HEDGING OF OPTIONS
UNDER MEAN-SQUARE CRITERION
AND SEMI-MARKOV VOLATILITY

We consider a problem of hedging of the European call option for a model such that appreciation rate and volatility are functions of a semi-Markov process. In such a model, the market is incomplete.

1. Introduction. In famous Black–Scholes model, which is used for evaluation of option prices, it is supposed that the dynamic of stocks prices is set by the linear stochastic differential equation

\[ dS_t = aS_t \, dt + \sigma S_t \, dW_t, \]

where \( a \) and \( \sigma \) are deterministic functions (in the simplest case – constants).

We suppose that, in our model, the coefficients \( a \) and \( \sigma \), appreciation rate and volatility respectively, are dependent on a semi-Markov process \( Y_t \), which doesn’t depend on standard Wiener process \( W_t \). We consider the hedging problem of the European call option with terminal payment \( H = f(S_T) \).

Since the additional source of randomness exists (the semi-Markov process \( Y_t \)) in addition to the Wiener process \( W_t \), the market is incomplete and perfect hedging is not possible. We find a strategy, which locally minimizes the risk.

2. Description of the model and preliminary notions. Let \((\Omega, \mathcal{F}, \mathbb{P} = (\mathcal{F}_t), \mathbb{P})\) be a probability space. We suppose that risk assets (stocks) exist and their price evolution is given by the following stochastic differential equation [5]:

\[ dS_t = a(X_t)S_t \, dt + \sigma(X_t)S_t \, dW_t, \tag{1} \]

where \( W_t \) is a Wiener process, \( X_t \) is some observed variable, which is described by a semi-Markov process [4, 5] with the phase space \((X, \mathcal{X})\), \( X := X_{\nu_{\geq n}}, \nu(t) := \max\{n: \tau_n \leq t\}, \tau_n := \sum_{k=1}^{n} \theta_{k}, \theta_n : n \geq 0 \) is a Markov renewal process. \( \mathbb{P}\{\omega: x_n(t) = x\} = P(x, A) \cdot G_x(t), x \in X, A \subset X, t \geq 0 \). We suppose that \( G_x(t) \) is a differentiable function of \( t \) and \( g_x(t) := dG_x(t)/dt, \forall x \in X \). Coefficients \( a \) and \( \sigma \) are measurable functions on \( X \), \( \sigma > 0 \), processes \( W_t \) and \( X_t \) are independent, and filtration \( \mathcal{F} \) is generated by \( X_t \) and \( W_t \).

We solve the problem of hedging of the European call option which is sold at the moment \( t = 0 \), with terminal payment \( H = f(S_T) \) at a cancellation moment \( T \). It’s considered that \( EH^{2+\varepsilon} < +\infty, \varepsilon > 0 \).

Besides the risk assets, we have nonrisk assets (bond or bank account), and we suppose that its price is a constant and is equal to one (without of loss generality) at all moments (i.e., percentage rate is equal to zero).

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Stock exchange strategy (SES) \( \pi \) is a pair \((\gamma, \beta)\), where \( \gamma = (\gamma_t) \) is such a predictable process that
\[
E \int_0^T \gamma_t \sigma^2(X_t) S_t^2 dt + E \left( \int_0^T \gamma_t |a(X_t)| S_t dt \right)^2 < +\infty.
\] (2)

\( \beta = (\beta_t) \) is a coordinated process, \( E \beta_t^2 < +\infty, \forall t \leq T \).

SES defines a portfolio with a number of units of risk assets \( \gamma_t \) (which the holder has at the moment \( t \)) and with a number of means that were invested in bonds at the moment \( t \).

The value process \( V(\pi) \) of the portfolio with respect to the strategy \( \pi \) is defined as
\[
V_t(\pi) = \gamma_t S_t + \beta_t.
\] (3)

and the cost process
\[
C_t(\pi) = V_t(\pi) - \int_0^t \gamma_s dS_s.
\] (4)

SES \( \pi \) is said to be \( H \)-admissible if \( V_t(\pi) \equiv H \). SES \( \pi \) is said to be self-financing (or mean-value self-financing) if the cost process \( C_t(\pi) \) is a constant time (or martingale).

The residual risk is defined by the formula
\[
R_t(\pi) := E \left\{ |C_t(\pi) - C_0(\pi)|^2 / T_t \right\}.
\] (5)

The SES \( H \)-admissible strategy \( \pi^* \) is called a risk-minimizing if, for any \( H \)-admissible SES \( \pi \) and for any \( t \),
\[
R_t(\pi^*) \leq R_t(\pi).
\]

The purpose of this article is to find a risk-minimizing \( H \)-admissible SES.

It was stated in [2] that existence of a risk-minimizing \( H \)-admissible SES is equivalent to the existence of the expansion of the terminal payment \( H \) in the form
\[
H = H_0 + \int_0^T \gamma_{Ht}^t dS_t + L_{Ht}^t \quad (P - a.s.),
\] (6)

where \( H_0 \in L^2(\mathcal{F}_0, P) \), \( \gamma_{Ht}^t \) satisfies (2), and \( L_{Ht}^t \) is a square-integrable martingale, which is orthogonal to the martingale component \( S \). Then the \( \gamma \)-component of a risk-minimizing strategy \( \pi \) is \( \gamma = \gamma_{Ht}^t \) and \( C_\pi = H_0 + L_{Ht}^t \).

To obtain expansion (6), it is introduce a minimal martingale measure \( \tilde{\mathcal{P}} \) [2].

For our model (1), the minimal martingale measure is \( \tilde{\mathcal{P}} = \rho_T \mathcal{P} \), where the density \( \rho_T \) is defined by the equality
\[
\rho_T = \exp \left\{ -\int_0^T \frac{a(X_t)}{\sigma(X_t)} dW_t - \frac{1}{2} \int_0^T \frac{a^2(X_t)}{\sigma^2(X_t)} dt \right\}.
\] (7)

In our situation, when the process \( S_t \) has continuous paths, the process \( a(X_t)/\sigma(X_t) \) is bounded and \( H \in L^2(\mathcal{P}), \epsilon > 0 \), and the desired expansion (6) can
be obtained from Kunita–Watanabe’s expansion (as \( t = T \)) with respect to the measure \( \tilde{P} \):

\[
\tilde{E}(H/F_t) = \tilde{E}H + \int_0^T \tilde{\gamma}_t^H dS_t + \tilde{I}_t^H.
\]

(8)

In such a way, it is needed to find the Kunita–Watanabe’s expansion. Let’s introduce some notations.

A jump measure for \( X_t \) has the following form:

\[
\mu([0,t] \times A) = \sum_{n \geq 0} I(X_n \in A, \tau_n \leq t), \quad A \in \mathcal{F}, \quad t \geq 0.
\]

(9)

It’s known [4], that F-dual predictable projection for \( \mu \) has the form:

\[
v(dt, dy) = \sum_{n \geq 0} I(\tau_n < t \leq \tau_{n+1}) \frac{P(X_n, dy)}{G_x(t)} dt.
\]

(10)

where \( G_x(t) := 1 - G_x(t), \) \( g_x(t) := dG_x(t)/dt, \) \( \forall x \in X, \) \( t \geq 0. \)

For given \( H \in L_2(\tilde{P}), \) we find the expansion

\[
\tilde{E}(H/F_t) = \tilde{E}H + \int_0^T \tilde{\gamma}_r^H dS_r + \int_0^T \int_X \tilde{\psi}(r, y)(\mu - v)(ds, dy).
\]

(11)

We note that the last integral in (11) is \( \tilde{P} \)-orthogonal to \( S \) (it’s an \( (F, \tilde{P}) \)-martingale), and the uniqueness of the Kunita–Watanabe’s expansion is a guarantee that (11) is the desired expansion of (8) for our case.

Finally, a risk-minimizing \( H \)-admissible strategy \( \pi^* \) is defined by \( \pi^* = (\gamma^*, \beta^*), \) where \( \gamma^* = \tilde{\gamma}_t^H, \) and \( \beta^* \) is such that \( V_t(\pi^*) = \tilde{E}(H/F_t), \) i.e.,

\[
\beta^*_t = \tilde{E}(H/F_t) - \tilde{\gamma}_t^H S_t.
\]

(12)

In the next section we will obtain an exact representation for \( \pi^* = (\gamma^*, \beta^*). \)

3. The result. Let \( f(z) \) be a function such that \( |f(z)| \leq c \cdot (1 + |z|)^m, \) for some \( m \geq 0. \) Let’s consider a function \( u(t, z, x) \) on \( [0, T] \times \mathbb{R}_+ \times X \) such that it is a solution of the Cauchy problem:

\[
\begin{aligned}
&u_t(t, z, x) + \frac{1}{2}\sigma^2(x) \cdot z^2 \cdot u_{zz}(t, z, x) + Au(t, z, x) = 0, \\
&u(T, z, x) = f(z),
\end{aligned}
\]

(13)

where

\[
Au(t, z, x) := \frac{g_x(t)}{G_x(t)} \int_X P(x, dy)[u(t, z, y) - u(t, z, x)].
\]

(14)

Theorem 1. The risk-minimizing \( H \)-admissible stock exchange strategy \( \pi^* = (\gamma^*, \beta^*), \) is given by the following formula:

\[
\begin{aligned}
\gamma^*_t &= u_z(t, S_t, X_t), \\
\beta^*_t &= V_t(\pi^*) - \gamma^*_t \cdot S_t.
\end{aligned}
\]

(15)

where
\[ V_f(\pi^*) = \mathbb{E} f(S_T) + \int_0^T \int_0^X u_z(r, S_r, X_r) dS_r + \int_0^T \int_0^X \psi(r, y)(\mu - \nu)(dr, dy). \quad (16) \]

\[ \psi(r, y) = u(r, S_r, y) - u(r, S_r, X_r). \]

The residual risk process has the form

\[ \mathcal{R}_d(\pi^*) = \mathbb{E} \left( \int_0^T \left[ A u^2(r, S_r, X_r) - 2u(r, S_r, X_r) A u(r, S_r, X_r) \right] ds / \mathcal{F}_t \right). \]

In particular, the residual risk at the moment \( t = 0 \) is equal to

\[ \mathcal{R}_d(\pi^*) = \mathbb{E} \left( \int_0^T \left[ A u^2(r, S_r, X_r) - 2u(r, S_r, X_r) A u(r, S_r, X_r) \right] ds \right). \quad (17) \]

where the operator \( A \) was defined in (14).

4. Proof. By applying Itô's formula to the solution of (13) we obtain:

\[ f(S_T) = u(T, S_T, X_T) = u(0, z, y) + \int_0^T u_z(r, S_r, X_r) dS_r + \]

\[ + \int_0^T \left[ u_t(r, S_r, X_r) + \frac{1}{2} \sigma^2(X_r)(S_r)^2 u_{zz}(r, S_r, X_r) \right] dr + \]

\[ + \sum_{r \leq T} \left[ u(r, S_T, X_T) - u(r, X_T, X_T) \right]. \quad (18) \]

We note that, for any function \( h \) on \([0, T] \times X\), right-continuous and left-limit of \( t \), we have

\[ \sum_{r \leq T} \left[ h(r, X_r) - h(r, X_T, X_T) \right] = \int_0^T \int_0^X \left[ h(r, y) - h(r, X_T, X_T) \right] \mu(dr, dy) = \]

\[ = \int_0^T \int_0^X \left[ h(r, y) - h(r, X_T, X_T) \right] (\mu - \nu)(dr, dy) + \]

\[ + \int_0^T \int_0^X \left[ h(r, y) - h(r, X_T, X_T) \right] \nu(dr, dy) = \]

\[ = \int_0^T \int_0^X \left[ h(r, y) - h(r, X_T, X_T) \right] (\mu - \nu)(dr, dy) + \]

\[ + \int_0^T \int_0^X \frac{h(r, X_T)}{\Gamma_X(r)} P(X_T, dy) \left[ h(r, y) - h(r, X_T, X_T) \right] dr = \]

\[ = \int_0^T \int_0^X \left[ h(r, y) - h(r, X_T, X_T) \right] (\mu - \nu)(dr, dy) + \int_0^T A h(r, X_T) dr \quad (19) \]

(see, (10) and (14).

Hence, from (13), (18), and (19) we obtain
\[ f(S_t) = u(T, S_T, X_T) = u(0, z, v) + \int_0^T u_x(r, S_r, X_r) dS_r + \]
\[ + \int_0^T \int \big[ u(r, S_r, X_r) - u(r, S_r, X_r^-) \big] (\mu - v) (dr, dy). \]

and relations (15), (16) are valid.

Residual risk process can be expressed in the following way (see (11)):

\[ \mathcal{R}_t(\pi^*) = \mathbb{E} \left( \left[ \int_0^T \int_X \psi(r, y)(\mu - v)(dr, dy) \right]^2 / \mathcal{F}_T \right) = \]
\[ = \mathbb{E} \left( \int_0^T \int_X \left[ u(r, S_r, y) - u(r, S_r, X_r^-) \right] (dr, dy) / \mathcal{F}_T \right) = \]
\[ = \mathbb{E} \left( \int_0^T \int_X \frac{\pi_{X_r} (r)}{\pi_{X_r}} P(X_{r^-}, dy) \left[ u(r, S_r, y) - u(r, S_r, X_r^-) \right]^2 / \mathcal{F}_T \right) = \]
\[ = \mathbb{E} \left( \int_0^T \left[ (\lambda u^2(r, S_r, X_r) - 2u(r, S_r, X_r^+) u(r, S_r, X_r^-) \right] dr / \mathcal{F}_T \right). \]

and the theorem is proved.

We must only prove that the solution of Cauchy problem (15) exists. We do it in the next section.

5. Random evolution approach. Let \( \tilde{S}_t \) be a solution of the stochastic differential equation

\[ d\tilde{S}_t = \sigma(X_t) \tilde{S}_t dW_t, \quad \tilde{S}_0 = z. \]

(21)

This solution has the form

\[ \tilde{S}_t = z \exp \left\{ \int_0^t \sigma(X_r) dW_r - \frac{1}{2} \int_0^t \sigma^2(X_r) dr \right\}. \]

(22)

We note that \( \tilde{S}_t \) is a continuous semi-Markov random evolution [5: 4, p. 77], \( V^*(t) \):

\[ V^*(t)f(z) := \mathbb{E} [ f(\tilde{S}_t) / X(s), 0 \leq s \leq t]. \]

(23)

i.e., random evolution underlying the semi-Markov process \( X_t \). This evolution is generated by the following generating operators:

\[ \Gamma(x)f(z) := \frac{1}{2} \sigma^2(x) z^2 f''(z) \quad \forall f(z) \in C^2(\mathbb{R}). \]

(24)

Further, let's consider the following process \( (X_t, t - \tau_{t^*}) \). It is a Markov process on \( X \times \mathbb{R}_+ \) with the infinitesimal operator \( \tilde{A} = A + d/dt \), where \( A \) is defined in (14).

The expectation for the random evolution \( V^*(t) \) of Markov process \( (X_t, t - \tau_{t^*}) \) satisfies the following equation:

\[ \psi(t, z, x) := \mathbb{E} [ V^*(t)f(z, X_t, t - \tau_{t^*})]. \]
where $\Gamma(x)$ is defined in (24).

Let $(\tilde{S}_t^x, X_t^x, t - \tau_{\text{exit}})$ be a Markov process with the initial point $(z, x, 0)$ with the first component $\tilde{S}_t^x$ in (21). $X_t^x$ be a semi-Markov process. From (21)-(25) we obtain the following result:

**Lemma 1.** Function

$$u(t, z, x) := E f(\tilde{S}_{T-t}^x)$$

is a solution of problem (13).

**Proof.** From (22) we have

$$u(T-t, z, x) = E f(\tilde{S}_{T-t}^x) = \int f(y) y^{-1} h(y; t, z, x) dy$$

where

$$h(y; t, z, x) = \int \varphi \left( \frac{\xi}{z} + \frac{1}{2} \frac{\xi}{z}, \ln \frac{y}{z} + \frac{1}{2} \frac{\xi}{z} \right) F_t^x(d\xi) = E \varphi \left( z^x, \ln \frac{y}{z} + \frac{1}{2} z^x \right)$$

$$\varphi(t, x) = (2\pi t)^{-1/2} \exp \left\{-x^2 / (2t) \right\}. \quad F_t^x$$ is a distribution of the random variable

$$Z_t^x = \int_0^t \sigma^2(X_r^x) dr$$

Let $\nu$ be a solution of the equation

$$\nu_t(t, \eta, x) = \sigma^2(x) \nu_x(t, \eta, x) + \tilde{A} \nu(t, \eta, x)$$

with the initial condition $\nu(0, \eta, x) = g(\eta)$. From Ito's formula we have that

$$\nu(t, \eta, x) = E g(\eta + Z_t^x)$$

By substituting this formula $g(\eta) = \varphi(\eta - \ln (y/z), (\ln (y/z) + \frac{\eta}{2})$ we obtain that

$$h(y; t, z, x) = \nu \left( \ln \frac{y}{z}, t, x \right)$$

From (30) and (32) we have:

$$h_t(y; t, z, x) = \sigma^2(x) E \left[ \frac{\varphi}{z} \left( \frac{y}{z}, \ln \frac{y}{z} + \frac{1}{2} Z_t^x \right) \right] +$$

$$+ \frac{1}{2} \varphi \left( \frac{y}{z}, \ln \frac{y}{z} + \frac{1}{2} Z_t^x \right) \right] + \tilde{A} h(y; t, z, x)$$

Differentiation of (28) gives the equality

$$h_{zz} = z^{-2} E [\varphi_{zz} + \phi_z]$$

Since $\varphi$ satisfies the heat equation $\varphi_t = 1/2 \varphi_{zz}$

$$h_t = \frac{1}{2} \sigma^2(x) z^2 h_{zz} + \tilde{A} h.$$
Hence, the function \( u \) in (26) is a solution of (13).

**Remark 1.** Let's define the following process:

\[
m_t = f(\bar{S}, X_t, r - \tau_{\nu(t)}) - f(\bar{S}, X_t, r) - \int_0^t \left( A + \frac{d}{dr} \right) f(\bar{S}, X_r, r - \tau_{\nu(r)}) \, dr.
\]  

(33)

It's an \( \mathcal{F}_t \) martingale, where \( \mathcal{F}_t = \sigma \{ X_r, W_r; 0 \leq s \leq t \} \). Its quadratic variation is equal to

\[
\langle m_t \rangle = \int_0^t \left[ \left( A + \frac{d}{dr} \right) f^2(\bar{S}, X_r, r - \tau_{\nu(r)}) \right. \\
- 2 f(\bar{S}, X_r, r - \tau_{\nu(r)}) \left( A + \frac{d}{dr} \right) f(\bar{S}, X_r, r - \tau_{\nu(r)}) \left. \right] \, dr = \\
= \int_0^t \left[ A f^2(\bar{S}, X_r, r - \tau_{\nu(r)}) - 2 f(\bar{S}, X_r, r - \tau_{\nu(r)}) A f(\bar{S}, X_r, r - \tau_{\nu(r)}) \right] \, dr + \beta(t).
\]

In such a way, from (17) and (34), it follows that \( \mathcal{R}_n(\pi^*) = \langle m_T \rangle \) with the function \( u \) replacing \( f \) in (33).

**Remark 2.** In the Markov case, the operator \( A \) in (14) has the following form:

\[
A f(x) = \lambda(x) \int_{\mathbb{X}} P(x, dy) [f(y) - f(x)],
\]  

(35)

where \( \lambda(x) \) are the intensities of jumps of the jump Markov process \( X_t \). In this case

\[
Q(x, A, t) = P(x, dy)(1 - e^{-\lambda(y)t}),
\]

\[
G_t(v) = 1 - e^{-\lambda(v)t},
\]

\[
\overline{G}_t(v) = e^{-\lambda(v)t},
\]

\[
g_t(v) = \lambda(v)e^{-\lambda(v)t},
\]

and

\[
\frac{g_t(v)}{G_t(v)} = \lambda(v).
\]

In this way, the operator \( A \) in (35) is an infinitesimal operator of the jump Markov process \( X_t \).

**Corollary 1.** Initial capital for hedging strategy in our model is defined by the formula:

\[
V_0(\pi) = \mathbb{E} f(S_T) = \int \left( \int f(y) \phi \left( \eta, \ln \frac{y}{z} + \frac{1}{2} \eta \right) \, d\eta \right) F_0^2(d\eta).
\]

In particular, for the European call options \( f(y) = (y - K)^+ \), we have

\[
V_0(\pi) = \int C_{BS}(\bar{S}, T)^{1/2} \, dF_T(dz),
\]

where \( C_{BS}(\bar{S}, T) \) is the Black–Scholes price for call option with volatility \( \bar{S} \).
i.e., $C_{GS}((z/T)^{1/2}, T) = S_0 \Phi(d_+) - K \Phi(d_-)$, where

$$d_+ = \left[ \ln \frac{S_0}{K} + \frac{S_0}{2} \right] / \sqrt{S_0}.$$  

**Corollary 2.** Let $X = (1, 2)$, and $\nu(t)$ be a counting process for $X_t$, then

$$Z^*_T = \int_0^T [\sigma^2(1)I(X_t = 1) + \sigma^2(2)I(X_t = 2)] \, dt = aT + bI_t.$$

where

$$I_t = \int_0^T (-1)^{\nu(t)} \, dt,$$

$$a = \frac{1}{2} (\sigma^2(1) + \sigma^2(2)),$$

$$b = \frac{1}{2} (\sigma^2(1) - \sigma^2(2)).$$


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