^{(\nu i)} & \ldots & \mathcal{T}_{2\mu_i}^{(\nu i)} \\ \ldots & \ldots & \ldots & \ldots \\ \mathcal{T}_{\mu_i 1}^{(\nu i)} & \mathcal{T}_{\mu_i 2}^{(\nu i)} & \ldots & \mathcal{T}_{\mu_i \mu_i}^{(\nu i)} \end{Vmatrix} \right\} x^{(i)}. \quad (21)$$

Theorem 5. *The solution of centralized system* (13) *can be represented in the form*

$$x^{(i)} = \begin{Vmatrix} e^{A_i(t-t_0)} & 0 & \ldots & 0 \\ 0 & e^{A_i(t-t_0)} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & e^{A_i(t-t_0)} \end{Vmatrix} \eta^{(i)}(\tau), \quad (22)$$

where the vector $\eta^{(i)}(\tau) = \operatorname{col}\|\eta_{1r_i}(\tau), \ldots, \eta_{r_i r_i}(\tau)\|$ *is a function of slow variables* $\tau \equiv \varepsilon t$ *and is determined by integration of the system of equations with slow time*

$$\frac{d\eta^{(i)}}{d\tau} = \left(\sum_{\nu=1}^{\infty} \varepsilon^{\nu-1} \mathcal{M}_{ii}^{(\nu)} \right) \eta^{(i)}. \quad (23)$$

Thus, integration of (13) *is reduced to integration of* m *independent subsystems*

$$\frac{dy^{(i)}}{dt} = \mathcal{A}_i y^{(i)}$$

of orders k_i, $i = \overline{1,m}$, *and to integration of* m *independent subsystems of orders* r_i, $i = \overline{1,m}$ *(see* (23)*), which depend on slow time* τ.

Proof. Represent the relations (22) in the compact form

$$x^{(i)} = e^{A_{ii}(t-t_0)} \eta^{(i)}. \quad (24)$$

Let the vector $\eta^{(i)}$ be constant. Formula (24) represents the solution of the system of zero approximation

$$\frac{dx^{(i)}}{dt} = \mathcal{A}_{ii} x^{(i)}.$$

Suppose that $\eta^{(i)}$ are new variables in formula (24). Upon the change of variables in (21) and after simple calculations, we obtain

$$\frac{d\eta^{(i)}}{dt} = \left(e^{A_{ii}(t-t_0)} \left(\sum_{\nu=1}^{\infty} \varepsilon^{\nu} \mathcal{M}_{ii}^{(\nu)} \right) e^{-A_{ii}(t-t_0)} \right) \eta^{(i)}. \quad (25)$$

Since $\mathcal{A}_{ii}$ and $\mathcal{M}_{ii}^{(\nu)}$ commmute, (25) can be represented in the form

$$\frac{d\eta^{(i)}}{d\tau} = \left(\sum_{\nu=1}^{\infty} \varepsilon^{\nu} \mathcal{M}_{ii}^{(\nu)} \right) \eta^{(i)}.$$

Introducing the slow time $\tau = \varepsilon t$, we obtain (25).

So far, we have not assumed that the characteristic numbers of the matrices $\mathcal{A}_i$, $i = \overline{1,m}$, involved in the system of zero approximation are known. If they are really known, we can formulate one more result on the decomposition of the centralized system (21). Since (21) involves subsystems identical in structure, we can simplify our calculations by introducing a representative system in the form

$$\frac{d\tilde{x}}{dt} = \left\{ \begin{Vmatrix} \mathcal{P} & 0 & \ldots & 0 \\ 0 & \mathcal{P} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \mathcal{P} \end{Vmatrix} + \sum_{\nu=1}^{\infty} \varepsilon^{\nu} \begin{Vmatrix} \mathcal{T}_{11}^{(\nu)} & \mathcal{T}_{12}^{(\nu)} & \ldots & \mathcal{T}_{1\mu}^{(\nu)} \\ \mathcal{T}_{21}^{(\nu)} & \mathcal{T}_{22}^{(\nu)} & \ldots & \mathcal{T}_{2\mu}^{(\nu)} \\ \ldots & \ldots & \ldots & \ldots \\ \mathcal{T}_{\mu 1}^{(\nu)} & \mathcal{T}_{\mu 2}^{(\nu)} & \ldots & \mathcal{T}_{\mu\mu}^{(\nu)} \end{Vmatrix} \right\} \tilde{x}, \tag{26}$$

where $\mathcal{P}$ are matrices of dimension $k \times k$ with multiplicity involving μ in the coefficient matrix, $\mathcal{T}_{ij}^{(\nu)}$ are square matrices of dimensions $k \times k$, $x = \operatorname{col} \| \tilde{x}_1, \ldots, \tilde{x}_s \|$, $s = k\mu$.

The matrices $\mathcal{P}_d = \operatorname{diag} \| \mathcal{P}, \ldots, \mathcal{P} \|$ and $\mathcal{Z}_\nu = \left\{ \mathcal{T}_{ij}^{(\nu)} \right\}$, $i, j = \overline{1,\mu}$, are supposed to be commutative, i. e.

$$\mathcal{P}_d \mathcal{Z}_\nu \equiv \mathcal{Z}_\nu \mathcal{P}_d, \quad \nu = 1, 2, \ldots \tag{27}$$

Theorem 6. *Let the matrix* $\mathcal{P}$ *in* (26) *have the characteristic numbers* λ_j *of multiplicities* m_j, $j = 1, \ldots, p$; *then we can indicate the change of variables* $\tilde{x} = \mathcal{L}\tilde{z}$, *which decompose this system into independently integrated systems of dimensions* $m_1\mu, \ldots, m_p\mu((m_1 + \ldots + m_p)\mu = k\mu)$

$$\frac{d\tilde{z}}{dt} = \left\{ \begin{Vmatrix} \mathcal{P}_1 & 0 & \ldots & 0 \\ 0 & \mathcal{P}_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \mathcal{P}_p \end{Vmatrix} + \sum_{\nu=1}^{\infty} \varepsilon^{\nu} \begin{Vmatrix} \mathcal{R}_1^{(\nu)} & 0 & \ldots & 0 \\ 0 & \mathcal{R}_2^{(\nu)} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \mathcal{R}_p^{(\nu)} \end{Vmatrix} \right\} \tilde{z},$$

where

$$\mathcal{P}_i = \operatorname{diag} \Big\| \underbrace{\{ \lambda_i, \ldots, \lambda_i \}}_{m_i\mu} \Big\|, \; \mathcal{R}_i^{(\nu)}$$

are square matrices of the dimensions $\mu_i\mu \times \mu_i\mu$, $i = \overline{1,p}$.

Proof. Denote the root subspace of the matrix $\mathcal{P}_d$ by $P(\lambda_i)$. It is determined by various solutions of the equation $(\mathcal{P}_d - \lambda_i \mathcal{E})\xi = 0$, $\xi = \operatorname{col} \| \xi_1, \ldots, \xi_s \|$, $s = k\mu$. Identity (27) implies

$$(\mathcal{P}_d - \lambda_i \mathcal{E}) \mathcal{Z}_\nu \equiv \mathcal{Z}_\nu (\mathcal{P}_d - \lambda_i \mathcal{E}).$$

Multiplying both sides of the matrix identity by the vector $\xi \in P(\lambda_i)$, we obtain $(\mathcal{P}_d - \lambda_i \mathcal{E}) \mathcal{Z}_\nu \xi \equiv 0$. Therefore, the subspace $P(\lambda_i)$ is invariant under the matrix $\mathcal{Z}_\nu$, $\nu = 1, 2, \ldots$. The reducing matrix $\mathcal{L}$ is composed of the vectors of the root subspaces $P(\lambda_i)$, $i = \overline{1,p}$.

1. 4. A model of a mechanical system. Let us assume that oscillating masses of the algebraically reducible mechanical system are subjected to small perturbations $\varepsilon\Delta_1$, $\varepsilon\Delta_1$ (Fig. 1). In this case, additional terms proportional to the parameter ε and characterizing the presence of perturbation factors, appear in the matrix of the system:

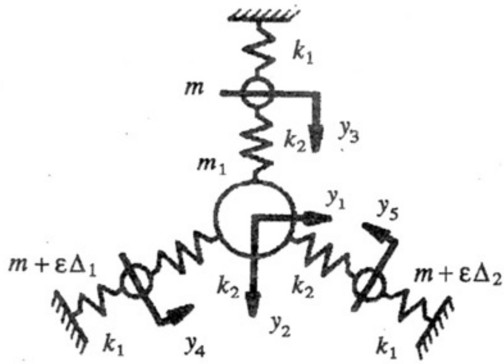

Fig. 1. Model of a mechanical system with disturbed symmetry.

$$\begin{Vmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \\ \ddot{y}_5 \end{Vmatrix} + \left\{ \begin{Vmatrix} 3a_1/2 & 0 & 0 & -\sqrt{3}a_1/2 & \sqrt{3}a_1/2 \\ 0 & 3a_1/2 & -a_1 & a_1/2 & a_1/2 \\ 0 & -a_2 & a_2+a_3 & 0 & 0 \\ -\sqrt{3}a_2/2 & a_2/2 & 0 & a_2+a_3 & 0 \\ \sqrt{3}a_2/2 & 3a_2/2 & 0 & 0 & a_2+a_3 \end{Vmatrix} + \right.$$

$$\left. + \varepsilon \begin{Vmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ c_{31} & c_{32} & 0 & c_{34} & 0 \\ c_{41} & c_{42} & 0 & 0 & c_{44} \end{Vmatrix} + \ldots \right\} \begin{Vmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{Vmatrix} = 0, \tag{28}$$

where

$$c_{31} = \frac{\sqrt{3}}{2} a_2 \frac{\Delta_1}{m}; \quad c_{32} = -\frac{1}{2} a_2 \frac{\Delta_1}{m}; \quad c_{34} = -(a_2+a_3)\frac{\Delta_1}{m};$$

$$c_{41} = -\frac{\sqrt{3}}{2} a_2 \frac{\Delta_2}{m}; \quad c_{42} = -\frac{1}{2} a_2 \frac{\Delta_2}{m}; \quad c_{44} = -(a_2+a_3)\frac{\Delta_2}{m}.$$

Only one matrix of perturbations for the first power of the parameter ε is written out in (28). Upon the change of variables $y = S\,y'$, $y' = \mathrm{col}\,\| y_1', y_2', y_3', y_4', y_5', \|$,

$$\begin{Vmatrix} \ddot{y}_1' \\ \ddot{y}_2' \\ \ddot{y}_3' \\ \ddot{y}_4' \\ \ddot{y}_5' \end{Vmatrix} + \left\{ \begin{Vmatrix} 3a_1/2 & 3a_1 & 0 & 0 & 0 \\ a_2/2 & a_2+a_3 & 0 & 0 & 0 \\ 0 & 0 & 3a_1/2 & 3a_1 & 0 \\ 0 & 0 & a_2/2 & a_2+a_3 & 0 \\ 0 & 0 & 0 & 0 & a_2+a_3 \end{Vmatrix} + \right.$$

$$\left. + \varepsilon \begin{Vmatrix} 0 & 0 & 0 & 0 & 0 \\ q_{21} & q_{22} & q_{23} & q_{24} & q_{25} \\ q_{31} & 0 & 0 & 0 & 0 \\ 0 & q_{42} & q_{43} & q_{44} & q_{45} \\ q_{51} & q_{52} & q_{53} & q_{54} & q_{55} \end{Vmatrix} + \ldots \right\} \begin{Vmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \\ y_5' \end{Vmatrix} = 0, \tag{29}$$

where

$$q_{21} = -\frac{a_2(\Delta_1 + \Delta_2)}{4m}; \quad q_{22} = -\frac{(a_2 + a_3)(\Delta_1 + \Delta_2)}{2m}; \quad q_{23} = \frac{a_2(\Delta_1 - \Delta_2)}{4\sqrt{3}m};$$

$$q_{24} = q_{25} = \frac{(a_2 + a_3)(\Delta_1 - \Delta_2)}{2\sqrt{3}m}; \quad q_{31} = \frac{\sqrt{3}a_2(\Delta_1 - \Delta_2)}{12m};$$

$$q_{42} = -\frac{\sqrt{3}(a_2 + a_3)(\Delta_1 - \Delta_2)}{6m}; \quad q_{43} = -\frac{a_2(\Delta_1 + \Delta_2)}{12m};$$

$$q_{44} = q_{45} = -\frac{(a_2 + a_3)(\Delta_1 + \Delta_2)}{6m}; \quad q_{51} = \frac{\sqrt{3}a_2(\Delta_1 - \Delta_2)}{6m};$$

$$q_{52} = -\frac{\sqrt{3}(a_2 + a_3)(\Delta_1 - \Delta_2)}{3m}; \quad q_{53} = -\frac{a_2(\Delta_1 + \Delta_2)}{6m};$$

$$q_{54} = q_{55} = -\frac{(a_2 + a_3)(\Delta_1 + \Delta_2)}{3m}.$$

By introducing the new variables

$$y_1' = x_1', \quad y_2' = x_2', \quad \dot{y}_1' = x_3', \quad \dot{y}_2' = x_4', \quad y_3' = x_5', \quad y_4' = x_6',$$
$$\dot{y}_3' = x_7', \quad \dot{y}_4' = x_8', \quad y_5' = x_9', \quad \dot{y}_5' = x_{10}',$$

we pass from second-order system (29) to a normal system of differential equations. Finally we obtain

$$\left\| \begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \\ \dot{x}_8 \\ \dot{x}_9 \\ \dot{x}_{10} \end{array} \right\| = \left\{ \left\| \begin{array}{cccc|cccc|cc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3a_1/2 & -3a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_2/2 & -b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3a_1/2 & -3a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_2/2 & -b_1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b_1 & 0 \end{array} \right\| - \right.$$

$$\left. -\varepsilon \left\| \begin{array}{cccc|cccc|cc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q_{21} & q_{22} & 0 & 0 & q_{23} & q_{24} & 0 & 0 & q_{25} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_{43} & q_{44} & 0 & 0 & q_{45} & 0 \\ \hline q_{51} & q_{52} & 0 & q_{53} & q_{54} & 0 & 0 & 0 & q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right\| + \ldots \right\} \left\| \begin{array}{c} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \\ x_6' \\ x_7' \\ x_8' \\ x_9' \\ x_{10}' \end{array} \right\|, \qquad (30)$$

where $b_1 = a_2 + a_3$.

Denote the matrix of zero approximation by $\mathcal{A}$ and the perturbation matrix by $\tilde{\mathcal{A}}$. Upon introducing the notations

$$\mathcal{M} = \begin{Vmatrix} -3a_1/2 & -a_2/2 \\ -3a_1 & -b_1 \end{Vmatrix}, \quad \mathcal{A}_1 = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3a_1/2 & -3a_1 & 0 & 0 \\ -a_2/2 & -b_1 & 0 & 0 \end{Vmatrix},$$

$$\mathcal{A}_2 = \begin{Vmatrix} 0 & 1 \\ -b_1 & 0 \end{Vmatrix}, \tag{31}$$

we can write the matrix of zero approximation in the following way:

$$\mathcal{A} = \begin{Vmatrix} \mathcal{A}_1 & 0 & 0 \\ 0 & \mathcal{A}_1 & 0 \\ 0 & 0 & \mathcal{A}_2 \end{Vmatrix}.$$

Using the described theory, we can find the centralized system of the first approximation for (30). The matrix equation for determining elements of the algebra of the centralizer $\mathcal{B}_0^{(1)}$ is represented in the form

$$\begin{Vmatrix} \mathcal{A}_1 & 0 & 0 \\ 0 & \mathcal{A}_1 & 0 \\ 0 & 0 & \mathcal{A}_2 \end{Vmatrix} \begin{Vmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{Vmatrix} =$$

$$= \begin{Vmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{Vmatrix} \begin{Vmatrix} \mathcal{A}_1 & 0 & 0 \\ 0 & \mathcal{A}_1 & 0 \\ 0 & 0 & \mathcal{A}_2 \end{Vmatrix}, \tag{32}$$

where

$$\chi_{11}, \ \chi_{12}, \ \chi_{21}, \ \chi_{22} \in R^{(1,4)},$$

$$\chi_{33} \in R^{(2,2)}, \ \chi_{13}, \chi_{23} \in R^{(4,2)}, \ \chi_{31}, \chi_{32} \in R^{(2,4)}.$$

This equation decomposed into nine independent matrix equations

$$\mathcal{A}_1\chi_{11} = \chi_{11}\mathcal{A}_1, \ \mathcal{A}_1\chi_{12} = \chi_{12}\mathcal{A}_1, \ \mathcal{A}_1\chi_{13} = \chi_{13}\mathcal{A}_2,$$

$$\mathcal{A}_1\chi_{21} = \chi_{21}\mathcal{A}_1, \ \mathcal{A}_1\chi_{22} = \chi_{22}\mathcal{A}_1, \ \mathcal{A}_1\chi_{23} = \chi_{23}\mathcal{A}_2,$$

$$\mathcal{A}_2\chi_{31} = \chi_{31}\mathcal{A}_2, \ \mathcal{A}_2\chi_{32} = \chi_{32}\mathcal{A}_2, \ \mathcal{A}_2\chi_{33} = \chi_{33}\mathcal{A}_2.$$

By virtue of the assumption that the matrices $\mathcal{A}_1$ and $\mathcal{A}_2$ have no common characteristic numbers, $\chi_{13}, \chi_{23}, \chi_{31}, \chi_{32}$ are zero. The remaining five equations can be decomposed into two groups. The equations for $\chi_{11}, \chi_{12}, \chi_{21}, \chi_{22}$ are equivalent to solution of the equation

$$\mathcal{A}_1\chi = \chi\mathcal{A}_1, \quad \chi \in R^{(4,4)}. \tag{33}$$

The second group is composed of the equation

$$\mathcal{A}_2\chi_{33} = \chi_{33}\mathcal{A}_2. \tag{34}$$

Under the assumption that the matrix $\mathcal{M}$ has various characteristic numbers, we can find a general solution of equation (33). Talking into account the structure of matrix $\mathcal{A}_1$ (see formulae (31)), equation (33) can be represented in the form

$$\begin{Vmatrix} 0 & \mathcal{M} \\ \mathcal{E}_2 & 0 \end{Vmatrix} \begin{Vmatrix} \chi_1 & \chi_2 \\ \chi_3 & \chi_4 \end{Vmatrix} = \begin{Vmatrix} \chi_1 & \chi_2 \\ \chi_3 & \chi_4 \end{Vmatrix} \begin{Vmatrix} 0 & \mathcal{M} \\ \mathcal{E}_2 & 0 \end{Vmatrix}, \tag{35}$$

where $\chi_1, \chi_2, \chi_3, \chi_4 \in R^{(2,2)}$, $\mathcal{E}_2$ is the unit matrix. System (35) is decomposed into four matrix equations

$$\chi_2 = \chi_3 \mathcal{M}, \quad \chi_4 = \chi_1, \quad \mathcal{M}\chi_1 = \chi_4 \mathcal{M}, \quad \mathcal{M}\chi_3 = \chi_2.$$

The system of these equations is reduced to finding the matrices χ_1, χ_2 from the identical equations

$$\mathcal{M}\chi_1 = \chi_1 \mathcal{M}, \quad \mathcal{M}\chi_3 = \chi_3 \mathcal{M}. \tag{36}$$

Under the assumption that $\mathcal{M}$ has different characteristic numbers, we can easily obtain the general solution of (36):

$$\chi_1 = \mu_1 \mathcal{E}_2 + \mu_2 \mathcal{M}, \quad \chi_3 = \mu_3 \mathcal{E}_2 + \mu_4 \mathcal{M}, \quad \mu_1, \mu_2, \mu_3, \mu_4 \in P.$$

Analogously, the general solution of equation (34) has the form

$$\chi_{32} = a_1 \mathcal{E}_2 + a_2 \mathcal{A}_2, \quad a_1, a_2 \in P.$$

Thus, the general solution χ of equation (33) depends on four arbitrary parameters and can be represented by the matrix

$$\chi = \begin{Vmatrix} \mu_1 \mathcal{E}_2 + \mu_2 \mathcal{M} & \mu_3 \mathcal{M} + \mu_4 \mathcal{M}^2 \\ \mu_3 \mathcal{E}_2 + \mu_4 \mathcal{M} & \mu_1 \mathcal{E}_2 + \mu_2 \mathcal{M} \end{Vmatrix}.$$

Coming back to equation (32) and considering the structure of the matrices χ_{33} and $\tilde{\mathcal{A}}$, we can write the general element $\mathcal{Z}$ of the algebra of the centralizer $\mathcal{B}_0^{(1)}$ in the following way:

$$\mathcal{Z} = \left\Vert \begin{array}{cc|cc|c} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 \\ \hline a_{31} & a_{32} & a_{33} & a_{34} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ \hline 0 & 0 & 0 & 0 & a_{55} \end{array} \right\Vert,$$

where

$$a_{11} = \mu_1 \mathcal{E}_2 + \mu_2 \mathcal{M}, \quad a_{12} = \mu_3 \mathcal{M} + \mu_4 \mathcal{M}^2, \quad a_{13} = \mu_9 \mathcal{E}_2 + \mu_{10} \mathcal{M},$$
$$a_{14} = \mu_{11} \mathcal{M} + \mu_{12} \mathcal{M}^2, \quad a_{21} = \mu_3 \mathcal{E}_2 + \mu_4 \mathcal{M}, \quad a_{22} = \mu_1 \mathcal{E}_2 + \mu_2 \mathcal{M},$$
$$a_{23} = \mu_{11} \mathcal{E}_2 + \mu_{12} \mathcal{M}, \quad a_{24} = \mu_9 \mathcal{E}_2 + \mu_{10} \mathcal{M}, \quad a_{31} = \mu_5 \mathcal{E}_2 + \mu_6 \mathcal{M},$$
$$a_{32} = \mu_7 \mathcal{M} + \mu_8 \mathcal{M}^2, \quad a_{33} = \mu_{13} \mathcal{E}_2 + \mu_{14} \mathcal{M}, \quad a_{34} = \mu_{15} \mathcal{M} + \mu_{16} \mathcal{M}^2,$$
$$a_{41} = \mu_7 \mathcal{E}_2 + \mu_8 \mathcal{M}, \quad a_{42} = \mu_5 \mathcal{E}_2 + \mu_6 \mathcal{M}, \quad a_{43} = \mu_{15} \mathcal{E}_2 + \mu_{16} \mathcal{M},$$
$$a_{44} = \mu_{13} \mathcal{E}_2 + \mu_{14} \mathcal{M}, \quad a_{55} = \mu_{17} \mathcal{E}_2 + \mu_{18} \mathcal{A}_2.$$

Let $\mu_i = 1$, $\mu_j = 0$, $i \neq j$, where i, j run over the values from 1 to 18. We obtain 18 independent elements in the algebra of the centralizer $\mathcal{B}_0^{(1)}$ and denote them by $\mathcal{Z}_1, \ldots, \mathcal{Z}_{18}$. The projection of the matrix $\tilde{\mathcal{A}}$ onto the kernel of the operator $G_{\mathcal{A}}$ should be found in the form of the sum

$$\tilde{\mathcal{A}}_N = \sum_{i=1}^{18} \alpha_i Z_i. \tag{37}$$

The coefficients $\alpha_{01}, \ldots, \alpha_{018}$, are determined from the system of equations

$$\begin{gathered}
2m_1\alpha_{01} + 2m_2\alpha_{04} = p_1; \quad 4\alpha_{01} + 2m_1\alpha_{02} = 0; \\
2m_2\alpha_{03} + 2m_3\alpha_{04} = q_1; \quad 2m_1\alpha_{01} + 2m_2\alpha_{02} = 0; \\
2m_1\alpha_{011} + 2m_2\alpha_{012} = p_2; \quad 4\alpha_{09} + 2m_1\alpha_{010} = 0; \\
2m_2\alpha_{011} + 2m_3\alpha_{012} = q_2; \quad 2m_1\alpha_{09} + 2m_2\alpha_{010} = 0; \\
2m_1\alpha_{015} + 2m_2\alpha_{016} = p_3; \quad 4\alpha_{013} + 2m_1\alpha_{014} = 0; \\
2m_2\alpha_{015} + 2m_3\alpha_{016} = q_3; \quad 2m_1\alpha_{013} + 2m_2\alpha_{014} = 0; \\
2\alpha_{017} = q_{55}; \quad -2(a_2 + a_3)\alpha_{018} = 0,
\end{gathered} \tag{38}$$

where

$$p_1 = -\frac{1}{2}a_2 q_{21} - q_{22}(a_1 + a_2); \quad p_2 = -\frac{1}{2}a_2 q_{23} - q_{24}(a_1 + a_2);$$

$$p_3 = -\frac{1}{2}a_2 q_{43} - q_{44}(a_1 + a_2); \quad m_1 = \operatorname{tr}\mathcal{M}; \quad m_2 = \operatorname{tr}\mathcal{M}^2; \quad m_3 = \operatorname{tr}\mathcal{M}^3;$$

$$q_1 = \frac{1}{2}a_2\left(\frac{3}{2}a_1 q_{21} - 3a_1 q_{22}\right) + (a_2 + a_3)\left(\frac{1}{2}a_2 q_{21} + q_{22}(a_1 + a_2)\right);$$

$$q_2 = \frac{1}{2}a_2\left(\frac{3}{2}a_1 q_{23} + 3a_1 q_{24}\right) + (a_2 + a_3)\left(\frac{1}{2}a_2 q_{23} + q_{24}(a_1 + a_2)\right);$$

$$q_3 = \frac{1}{2}a_2\left(\frac{3}{2}a_1 q_{43} + 3a_1 q_{44}\right) + (a_2 + a_3)\left(\frac{1}{2}a_2 q_{43} + q_{44}(a_1 + a_2)\right).$$

It is easy to solve the system of algebraic equations (38):

$$\alpha_{03} = \frac{1}{2}\frac{m_3}{m_1 m_2 - m_2^2}p_1 - \frac{1}{2}\frac{m_2}{m_1 m_3 - m_2^2}q_1;$$

$$\alpha_{043} = -\frac{1}{2}\frac{m_2}{m_1 m_3 - m_2^2}p_1 + \frac{1}{2}\frac{m_1}{m_1 m_3 - m_2^2}q_1.$$

Analogous expressions can be obtained for α_{011}, α_{012} and α_{015}, α_{016}:

$$\alpha_{017} = \frac{1}{2}q_{55}; \quad \alpha_{018} = 0.$$

The other coefficients are identically equal to zero. With the help of the coefficients in expansion (37), we can obtain a centralized system in the first approximation:

$$\frac{dx}{dt} = \left\{ \left\| \begin{array}{cc|cc|c} 0 & \mathcal{M} & 0 & 0 & 0 \\ \mathcal{E}_2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathcal{M} & 0 \\ 0 & 0 & \mathcal{E}_2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \mathcal{A}_2 \end{array} \right\|^T - \varepsilon \left\| \begin{array}{cc|cc|c} 0 & b_{12} & 0 & b_{14} & 0 \\ b_{21} & 0 & b_{23} & 0 & 0 \\ \hline 0 & 0 & 0 & b_{34} & 0 \\ 0 & 0 & b_{43} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & b_{55} \end{array} \right\|^T \right\}, \tag{39}$$

where

$$b_{12} = \alpha_{03}\mathcal{M} + \alpha_{04}\mathcal{M}^2, \quad b_{14} = \alpha_{011}\mathcal{M} + \alpha_{012}\mathcal{M}^2,$$

$$b_{21} = \alpha_{03}\mathcal{E}_2 + \alpha_{04}\mathcal{M}, \quad b_{23} = \alpha_{011}\mathcal{E}_2 + \alpha_{012}\mathcal{M},$$

$$b_{34} = \alpha_{015}\mathcal{M} + \alpha_{016}\mathcal{M}^2, \quad b_{43} = \alpha_{015}\mathcal{E}_2 + \alpha_{016}\mathcal{M}, \quad b_{55} = \alpha_{017}\mathcal{E}_2.$$

Comparing (39) with the initial perturbed (30) we can see that the centralized system decomposes into two independent subsystems of orders 8×8 and 2×2. The system of order 8×8, in its turn, decomposes into two successively integrated subsystems of orders 4×4 and 4×4.

Under the assumption that the characteristic numbers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of the matrix $\mathcal{A}_1$ (see formulae (31)) are known, (39) admits further decomposition into independently integrable subsystems. In fact, let $\eta^{(j)} = \mathrm{col}\,\|\eta_1^{(j)}, \eta_2^{(j)}, \eta_3^{(j)}, \eta_4^{(j)}\|$, $j = \overline{1,4}$, be the eigenvectors of the matrix $\mathcal{A}_1$. Introduce the notations $\zeta_1^{(j)} = \|\eta_1^{(j)}, \eta_2^{(j)}\|$, $\zeta_2^{(j)} = \|\eta_3^{(j)}, \eta_4^{(j)}\|$. Then the relation

$$\begin{Vmatrix} 0 & \mathcal{E}_2 \\ \mathcal{M} & 0 \end{Vmatrix} \begin{Vmatrix} \zeta_1^{(j)} \\ \zeta_2^{(j)} \end{Vmatrix} = \lambda_j \begin{Vmatrix} \zeta_1^{(j)} \\ \zeta_2^{(j)} \end{Vmatrix}$$

is valid.

To simplify our calculations we assume that the characteristic numbers of the matrix $\mathcal{M}$ vary. In this case, the characteristic numbers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of the matrix $\mathcal{A}_1$ also vary.

Introduce the matrix

$$S = \begin{Vmatrix}
\eta_1^{(1)} & \eta_2^{(1)} & \lambda_1\eta_1^{(1)} & \lambda_1\eta_2^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \eta_1^{(1)} & \eta_2^{(1)} & \lambda_1\eta_1^{(1)} & \lambda_1\eta_2^{(1)} & 0 & 0 \\
\eta_1^{(2)} & \eta_2^{(2)} & \lambda_2\eta_1^{(2)} & \lambda_2\eta_2^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \eta_1^{(2)} & \eta_2^{(2)} & \lambda_2\eta_1^{(2)} & \lambda_2\eta_2^{(2)} & 0 & 0 \\
\eta_1^{(3)} & \eta_2^{(3)} & \lambda_3\eta_1^{(3)} & \lambda_3\eta_3^{(3)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \eta_1^{(3)} & \eta_2^{(3)} & \lambda_3\eta_1^{(3)} & \lambda_3\eta_3^{(3)} & 0 & 0 \\
\eta_1^{(4)} & \eta_2^{(4)} & \lambda_4\eta_1^{(4)} & \lambda_4\eta_4^{(4)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \eta_1^{(4)} & \eta_2^{(4)} & \lambda_4\eta_1^{(4)} & \lambda_4\eta_4^{(4)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{Vmatrix}.$$

Upon the change of variables $x = S\,y$, the centralized system (39) falls into five independently integrable subsystems of the second order

$$\frac{dy}{dt} = \left\{ \begin{Vmatrix}
\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(a_2 + a_3) & 0
\end{Vmatrix} - \right.$$

$$
-\varepsilon \left\| \begin{array}{cc|cc|cc|cc|cc}
b_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b_{12} & b_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & b_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b_{34} & b_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & b_{55} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b_{56} & b_{66} & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & b_{77} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_{78} & b_{88} & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{017} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{017}
\end{array} \right\| \Bigg\} y,
$$

where

$$b_{11} = \lambda_1\alpha_{03} + \lambda_1^3\alpha_{04}, \quad b_{12} = \lambda_1\alpha_{011} + \lambda_1^3\alpha_{012},$$

$$b_{22} = \lambda_1\alpha_{015} + \lambda_1^3\alpha_{016},$$

$$b_{33} = \lambda_2\alpha_{03} + \lambda_2^3\alpha_{04}, \quad b_{34} = \lambda_2\alpha_{011} + \lambda_2^3\alpha_{012},$$

$$b_{44} = \lambda_2\alpha_{015} + \lambda_2^3\alpha_{016},$$

$$b_{55} = \lambda_3\alpha_{03} + \lambda_3^3\alpha_{04}, \quad b_{56} = \lambda_3\alpha_{011} + \lambda_3^3\alpha_{012},$$

$$b_{66} = \lambda_3\alpha_{015} + \lambda_3^3\alpha_{016},$$

$$b_{77} = \lambda_4\alpha_{03} + \lambda_4^3\alpha_{04}, \quad b_{78} = \lambda_4\alpha_{011} + \lambda_4^3\alpha_{012},$$

$$b_{88} = \lambda_4\alpha_{015} + \lambda_4^3\alpha_{016}.$$

Each subsystem, it its turn, is reduced to quadratures. We should determine the matrix of the operator $\boldsymbol{S}$. This determination is also reduced to the solution of a system of linear inhomogeneous algebraic equations. We do not present these calculation here because of their awkwardness.

2. The general case of the structure of a matrix of zero approximation system. 2. 1. Formulation of the problem and realization of the algorithm. Consider the case when the matrix in equation (1) $\mathcal{A}$ is nondiagonable (defective). It should be considered while solving the operator equations

$$[\boldsymbol{U}, \boldsymbol{S}_\nu] = \boldsymbol{F}_\nu, \quad \nu = 1, 2, \ldots, \tag{40}$$

and equivalent matrix equations

$$\mathcal{F}_1\Gamma_\nu - \Gamma_\nu\mathcal{F} = Q_\nu, \quad \mathcal{F}_1 \equiv \mathcal{A}^T, \quad \mathcal{F} \equiv \mathcal{A}^T, \tag{41}$$

or

$$G_{\mathcal{F}}\hat{\Gamma}_\nu = \hat{Q}_\nu, \quad G_{\mathcal{F}} = \mathcal{F}_1 \otimes \mathcal{E}_n - \mathcal{E}_n \otimes \mathcal{F}^T,$$

which are equivalent to (40).

The matrix $G_{\mathcal{F}}$ will also be nondiagonable. So the kernel $N_{\mathcal{F}}$ of $G_{\mathcal{F}}$, defined as a solution of the homogeneous equation $G_{\mathcal{F}}\hat{\chi} = 0$, will not coincide with the kernel $\hat{N}_{\mathcal{F}}^2$ of $\hat{G}_{\mathcal{F}}^2$ which is defined as a solution of the homogeneous equation $G_{\mathcal{F}}^2\,\hat{\chi} = 0$.

To preserve the symmetry of notation, we take $\hat{N}_{\mathcal{F}}^{(1)}$ for the kernel $\hat{N}_{\mathcal{F}}$ of the matrix $G_{\mathcal{F}}$. Thus, $N_{\mathcal{F}}^{(1)} \subset N_{\mathcal{F}}^{(2)}$.

It is evident that we consider the subspaces $\hat{N}_{\mathcal{F}}^{(j)}$, $j = \overline{1, r}$.

$$G_{\mathcal{F}}^{j} \hat{\chi} = 0. \tag{42}$$

As is known from linear algebra, the chain of subspaces

$$\hat{N}_{\mathcal{F}}^{(1)} \subset \hat{N}_{\mathcal{F}}^{(2)} \subset \ldots \subset \hat{N}_{\mathcal{F}}^{(j)} \subset \hat{N}_{\mathcal{F}}^{(j+1)}$$

is broken when the equality $\hat{N}_{\mathcal{F}}^{(j)} = \hat{N}_{\mathcal{F}}^{(j+1)}$ is fulfilled. The number j should not exceed the multiplicity of the zero characteristic number of $G_{\mathcal{F}}$.

Let j be the power in equation (42) and vector $\hat{\chi}$ be its solution. The smallest power index j is called the height of vector $\hat{\chi}$. For the linear space $R^{(n,n)}$, the set of equations (42) can be represented in the form

$$[\mathcal{F}, \chi] = 0, [\mathcal{F}, [\mathcal{F}, \chi]] = 0, \ldots, \underbrace{[\mathcal{F}, \ldots, [\mathcal{F}, \chi}_{j}]] = 0. \tag{43}$$

It is evident that if j is the height of the vector $\hat{\chi}$, then $j-1$ is the height of the vector $G_{\mathcal{F}}\hat{\chi}$. In algebraic Lie notations, its means that, if the element $\chi \in N_{\mathcal{F}}^{(j)}$, then the element $[\mathcal{F}, \chi] \in N_{\mathcal{F}}^{(j-1)}$. It is easy to show that, if $\chi' \in N_{\mathcal{F}}^{(j')}$, $\chi'' \in N_{\mathcal{F}}^{(j'')}$, then $[\chi', \chi''] \in N_{\mathcal{F}}^{(j_0-1)}$, where $j_0 = \max(j', j'')$. Really, Jacobian identity implies $[\mathcal{F}, [\chi', \chi'']] + [\chi'', [\mathcal{F}, \chi']] + [\chi', [\chi'', \mathcal{F}]] \equiv 0$, that $[\chi'', [\mathcal{F}, \chi']]$, $[\chi', [\chi'', \mathcal{F}]] \in N_{\mathcal{F}}^{(j_0-2)}$ and, hence, $[\chi', \chi''] \in N_{\mathcal{F}}^{(j_0)}$.

Denote the maximum height of the vectors of the matrix $G_{\mathcal{F}}$ by r and the number of independent solutions of the equation $G_{\mathcal{F}}\hat{\chi} = 0$ by k. The obtained result can be formulated as a statement.

Theorem 7. *Let the matrix $G_{\mathcal{F}}$ with the maximum height of the vectors r correspond to $\boldsymbol{U}$ in the linear space $\hat{R}^{(n,n)}$. Assume that the dimension of the kernel $G_{\mathcal{F}}^{r}$ is equal to k. Then all the operators $\boldsymbol{X} \in \mathcal{B}(V_{\otimes 1})$, satisfying the equation*

$$\underbrace{[\boldsymbol{U}, [\boldsymbol{U} \ldots [\boldsymbol{U}}_{k}, \boldsymbol{X}] \ldots] = 0,$$

yield a finite dimensional Lie algebra of rank k.

The way of constructing the centralized system is analogous to the rule described in Section 1 of this chapter. The right-hand side of equation (41) is expanded into the sum:

$$\hat{Q}_{\nu} = \hat{Q}_{\nu N}^{(r)} + \hat{Q}_{\nu T}^{(r)}, \quad \hat{Q}_{\nu N}^{(r)} \in \hat{N}_{\mathcal{F}}^{(r)}, \quad \hat{Q}_{\nu T}^{(r)} \in \hat{T}_{\mathcal{F}}^{(r)}. \tag{44}$$

By definition

$$\operatorname{pr} \boldsymbol{F}_{\nu} = \boldsymbol{N}_{\nu}, \tag{45}$$

where $\boldsymbol{N}_{\nu} = \hat{\hat{x}}_{m_1} \hat{Q}_{\nu N}^{(r)} \partial$ can be taken as a projection of the right-hand side of (45). The introduced definition implies that the matrix $Q_{\nu N}^{(r)}$ satisfies the identity

$$\underbrace{[\mathcal{F},\dots[\mathcal{F},}_{r}Q_{\nu N}^{(r)}]\dots] \equiv 0,$$

and the operator N_ν satisfies

$$\underbrace{[U,\dots[U,}_{r}N_\nu]\dots] \equiv 0.$$

Obtaining the component $Q_{\nu N}^{(r)}$ in (44) and the solution Γ_ν of the equation

$$G_{\mathcal{F}}\hat{\Gamma}_\nu = \hat{Q}_\nu - \hat{Q}_{\nu N}^{(r)}$$

is analogous to the case of a diagonable matrix.

A centralized system with a nondiagonable matrix $\mathcal{A}$, in essence all the properties of a centralized system with the diagonable matrix $\mathcal{A}$. This is explained by the representing $\mathcal{A}$ as the sum:

$$\mathcal{A} = \mathcal{A}_d + \mathcal{A}_n, \tag{46}$$

where $\mathcal{A}_d$, $\mathcal{A}_n$ are diagonable and nilpotent components of $\mathcal{A}$.

Formula (46) implies the representation of the operator U as the sum: $U = U_d + U_n$, where

$$U_d = \hat{\hat{x}}_{m_1}\mathcal{F}_d\partial; \quad U_n = \hat{\hat{x}}_{m_1}\mathcal{F}_n\partial; \quad \mathcal{F}_d = \mathcal{A}_d^T, \quad \mathcal{F}_n = \mathcal{A}_n^T, \quad \mathcal{F} = \mathcal{A}^T,$$

$$\mathcal{A}_d = \sum_{j=0}^{k'} c_j'\mathcal{A}^j, \quad \mathcal{A}_n = \sum_{j=0}^{k''} c_j''\mathcal{A}^j, \quad k', k'' \in Z^+.$$

The operator U_d is semi-simple component and U_n is nilpotent component of U. The following important statement is valid with respect to U_d.

Theorem 8. *The semi-simple component of* U *commutes with* $N_1, \dots, N_\nu$ *involved in the associated centralized system operator* $U_0 = U + \varepsilon N_1 + \dots + \varepsilon^\nu N_\nu$:

$$[U_d, N_\nu] \equiv 0, \quad \nu = 1, 2, \dots, \tag{47}$$

where the N_ν *are defined by the equalities* (45).

Proof. Represent the identity (47) as an equivalent matrix identity

$$[\mathcal{F}_d, \chi] \equiv 0, \quad \chi \in \hat{N}_{\mathcal{F}}^{(r)}.$$

The proof will be carried by induction for the elements of the subspaces $N_{\mathcal{F}}^{(j)}$, $j = 1, \dots, r$, defined by the sequence of equations (43). Let $\chi \in N_{\mathcal{F}}^{(1)}$ be determined as a solution of the equation $[\mathcal{F}, \chi] = 0$. Then the matrix χ is commutes with $\mathcal{F}$, i.e. $\mathcal{F}\chi = \chi\mathcal{F}$. Hence, $\mathcal{F}^2\chi = \mathcal{F}\chi\mathcal{F} = \chi\mathcal{F}^2$, and χ is also commutes with the matrix $\mathcal{F}^2$. This implies that any entire positive power $\mathcal{F}^p$ commutes with χ. The sum $\alpha_1\mathcal{F}^{p1} + \alpha_2\mathcal{F}^{p2}$, $p_1, p_2 \in Z^+$, also commutes with χ: $[\alpha_1\mathcal{F}^{p1} + \alpha_2\mathcal{F}^{p2}, \chi] \equiv \alpha_1[\mathcal{F}^p, \chi] + \alpha_2[\mathcal{F}^{p2}, \chi] \equiv 0$. Hence,

$$[\mathcal{F}_d, \chi] \equiv \left[\sum_{j=0}^{k'} c_j\mathcal{F}^j, \chi\right] \equiv 0, \quad \chi \in N_{\mathcal{F}}^{(1)}.$$

Further, we use the induction method. Assume that for the elements of the space $N_{\mathcal{F}}^{(j)}$, defined to be solutions of the equation

$$\underbrace{[\mathcal{F},\ldots[\mathcal{F},\chi]}_{j}\ldots] = 0, \quad \chi \in N_{\mathcal{F}}^{(j)}, \tag{48}$$

the theorem is fulfilled and the identity to be proved holds for $r = j$: $[\mathcal{F}_d, \chi] \equiv 0$, $\chi \in$ $\in N_{\mathcal{F}}^{(j)}$. For elements of the space $N_{\mathcal{F}}^{(j+1)}$, the defining equation has the form

$$\underbrace{[\mathcal{F},\ldots[\mathcal{F},\chi]}_{j+1}\ldots] = 0, \quad \chi \in N_{\mathcal{F}}^{(j+1)}, \tag{49}$$

If the condition (49) is fulfilled for χ, then condition (48) is fulfilled for $\mathcal{Y} = [\mathcal{F}, \chi]$. By assumption $[\mathcal{F}_d, \mathcal{Y}] \equiv [\mathcal{F}_d, [\mathcal{F}, \chi]] \equiv 0$. Let us prove, that the identity

$$[\mathcal{F}_d, [\mathcal{F}, \chi]] \equiv 0, \quad p \in Z^+, \tag{50}$$

holds for $\mathcal{F}^p$, where p is an arbitrary positive number.

For the Poisson bracket $[\mathcal{F}^p, \chi]$, we use the formula

$$[\mathcal{F}^p, \chi] = \sum_{j=0}^{p-1} \mathcal{F}^{p-j-1}[\mathcal{F}, \mathcal{Y}]\mathcal{F}^j. \tag{51}$$

The validity of this formula is checked by substituting $[\mathcal{F}^p, \chi] = \mathcal{F}^p\chi - \chi\mathcal{F}^p$, $[\mathcal{F}, \chi] = \mathcal{F}\chi - \chi\mathcal{F}$. Multiply both sides of (51) by $\mathcal{F}^p$. By assumption, $\mathcal{F}_d[\mathcal{F}, \chi] \equiv [\mathcal{F}, \chi]\mathcal{F}_d$. Hence

$$\mathcal{F}_d[\mathcal{F}^p, \chi] \equiv \sum_{j=0}^{p-1} \mathcal{F}_d\mathcal{F}^{p-j-1}[\mathcal{F}_d, \chi]\mathcal{F}^j \equiv \sum_{j=0}^{p-1} \mathcal{F}^{p-j-1}[\mathcal{F}, \chi]\mathcal{F}^j\mathcal{F}_d = [\mathcal{F}^p, \chi]\mathcal{F}_d.$$

Hence, the desired identity (50) is proved. Let us write it for $p = 1$

$$[\mathcal{F}_d, [\mathcal{F}, \chi]] \equiv 0. \tag{52}$$

Denote the matrix of the equation $[\mathcal{F}_d, \chi] = 0$ in the linear space $\hat{R}^{(n,n)}$ by $G_{\mathcal{F}_d}$. We can write identity (52) in equivalent form $G^2_{\mathcal{F}_d}\hat{\chi} \equiv 0$. By virtue of the diagonability of the matrix $G_{\mathcal{F}_d}$, equations $G_{\mathcal{F}_d}\hat{\chi} = 0$ and $G^2_{\mathcal{F}_d}\hat{\chi} = 0$ are equivalent to $[\mathcal{F}_d, \chi] \equiv 0$, $\chi \in N_{\mathcal{F}}^{(j+1)}$.

2.2. Basic theorems. The operator $\boldsymbol{U}$ of the centralized system can be represented in the following way:

$$\boldsymbol{U}_0 = \boldsymbol{U}_d + \boldsymbol{U}_n + \varepsilon \boldsymbol{N}_1 + \ldots + \varepsilon^{\nu}\boldsymbol{N}_{\nu} + \ldots$$

The centralized system can be represented in the form

$$\frac{dx}{dt} = (\mathcal{F}_d + \mathcal{F}_n + \varepsilon\mathcal{M}_{\nu 1} + \varepsilon^2\mathcal{M}_{\nu 2} + \ldots)x, \tag{53}$$

where $\mathcal{M}_{\nu N} = (Q_{\nu N}^{(r)})^T$, T denotes the transposition.

Talking into account the fact about the commutatively of the operators $\boldsymbol{U}_d$ and $\boldsymbol{U}_n$ with $\boldsymbol{U}_\nu$, $\nu = 1, 2, \ldots$, we can relate $\boldsymbol{U}_n$ to the algebra of the centralizer. Here, Theorem 1 concerning the structure of the centralized system, can be automatically applied.

Now formulate an analogy of Theorem 1 for the considered case of a matrix $\mathcal{F}$ of general structure based on Theorem 8.

Theorem 9. *Let the matrix* $\mathcal{A}$ *be reduced to the normal Jordan form by the invertible matrix* $\mathcal{L}$

$$\mathcal{L}^{-1}\mathcal{A}\mathcal{L} = \operatorname{diag}\{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_m\},$$

where $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_m$ *are Jordan blocks of dimensions* $r_1 \times r_1, r_2 \times r_2, \ldots, r_m \times r_m$, *respectively. Then, upon the change of variables* $x = \mathcal{L}z$, *the centralized system is transformed to the block diagonal form*

$$\frac{dz}{dt} = \left\{ \begin{Vmatrix} \mathcal{E}_1 & 0 & \ldots & 0 \\ 0 & \mathcal{E}_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \mathcal{E}_m \end{Vmatrix} + \sum_{\nu=1}^{\infty} \varepsilon^{\nu} \begin{Vmatrix} Q_{\nu 1} & 0 & \ldots & 0 \\ 0 & Q_{\nu 2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & Q_{\nu m} \end{Vmatrix} \right\} z,$$

where $Q_{\nu 1}, Q_{\nu 2}, \ldots, Q_{\nu m}$ *are square blocks of dimensions* $r_1 \times r_1, r_2 \times r_2, \ldots, r_m \times r_m$.

The fact that $\mathcal{A}$ is nondiagonable matrix is considered in the following statement.

Theorem 10. *The solution of the centralized* (53) *can be represented as the product:*

$$x(t) = \sum_{j=0}^{\mu} \frac{t^j \mathcal{A}_n^j}{j!} e^{\mathcal{A}_d t} \eta(\tau), \tag{54}$$

where μ *is the smallest integer such that* $\mathcal{A}_n^{\mu} \equiv 0$ *and the vector* $\eta(\tau) = \operatorname{col}\|\eta_1(\tau), \ldots, \eta_n(\tau)\|$ *is the solution of the system of equations*

$$\frac{d\eta}{dt} = (\mathcal{M}_{\nu 1} + \varepsilon \mathcal{M}_{\nu 2} + \ldots)\eta, \quad \eta(0) = x(0), \quad \tau = \varepsilon t. \tag{55}$$

Proof. Let us make the substitution $x = \exp(\mathcal{A}t)\eta$ in (53). Then

$$\frac{d\eta}{d\tau} = \exp(-\mathcal{A}t)(\varepsilon \mathcal{M}_{\nu 1} + \varepsilon^2 \mathcal{M}_{\nu 2} + \ldots)\exp(\mathcal{A}t)\eta.$$

Because of the commutatively of $\mathcal{A}$ and $\mathcal{M}$, we obtain (55). On account of $[\mathcal{A}_d, \mathcal{A}_n] \equiv 0$, the relation $\exp((\mathcal{A}_d + \mathcal{A}_n)t) = \exp \mathcal{A}_d t.\ \exp \mathcal{A}_n t$ takes place, and as a corollary of this, formula (54) is true. Therefore, if the matrix is not diagonable, the solution contains terms which are proportional to powers of an independent variable.

3. Method of local asymptotic decomposition. Consider the special case of the choice of the operator P which is a projection on the algebra $\mathcal{B}$ of a zero approximation system, supposing that the system is decomposed (see also [4]). In this case, in the operators $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_m$ associated with the zero approximation system, the variables are separated. For example, the case where a full decomposition vector of variables $x = \|x_1, \ldots, x_n\|$ can be split into subgroups $x_{\nu_j} = \|x_{i\nu_j}, \ldots, x_{\nu_j \nu_j}\|$, $j = \overline{1, g}$, $\nu_1 + \ldots + \nu_g = n$ so that the operator $\boldsymbol{U}$ can be represented as a sum

$$\boldsymbol{U} = \boldsymbol{U}_{\nu_1} + \ldots + \boldsymbol{U}_{\nu_g},$$

where

$$\boldsymbol{U}_{\nu_j} = \omega_{1\nu_j}(x_{\nu_j})\frac{\partial}{\partial x_{1\nu_j}} + \ldots + \omega_{\nu_j \nu_j}(x_{\nu_j})\frac{\partial}{\partial x_{\nu_j \nu_j}}, \quad i = \overline{1, g}. \tag{56}$$

Then the operation of the projection of an arbitrary operator $\boldsymbol{F}$ onto the algebra $\mathcal{B}$

consists of leaving, in the expansions of the coefficients, only those variables from the group where we take the derivatives with respect to (56). As a result, in the decomposed system, the variables are separated as in (56).

By implication, the integration of the original perturbed system is replaced by integration of a sequence of subsystems with separated variables. Since the theorem of solvability of operator equation

$$[U_1, S] = F_1 - PF_1, \dots, [U_r, S] = F_r - PF_1$$

has a local character, the considered method is called the local decomposition method. Recall that the question of proof of the algorithm is open and, therefore, all the calculations are formal.

Consider some examples in which the local asymptotic decomposition method is used.

3.1. Dynamics of flying apparatus. The problem of local asymptotic decomposition of perturbed movement of flying apparatus is considered (see Fig. 2).

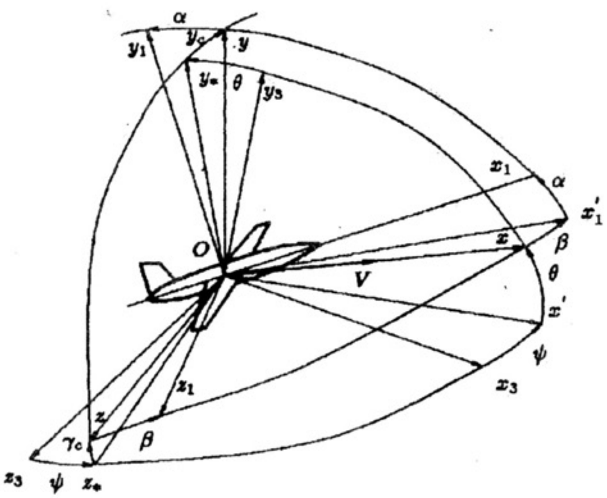

Fig. 2. System of coordinates for describing the dynamic motions of an airplane.

Suppose that an airplane-type flying apparatus has the vertical symmetry plane x_1Oy_1. Denote the stationary coordinate system $Ax_3y_3z_3$. The coordinate system connected with the flying apparatus is denoted by $Ox_1y_1z_1$ and the semi-velocity system is denoted by $O\,x^*y^*z^*$. The equations of movement of relative semivelocity coordinate system have 13 variables.

Longitudinal variables: V — speed of center of gravity; θ — angle of inclination of trajectory with respect to horizontal; ω_z — projection of angle speed vector on axis Qz; ν — pitch angle; x — coordinate of center of gravity along axes Ox_1; H — height of flight; m — mass of apparatus.

Profile variables: Ψ — angle of trajectory rotation; ω_x — projection of angle of speed vector on axis Ox_1; ω_y — projection of angle of speed vector on axis Oy_1; Ψ — angle of yaw; γ — angle of bank of flying apparatus; z — coordinate of center of gravity along axes Oz_1.

Longitudinal movement of the flying apparatus consists of the translational movement of the center of gravity along the axes. Ox_1 and Oy_1 (i. e. in symmetry plane Ox_1y_1) and rotational movement with respect to axes Oz_1. Lateral movement is added from the translational movement of the center of gravity of the flying apparatus along axes Oz_1 and rotary movement with respect to axes Ox_1 and Oy_1. The common movement of the flying apparatus consists of the two mentioned movements.

Granting that the above denotes equations of flying apparatus movement, we take the following form ([5]).

Equations for longitudinal movement are

$$\begin{gathered}
\frac{dV}{dt} = F_V(V, \theta, \upsilon, H, m, \Psi, \psi, \gamma); \\
\frac{d\theta}{dt} = F_\theta(V, \theta, \upsilon, H, m, \Psi, \psi, \gamma); \\
\frac{d\omega_z}{dt} = F_{\omega_z}(V, \theta, \upsilon, \omega_z, H, m, \Psi, \omega_x, \omega_y, \psi, \gamma); \\
\frac{d\upsilon}{dt} = F_\upsilon(\omega_z, \omega_y, \gamma), \quad \frac{dx}{dt} = F_x(V, \theta, \Psi); \\
\frac{dH}{dt} = F_H(V, \theta), \quad \frac{dm}{dt} = F_m(V, H, \gamma).
\end{gathered} \tag{57}$$

Equations for profile movement are

$$\begin{gathered}
\frac{d\Psi}{dt} = F_\Psi(V, \theta, H, \upsilon, m, \Psi, \psi, \gamma); \\
\frac{d\omega_x}{dt} = F_{\omega_x}(V, \theta, \omega_z, \upsilon, H, \Psi, \omega_x, \omega_y, \psi, \gamma); \\
\frac{d\omega_y}{dt} = F_{\omega_y}(V, \theta, \omega_z, \upsilon, H, \Psi, \omega_x, \omega_y, \psi, \gamma); \\
\frac{d\psi}{dt} = F_\psi(\omega_z, \upsilon, \omega_y, \gamma), \quad \frac{d\gamma}{dt} = F_\gamma(\omega_z, \upsilon, \omega_x, \omega_y, \gamma); \\
\frac{dz}{dt} = F_z(V, \theta, \Psi).
\end{gathered} \tag{58}$$

Explicit expressions for the right parts of equations (57) and (58) will not be stated here as they are contained in the work cited above. Later on, we use the vector form $d\eta/dt = F(t, \eta)$ for (57) and (58), where $\eta = (\eta_1, \eta_2, \ldots, \eta_{13})$; $F = (F_1, F_2, \ldots \ldots, F_{13})$ are variables and functions of the right parts of the movement equations in the same order as before.

Let $\eta = \eta_*$ be a certain programmed movement of flying apparatus and $d\eta_*/dt \equiv \equiv F(t, \eta_*)$. Consider movement in the neighborhood of this programmed movement $\eta = \eta_* + \varepsilon\Delta\eta$, where a small parameter $\varepsilon > 0$ characterized the smallness of the perturbed movement.

If, in the main programmed movement, there is no sliding $(\beta = 0)$, then, granting the symmetry of the flying apparatus, the system of perturbed movement equations can be represented in the reduced form:

$$\begin{gathered}
\frac{d\Delta\eta_i}{dt} = \left(\sum\nolimits_n \Delta\eta \frac{\partial}{\partial\eta}\right) F_i + \frac{\varepsilon}{2}\left(\sum \Delta\eta \frac{\partial}{\partial\eta}\right)^2 F_i + \varepsilon^2 \ldots; \\
\frac{d\Delta\eta_j}{dt} = \left(\sum\nolimits_\delta \Delta\eta \frac{\partial}{\partial\eta}\right) F_j + \frac{\varepsilon}{2}\left(\sum \Delta\eta \frac{\partial}{\partial\eta}\right)^2 F_j + \varepsilon^2 \ldots, \\
i = \overline{1, 7}, \quad j = \overline{8, 13}.
\end{gathered} \tag{59}$$

The following expression

$$\left(\sum \Delta\eta \frac{\partial}{\partial\eta}\right) F_i = \sum_{j=1}^{13} \Delta\eta_i \frac{\partial F_i(t, \eta_*)}{\partial\eta_j},$$

is used for simplification; indices n and δ denote similar sums but consist of only longitudinal or profile movement parameters. At $\varepsilon = 0$, the system of perturbed movement equations (59) is split into independent systems.

Using the described method, find the expression for $\boldsymbol{S}_1$ and the transformation (up to values of order ε, inclusively) has the following form

$$\eta_i = \left(1 - \sum_{k=1}^{13} \gamma_k \frac{\partial}{\partial\eta_k}\right), \quad i = \overline{1, 13}.$$

Further, the original system of (59) is split into two independent systems (up to values of order ε inclusively)

$$\frac{d\Delta\overline{\eta}_i}{dt} = \left(\sum\nolimits_n \Delta\overline{\eta}\frac{\partial}{\partial\overline{\eta}}\right) F_i + \varepsilon\left(\sum\nolimits_n \Delta\overline{\eta}\frac{\partial}{\partial\overline{\eta}}\right)^2 F_i, \quad i = \overline{1, 7},$$

$$\frac{d\Delta\overline{\eta}_j}{dt} = \left(\sum\nolimits_\delta \Delta\overline{\eta}\frac{\partial}{\partial\overline{\eta}}\right) F_j + \varepsilon\left(\sum\nolimits_\delta \Delta\overline{\eta}\frac{\partial}{\partial\overline{\eta}}\right)^2 F_j, \quad j = \overline{8, 13}.$$

3. 2. A model of gas dynamics. Consider one more example which is a hyperbolic system of nonlinear partial differential equations. The most studied among them are systems of quasi-linear equations with two independent variables. As is known, such systems describe nonstationary one-dimensional and supersonic two-dimensional stationary flow of compressible gases and liquides.

Consider equations describing the isoentropic flow of polytropic gas

$$\frac{\partial s}{\partial t} + (\alpha s + \beta r)\frac{\partial s}{\partial x} = \frac{\nu(\gamma - 1)(r^2 - s^2)}{4x};$$
$$\frac{\partial r}{\partial t} + (\alpha s + \beta r)\frac{\partial r}{\partial x} = -\frac{\nu(\gamma - 1)(r^2 - s^2)}{4x}. \tag{60}$$

Here, the following is assumed:

$$\alpha = \frac{1}{2} + \frac{\gamma - 1}{4} > \frac{1}{2} > 0, \quad \beta = \frac{1}{2} - \frac{\gamma - 1}{4};$$

the variables s, r are Riman invariants $s = u - \varphi(\rho)$, $r = u + \varphi(\rho)$, expressed in terms the movement parameters; u is flow speed; r is density; γ is an exponent characterizing the pressure of polytropic gas; and the parameter ν characterizes symmetry of the gas flow. Suppose $\nu = 0$ in equations (60) and receive plane-symmetrical movement equations

$$\frac{\partial s}{\partial t} + (\alpha s + \beta r)\frac{\partial s}{\partial x} = 0; \quad \frac{\partial r}{\partial t} + (\alpha s + \beta r)\frac{\partial r}{\partial x} = 0. \tag{61}$$

At $\gamma = 3$, equations (61) are separated into two independent quasi-linear equations

$$\frac{\partial s}{\partial t} + s\frac{\partial s}{\partial x} = 0; \quad \frac{\partial r}{\partial t} + r\frac{\partial r}{\partial x} = 0. \tag{62}$$

Consider movement equations similar to equations (62) which are plane-symmetrical and split. To this end, suppose that

$$\nu = \varepsilon, \quad \gamma = 3 + \varepsilon\Delta\gamma, \quad 0 < \varepsilon < 1. \tag{63}$$

If we substitute ν and γ, according to formulae (63), into equations (60), we obtain

$$\frac{\partial s}{\partial t} + s\frac{\partial s}{\partial x} = \varepsilon F_1(x, r, s); \quad \frac{\partial r}{\partial t} + r\frac{\partial r}{\partial x} = \varepsilon F_2(x, r, s), \tag{64}$$

where

$$F_1(x, r, s) = \frac{r^2 - s^2}{2x} - \frac{\Delta\gamma}{4}(s - r)\frac{\partial s}{\partial x} + \varepsilon\Delta\gamma\frac{r^2 - s^2}{4x};$$

$$F_2(x, r, s) = -\frac{r^2 - s^2}{2x} - \frac{\Delta\gamma}{4}(r - s)\frac{\partial r}{\partial x} - \varepsilon\Delta\gamma\frac{r^2 - s^2}{4x};$$

By $\varepsilon = 0$, (64) can be split into two independent quasi-linear equations of (62) type. At $\varepsilon \neq 0$, after a sequence of calculations, we obtain the following separated system (up to values of order ε inclusively) for new variables z_1, z_2, connected by the formulae of obtained change of variables:

$$\frac{\partial z_1}{\partial x_1} + z_1\frac{\partial z_1}{\partial x_2} = \varepsilon\left(\frac{z_1^2}{2x_2} + \frac{\Delta\gamma}{4}z_1\frac{\partial z_1}{\partial x_2}\right);$$

$$\frac{\partial z_2}{\partial x_1} + z_2\frac{\partial z_2}{\partial x_2} = -\varepsilon\left(\frac{z_2^2}{2x_2} + \frac{\Delta\gamma}{4}z_2\frac{\partial z_2}{\partial x_2}\right).$$

The above-mentioned examples illustrate the possibilities of a local asymptotic decomposition method for different classes of height-dimensional systems of differential equations. These problems are actual enough and can often used in various applications.

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