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TWO-PARAMETER LEVY PROCESSES: ITO FORMULA, SEMIGROUPS AND GENERATORS

ДВОПАРАМЕТРИЧНІ ПРОЦЕСИ: ФОРМУЛА ІТО, ПІВГРУПИ ТА ГЕНЕРАТОРИ

The random Levy fields, i.e., such fields that are continuous in probability, stationary, and have independent increments, are considered. It is proved that the trajectories of such fields have no more than one jump on every line that is parallel to axes. Ito's change of variables formula for Levy fields is established. The semigroups generated by Levy fields and their generators are considered.

Розглядаються випадкові поля Леві — стохастично неперервні поля з незалежними приростами. Доведено, що траєкторії такого поля мають не більше одного стрибка на кожній прямій, паралельній осям. Виведено формулу заміни змінних Іто для полів Леві. Розглянуто півгрупи, породжені полями Леві, та інфінітезімальні оператори цих груп.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(\mathcal{F}_t, t \in R_+^2)$ be the family of σ-fields, satisfying conditions $(F1)$–$(F4)$ [1].

We consider real stochastic processes $(x_t, \mathcal{F}_t, t \in R_+^2)$, which are zero on the axes. The increment of x in the rectangle $[s, t]$, $s < t$, will be denoted by $x]s, t]$. We use the notions of i-predictable and weak predictable stochastic processes in the plane [2], strong and weak martingale [1] and two-parameter process with independent increments [3–5].

***Definition* 1.** *Two-parameter process x is called stationary if the distributions of increments $x]s, t]$ and $x]s+a, t+a]$ coincide for any $a \in R_+^2$ such that $[s, t], [s+a, t+a] \in {}^{\bullet}R_+^2$.*

***Definition* 2.** *Two-parameter stochastic process is called a Levy process if it is continuous in probability, stationary and has independent increments.*

According to [3, 5], continuous in probability two-parameter stochastic process of semimartingale type with independent increments admits a representation

$$x_t = M_t + B_t + \int_{[0,t]} \int_{|u|<1} u(\mu(ds, du) + \hat{\nu}(ds, du)) + \\ + \int_{[0,t]} \int_{|u|\geq 1} u\mu(ds, du), \quad t \in R_+^2, \tag{1}$$

where M is a continuous strong martingale with independent increments, B is a nonrandom nondecreasing function, μ is the measure of jumps of x, $\hat{\nu}(t, A) = E\mu(t, A)$; moreover, the increments of M are normally distributed, $\hat{\nu}(\{t\}, R_0) \leq \leq 1$,

$$\int_{[0,t]} \int_{R_0} (u^2 \wedge 1)\, \hat{\nu}(ds, du) < \infty, \quad t \in R_+^2, \quad R_0 = R \setminus \{0\}.$$

Let x be a Levy process. Taking into account that it is stationary, it is easy to prove similarly to [6] that $M_t = B_t$ is a Brownian field, i.e., $\operatorname{cov}(B_t, B_s) = = \sigma^2 \min(t_1, s_1) \min(t_2, s_2)$, $\sigma > 0$, $B_t = a t_1 t_2$, $\mu(t, A)$ is a Poisson process in the plane [17] independent on B_t for any Borel set A, and

$$\hat{\nu}(t, A) = t\nu(A), \tag{2}$$

where $\nu(A)$ is the parameter of $\mu(t, A)$.

In addition, according to [3], there exists such a modification of separable continuous in probability two-parameter process with independent increments that has no discontinuities of the second kind, i.e., has the limits

$$x_t^{\pm\pm} = \lim_{s \to t,\ s \in Q_t^{\pm\pm}} x_s,$$

in all quadrants $Q_t^{\pm\pm}$ for all $t \in R_+^2$ [8], and also $x_t = x_t^{++}$. We shall consider such a modification of the Levy process. Denote by $\Box x_t$ the jump of x at the point t:

$$\Box x_t = x_t - x_t^{+-} - x_t^{-+} + x_t^{--}; \quad \Delta_t^1 x = x_t - x_t^{-+},$$

$$\Delta^2 x_t = x_t - x_t^{+-}, \quad \Delta_{t-}^1 x = x_t^{+-} - x_t^{--},$$

$$\Delta_{t-}^2 x = x_t^{-+} - x_t^{--}.$$

Lemma 1. *Let x be a Levy process. Then the trajectories of x a.s. have no more than one jump along any horizontal or vertical line, i.e.,*

$$P_1 = P\left\{ \sup_{0<s_1<t_1,\ 0<t_2} \min\left(|\Box x_{s_1 t_2}|, |\Box x_t|\right) > 0 \right\} = 0, \tag{3}$$

$$P_2 = P\left\{ \sup_{0<s_2<t_2,\ 0<s_1} \min\left(|\Box x_s|, |\Box x_{s_1 t_2}|\right) > 0 \right\} = 0. \tag{4}$$

Proof. We prove (3) ((4) is proved similarly). It is sufficient to establish that, for any $\varepsilon > 0$ and $T_i > 0$, $i = 1, 2$,

$$P_1(T, \varepsilon) = P\left\{ \sup_{0<s_1<t_1 \le T_1,\ 0<t_2 \le T_2} \min\left(|\Box x_{s_1 t_2}|; |\Box x_t|\right) > \varepsilon \right\} = 0.$$

Let

$$x_t^{\varepsilon} = \int_{|u|>\varepsilon} u\mu(t, du),$$

$$x_t^{\varepsilon,+} = \int_{u>\varepsilon} u\mu(t, du), \quad x_t^{\varepsilon,-} = -\int_{u<-\varepsilon} u\mu(t, du).$$

Denote

$$y_t^{\varepsilon} = \sup_{0<u_1<t_1} |\Box x_{u_1 t_2}^{\varepsilon}|.$$

The process y_t^{ε} is continuous in Q_t^{--} and Q_t^{-+}, so it is 1-predictable. In addition, y_t^{ε} is $\mathcal{F}_t$-adapted, so y_t^{ε} is weakly predictable. Further, $P_1(T, \varepsilon) \le P^+(T, \varepsilon) + P^-(T, \varepsilon)$, where

$$P^{\pm}(T, \varepsilon) = P\left\{ \sup_{0<t\le T} y_t^{\varepsilon} x_t^{\varepsilon,\pm} > 0 \right\}.$$

It is sufficient to prove that $P^+(T, \varepsilon) = 0$ (the equality $P^-(T, \varepsilon) = 0$ is proved similarly). The process $x_t^{\varepsilon,+}$ is nondecreasing and integrable; so, according to [2], it has dual weakly predictable projection $A_t^{\varepsilon,+}$, i.e., such a nondecreasing integrable weakly predictable process that, for any bounded weakly predictable process z,

$$E \int_{[0,t]} z_s \, dx_s^{\varepsilon,+} = E \int_{[0,t]} z_s \, dA_s^{\varepsilon,+}. \tag{5}$$

From (1) and (2),

$$A_t^{\varepsilon,+} = t \int_{u>\varepsilon} u \nu(du).$$

Note that $A_t^{\varepsilon,+}$ is continuous, the process y^ε differs from 0 only on a finite number of horizontal lines; therefore, from (5),

$$E \sup_{0<t\le T} y_t^\varepsilon x_t^{\varepsilon,+} \le E \int_{[0,T]} y_t^\varepsilon \, dx_s^{\varepsilon,+} =$$

$$= \lim_{A\to\infty} E \int_{[0,T]} \left(y_t^\varepsilon \wedge A\right) dx_t^{\varepsilon,+} = \lim_{A\to\infty} E \int_{[0,T]} \left(y_t^\varepsilon \wedge A\right) dA_t^{\varepsilon,+} = 0.$$

So, $P^+(T,\varepsilon) = 0$.

Now we shall use the notions of "pure" and "mixed" stochastic integrals according to semimartingales in the plane [9, 10]:

$$(f \circ x)_t = \int_{[0,t]} f_s \, dx_s,$$

$$(f \circ (x * x))_t = \int_{[0,t]} f(x_s) \, d_1 x_s \, d_2 x_s = \int_{[0,t]^2} f(x_{s\vee z}) I\{s \curlywedge z\} \, dx_s \, dx_z,$$

where the relation $s \curlywedge z$ means that $s_1 \le z_1$, $s_2 > z_2$, $s \vee z = (s_1 \vee z_1, s_2 \vee z_2)$.

We apply the Ito formula for semimartingales in the plane [9] to the Levy process. Denote

$$\lambda(ds,du) = \mu(ds,du) - \nu(ds)\,du, \quad F: R \to R, \quad F \in C^{(4)}(R),$$

$$G(x_t) = F(0) + \int_{[0,t]} F'(x_s) \, dB_s + a \int_{[0,t]} F'(x_s) \, ds +$$

$$+ \frac{1}{2}\sigma^2 \int_{[0,t]} F''(x_s) \, ds + \int_{[0,t]} \int_{|u|<1} F'(x_{s-}) u \lambda(ds,du) +$$

$$+ \int_{[0,t]} \int_{R_0} \left(F(x_{s-} + u) - F(x_{s-}) - F'(x_{s-}) u I\{|u|<1\}\right) \mu(ds,du),$$

$$\square_{uv} F(x_s) = F(x_{s-} + u + v) - F(x_{s-} + u) - F(x_{s-} + v) + F(x_{s-}),$$

$$H(x_t) = \int_{[0,t]} F''(x_s) \, d_1 x_s \, d_2 x_s +$$

$$+ \frac{1}{2}\sigma^2 \int_{[0,t]} F'''(x_s) \left[s_1 d_1 x_s \, d_2 s_2 + s_2 d_1 s_1 \, d_2 x_s\right] +$$

$$+ \frac{1}{4}\sigma^4 \int_{[0,t]} F^{\mathrm{IV}}(x_s) s_1 s_2 \, ds + \int_{[0,t]^2} \int_{R_0} \left(F'(x_{s\vee z-} + u) - F'(x_{s\vee z-}) - \right.$$

$$\left. - F''(x_{s\vee z-}) u\right) I\{s \curlywedge z\} \left(\mu(ds,du) \, dB_z + \right.$$

$$\left. + dB_s \mu(dz,du) + a \mu(ds,du) \, dz + a \, ds \, \mu(dz,du)\right) +$$

$$+ \int_{[0,t]^2} \int_{|u|<1} \int_{|v|\ge 1} \square_{uv} F(x_{s\vee z-}) I\{s \wedge z\} [\lambda(ds, du) \mu(dz, dv) +$$

$$+ \mu(ds, dv) \lambda(dz, du)] +$$

$$+ \int_{[0,t]^2} \int_{|u|<1} \int_{|v|\ge 1} \square_{uv} F(x_{s\vee z-}) I\{s \curlywedge z\} [\lambda(ds, du) \mu(dz, dv) + +$$

$$+ \int_{[0,t]^2} \int_{|u|<1} \int_{|v|\ge 1} \left(\square_{uv} F(x_{s\vee z-}) - (F'(x_{s\vee z-} + v) - F'(x_{s\vee z-}))u \right) \times$$

$$\times I\{s \curlywedge z\} [\nu(du)\, ds\, \mu(dz, dv) + \mu(ds, dv)\, \nu(du)\, dz +$$

$$+ \nu(du)\, ds\, \lambda(dz, dv) + \lambda(ds, dv)\, \nu(du)\, dz] +$$

$$+ \int_{[0,t]^2} \int_{|u|<1} \int_{|v|<1} \left(\square_{uv} F(x_{s\vee z-}) - (F'(x_{s\vee z-} + u) - F'(x_{s\vee z-}))v - \right.$$

$$\left. - (F'(x_{s\vee z-} + v) - F'(x_{s\vee z-}))u + F''(x_{s\vee z-})uv \right) \times$$

$$\times I\{s \curlywedge z\} \nu(du)\, \nu(dv)\, ds\, dz .$$

Theorem 1. *Let $F: R \to R$, $F \in C^{(4)}(R)$, x be a Levy process in the plane. Then*

$$F(x_t) = G(x_t) + H(x_t). \tag{6}$$

Proof. Let

$$x_t^\varepsilon = B_t + a t_1 t_2 + \int_{[0,t]} \int_{\varepsilon \le |u| < 1} u \lambda(ds, du) + \int_{|u|\ge 1} u \mu(t, du).$$

Then x_t^ε is the sum of continuous strong martingale and the process of bounded variation. According to [9], the Ito formula for x_t^ε has the form

$$F(x_t^\varepsilon) = F_\varepsilon^{\text{int}}(t) + F_\varepsilon^{\text{jum}}(t),$$

where $F_\varepsilon^{\text{int}}(t)$ is the "integral" part, $F_\varepsilon^{\text{jum}}(t)$ is the "jumping" part; more precisely,

$$F_\varepsilon^{\text{int}}(t) = F(0) + (F'(x^\varepsilon) \circ x^\varepsilon)_t + (F''(x^\varepsilon) \circ (x^\varepsilon * x^\varepsilon))_t +$$

$$+ \frac{1}{2} (F''(x^\varepsilon) \circ [B])_t + \frac{1}{2} (F'''(x^\varepsilon) \circ (x^\varepsilon * \langle B \rangle^2))_t +$$

$$+ \frac{1}{2} (F'''(x^\varepsilon) \circ (\langle B \rangle^1 * x^\varepsilon))_t + \frac{1}{4} (F^{\text{IV}}(x^\varepsilon) \circ (\langle B \rangle^1 * \langle B \rangle^2))_t,$$

where $[B]$ and $\langle B \rangle^i$, $i = 1, 2$, are the quadratic variation and i-characteristics of B respectively. For the Brownian field

$$[B]_t = \langle B \rangle_t^1 = \langle B \rangle_t^2 = \sigma^2 t_1 t_2,$$

so

$$F_\varepsilon^{\text{int}}(t) = F(0) + \int_{[0,t]} F'(x_s^\varepsilon)\, dx_s^\varepsilon + \int_{[0,t]} F''(x_s^\varepsilon)\, d_1 x_s^\varepsilon\, d_2 x_s^\varepsilon +$$

$$+ \frac{1}{2} \sigma^2 \int_{[0,t]} F'''(x_s^\varepsilon) [s_1 d_1 x_s^\varepsilon d_2 s_2 + s_1 d_1 s_1 d_2 x_s^\varepsilon] +$$

$$+\ \frac{1}{4}\sigma^4 \int_{[0,t]} F^{\mathrm{IV}}\left(x_s^{\varepsilon}\right) s_1 s_2\, ds.$$

Further, $F_{\varepsilon}^{\mathrm{jum}}(t) = A_t^1 + A_t^2 - A_t^3$, where

$$A_t^1 = \sum_{s_1 \le t_1} \left[F\left(x_{s_1 t_2}^{\varepsilon}\right) - F\left(x_{s_1 t_2}^{\varepsilon,-+}\right) - F'\left(x_{s_1 t_2}^{\varepsilon,-+}\right)\Delta_{s_1 t_2}^1 x^{\varepsilon} \right],$$

$$A_t^2 = \sum_{s_2 \le t_2} \left[F\left(x_{t_1 s_2}^{\varepsilon}\right) - F\left(x_{t_1 s_2}^{\varepsilon,+-}\right) - F'\left(x_{t_1 s_2}^{\varepsilon,+-}\right)\Delta_{t_1 s_2}^2 x^{\varepsilon} \right],$$

$$A_t^3 = \sum_{s \le t} \Big[\square_s F - F'\left(x_{s-}^{\varepsilon}\right)\square x_s^{\varepsilon} + F''\left(x_{s-}^{\varepsilon}\right)\Delta_{s-}^1 x^{\varepsilon}\, \Delta_{s-}^2 x^{\varepsilon} -$$

$$- \Delta_{s-}^1 F' \Delta_s^2 x^{\varepsilon} - \Delta_{s-}^2 F \Delta_s^1 x^{\varepsilon} \Big],$$

$$\square_s F = F\left(x_s^{\varepsilon}\right) - F\left(x_s^{\varepsilon,+-}\right) - F\left(x_s^{\varepsilon,-+}\right) + F\left(x_{s-}^{\varepsilon}\right), \quad x_{s-}^{\varepsilon} = x_s^{\varepsilon,--},$$

$$\Delta_{s-}^1 F' = F'\left(x_s^{\varepsilon,+-}\right) - F'\left(x_{s-}^{\varepsilon}\right), \quad \Delta_{s-}^2 F' = F'\left(x_s^{\varepsilon,-+}\right) - F\left(x_{s-}^{\varepsilon}\right).$$

We apply the generalized one-parameter Ito formula to $F(x_{s_1 t_2}^{\varepsilon})$, $F(x_{s_1 t_2}^{\varepsilon,-+})$, $F'(x_{s_1 t_2}^{\varepsilon,-+})\Delta_{s_1 t_2}^1 x^{\varepsilon}$, as to processes in t_2 with s_1 being fixed:

$$F(x_{s_1 t_2}^{\varepsilon}) = F(0) + \int_0^{t_2} F'\left(x_s^{\varepsilon,+-}\right) d_2 x_s^{\varepsilon} + \sum_{s_2 \le t_2} \left(\Delta_s^2 F - F'\left(x_s^{\varepsilon,+-}\right)\Delta^2 x_s^{\varepsilon}\right),$$

$$F(x_{s_1 t_2}^{\varepsilon,-+}) = F(0) + \int_0^{t_2} F'\left(x_{s-}^{\varepsilon}\right) d_2 x_s^{\varepsilon,-+} + \sum_{s_2 \le t_2} \left(\Delta_{s-}^2 F - F'\left(x_{s-}^{\varepsilon}\right)\Delta_{s-}^2 x^{\varepsilon}\right),$$

$$F'\left(x_{s_1 t_2}^{\varepsilon,-+}\right)\Delta^1 x_{s_1 t_2}^{\varepsilon} = \int_0^{t_2} \Delta_{s-}^1 x^{\varepsilon}\, d_2 F'\left(x_{s-}^{\varepsilon}\right) +$$

$$+ \int_0^{t_2} F'\left(x_{s-}^{\varepsilon}\right) d_2\left(\Delta^1 x_s^{\varepsilon}\right) + \sum_{s_2 \le t_2} \Delta_{s-}^2 F' \square x_s^{\varepsilon}.$$

According to the one-parameter Ito formula,

$$d_2 F'\left(x_{s-}^{\varepsilon}\right) = F''\left(x_{s-}^{\varepsilon}\right) d_2 x_{s-}^{\varepsilon,-+} + \left(\Delta_{s-}^2 F' - F''\left(x_{s-}^{\varepsilon}\right)\Delta_{s-}^2 x^{\varepsilon}\right).$$

Moreover,

$$\int_0^{t_2} F'\left(x_{s-}^{\varepsilon}\right) d_2\left(\Delta^1 x_s^{\varepsilon}\right) = \sum_{s_2 \le t_2} F'\left(x_{s-}^{\varepsilon}\right)\square x_s^{\varepsilon}.$$

According to lemma 1, $\Delta_{s-}^2 F' \square x_s^{\varepsilon} = 0$. Therefore

$$A_t^1 = \sum_{s_1 \le t_1} \int_0^{t_2} \left(F'\left(x_s^{\varepsilon,+-}\right) - F'\left(x_{s-}^{\varepsilon}\right) - F''\left(x_{s-}^{\varepsilon}\right)\Delta_{s-}^1 x^{\varepsilon}\right) d_2 x_s^{\varepsilon,-+} +$$

$$+ \sum_{s \le t} \Big(\square_s F - F'\left(x_{s-}^{\varepsilon}\right)\square x_s^{\varepsilon} - \Big(F'\left(x_s^{\varepsilon,+-}\right) - F'\left(x_{s-}^{\varepsilon}\right) -$$

$$- F''\left(x^{\varepsilon}_{s-}\right)\Delta^1 x^{\varepsilon}_{s-}\right)\Delta^2_{s-}x^{\varepsilon} - \Delta^1 x^{\varepsilon}_{s-}\Delta^2_{s-}F\Big).$$

Similarly,

$$A^2_t = \sum_{s_2 \le t_2} \int_0^{t_1} \left(F'\left(x^{\varepsilon,-+}_s\right) - F'\left(x^{\varepsilon}_{s-}\right) - F''\left(x^{\varepsilon}_{s-}\right)\Delta^2_{s-}x^{\varepsilon}\right) d_1 x^{\varepsilon,+-}_s +$$

$$+ \sum_{s \le t} \Big(\Box_s F - F'\left(x^{\varepsilon}_{s-}\right)\Box x^{\varepsilon}_s - \Big(F'\left(x^{\varepsilon,-+}_s\right) - F'\left(x^{\varepsilon}_{s-}\right) -$$

$$- F''\left(x^{\varepsilon}_{s-}\right)\Delta^2 x^{\varepsilon}_{s-}\Big)\Delta^1 x^{\varepsilon}_{s-} - \Delta^1_{s-}F\Delta^2_{s-}x^{\varepsilon}\Big).$$

Note that, according to Lemma 1, we can change in A^3_t $\Delta^1_{s-}F'\Delta^2_s x^{\varepsilon}$ and $\Delta^2_{s-}F'\Delta^1_s x^{\varepsilon}$ for $\Delta^1_{s-}F'\Delta^2_{s-}x^{\varepsilon}$ and $\Delta^2_{s-}F'\Delta^1_{s-}x^{\varepsilon}$, respectively. Therefore,

$$F^{\text{jum}}_{\varepsilon}(t) = \sum_{s_1 \le t_1} \int_0^{t_2} \left[\Delta^1_{s-}F' - F''\left(x^{\varepsilon}_{s-}\right)\Delta^1_{s-}x^{\varepsilon}\right] d_2 x^{\varepsilon}_{s-} +$$

$$+ \sum_{s_2 \le t_2} \int_0^{t_1} \left[\Delta^2_{s-}F' - F''\left(x^{\varepsilon}_{s-}\right)\Delta^2_{s-}x^{\varepsilon}\right] d_1 x^{\varepsilon}_{s-} +$$

$$+ \sum_{s \le t} \Big(\Box_s F - F'\left(x^{\varepsilon}_{s-}\right)\Box x^{\varepsilon}_s - F''\left(x^{\varepsilon}_{s-}\right)\Delta^1_{s-}x^{\varepsilon}\,\Delta^2_{s-}x^{\varepsilon} -$$

$$- \left[\Delta^1_{s-}F' - F''\left(x^{\varepsilon}_{s-}\right)\Delta^1_{s-}x^{\varepsilon}\right]\Delta^2_{s-}x^{\varepsilon} - \left[\Delta^2_{s-}F' - F''\left(x^{\varepsilon}_{s-}\right)\Delta^2 x^{\varepsilon}_{s-}\right]\Delta^1 x^{\varepsilon}_{s-}.$$

We transform

$$\sum_{s_1 \le t_1} \int_0^{t_2} (\cdot) \quad \text{and} \quad \sum_{s_2 \le t_2} \int_0^{t_1} (\cdot)$$

into the mixed stochastic integrals. Note that, according to Lemma 1,

$$\Box_s F = \Big(F\left(x^{\varepsilon}_{s-} + \Delta^1 x^{\varepsilon}_{s-} + \Delta^2 x^{\varepsilon}_{s-}\right) - F\left(x^{\varepsilon}_{s-} + \Delta^1 x^{\varepsilon}_{s-}\right) - F\left(x^{\varepsilon}_{s-} + \Delta^2 x^{\varepsilon}_{s-}\right) +$$

$$+ F\left(x^{\varepsilon}_{s-}\right)\Big) + \Big(F\left(x^{\varepsilon}_{s-} + \Box x^{\varepsilon}_s\right) - F\left(x^{\varepsilon}_{s-}\right)\Big)$$

and obtain

$$F^{\text{jum}}_{\varepsilon}(t) = \int_{[0,t]^2} \int_{|u|>\varepsilon} \Big(F'\left(x^{\varepsilon}_{s\vee z-} + u\right) - F'\left(x^{\varepsilon}_{s\vee z-}\right) -$$

$$- F''\left(x^{\varepsilon}_{s\vee z-}\right)u\Big) I\{s \wedge z\}\,\mu(ds, du)\,dx^{\varepsilon}_z + \int_{[0,t]^2} \int_{|v|>\varepsilon} \Big(F'\left(x_{s\vee z-} + v\right) -$$

$$- F'\left(x_{s\vee z-}\right) - F''\left(x_{s\vee z-}\right)v\Big) I\{s \wedge z\}\,dx^{\varepsilon}_s\,\mu(dz, dv) +$$

$$+ \int_{[0,t]^2} \int_{|u|,|v|\ge\varepsilon} \Box_{uv} F\left(x^{\varepsilon}_{s\vee z-}\right) I\{s \wedge z\}\,\mu(ds, du)\,\mu(dz, dv) +$$

$$+ \int_{[0,t]} \int_{|u|,|v|\ge\varepsilon} \Big(F\left(x^{\varepsilon}_{s-} + u\right) - F\left(x^{\varepsilon}_{s-}\right) - F'\left(x^{\varepsilon}_{s-}\right)u\Big)\mu(ds, du) -$$

$$- \int_{[0,t]^2} \int_{|u|,|v|\ge\varepsilon} F''\left(x^{\varepsilon}_{s\vee z-}\right) uv\, I\{s \wedge z\}\,\mu(ds, du)\,\mu(dz, dv).$$

Collecting "pure" integrals from $F_\varepsilon^{\text{int}}(t)$ and $F_\varepsilon^{\text{jum}}(t)$, we obtain that their sum equals

$$G\left(x_t^\varepsilon\right) = F(0) + \int_{[0,t]} F'\left(x_s^\varepsilon\right) dB_s + a \int_{[0,t]} F'\left(x_s^\varepsilon\right) ds +$$

$$+ \frac{1}{2}\sigma^2 \int_{[0,t]} F''\left(x_s^\varepsilon\right) ds + \int_{[0,t]} \int_{\varepsilon \le |u| < 1} F'\left(x_s^\varepsilon\right) u \lambda(ds, du) +$$

$$+ \int_{[0,t]} \int_{|u| \ge \varepsilon} \left(F\left(x_{s-}^\varepsilon + u\right) - F\left(x_{s-}^\varepsilon\right) - F'\left(x_{s-}^\varepsilon\right) u I\{|u|<1\}\right) \mu(ds, du).$$

If $\varepsilon \to 0$, then $G(x_t^\varepsilon) \xrightarrow{P} G(x_t)$. Further, collecting mixed integrals from $F_\varepsilon^{\text{int}}(t)$ and $F_\varepsilon^{\text{jum}}(t)$, we note that their sum equals $H(x_t^\varepsilon)$. Moreover, $H(x_t^\varepsilon) \xrightarrow{P} H(x_t)$, $\varepsilon \to 0$. Taking into account that $F(x_t^\varepsilon) \xrightarrow{P} F(x_t)$, $\varepsilon \to 0$, we obtain the proof.

Let us consider the semigroups connected with a two-parameter Levy process. If $f \in C_b(R)$ (i.e., f is continuous and bounded on R), then

$$T_t f(x) = E f(x_t + x), \quad x \in R.$$

Because x_t is stationary, we have that

$$T_{t_1+s_1 t_2} f(x) = E f(x_{s_1+t_1 t_2} + x) = E f(x_{s_1 t_2} + x_t + x) = T_{s_1 t_2} T_t f(x).$$

Similarly,

$$T_{t_1 t_2 + s_2} f(x) = T_{t_1 s_2} T_t f(x),$$

so T_t has coordinatewise semigroup properties.

***Definition* 3.** *The operator*

$$A f(x) = \lim_{t_1 \vee t_2 \to 0} (t_1 t_2)^{-1} (T_t f(x) - f(x)),$$

is called the generator of coordinatewise two-parameter semigroup.

We denote by D_A the domain of definition of A.

Theorem 2. *Let $E|x_t| < \infty$ for any $t \in R_+^2$. Then $C_b^{(2)}(R) \subset D(A)$ and, for any $f \in C_b^{(2)}(R)$,*

$$A f(x) = a f'(x) + \frac{1}{2}\sigma^2 f''(x) +$$

$$+ \int_{R_0} \left(f(x+u) - f(x) - f'(x) u I\{|u|<1\}\right) \nu(dx). \tag{4}$$

Proof. Let $f \in C_b^{(4)}(R)$. If $E|x_t| < \infty$, then

$$E \sum_{s \le t} |\square x_s| I\{|\square x_s| \ge 1\} = t \int_{|u| \ge 1} |u| \nu(du) < \infty$$

and so

$$\int_{|u| \ge 1} |u| \nu(du) < \infty.$$

Then, from the Theorem 2,

$$Ef(x_t+x) = f(x) + aE\int_{[0,t]} f'\big(x_s+x\big)\,ds +$$

$$+\frac{1}{2}\sigma^2 E\int_{[0,t]} f''\big(x_s+x\big)\,ds + E\int_{[0,t]}\int_{R_0}\Big(f\big(x_{s-}+u+x\big)-f\big(x_{s-}+x\big)-$$

$$-f'\big(x_{s-}+x\big)\,u\,I\{|u|<1\}\Big)\,\nu(du)\,ds + a^2E\int_{[0,t]^2} f''\big(x_{s\vee z}+x\big)\,I\{s\wedge z\}\,ds\,dz +$$

$$+2aE\int_{[0,t]^2} f''\big(x_{s\vee z}+x\big)\,I\{s\wedge z\}\int_{|u|\ge 1} u\,\nu(du)\,ds\,dz +$$

$$+\sigma^2E\int_{[0,t]^2} f'''\big(x_{s\vee z}+x\big)\,I\{s\wedge z\}\left(a\,ds\,dz + \int_{|u|\ge 1} u\,\nu(du)\,ds\,dz\right) +$$

$$+\frac{1}{4}\sigma^4E\int_{[0,t]^2} f^{\mathrm{IV}}\big(x_{s\vee z}+x\big)\,I\{s\wedge z\}\,ds\,dz +$$

$$+2aE\int_{[0,t]^2}\int_{R_0}\Big(f'\big(x_{s\vee z}+u+x\big)-f'\big(x_{s\vee z}+x\big)-$$

$$-f''\big(x_{s\vee z}+x\big)\,u\Big)\,I\{s\wedge z\}\,\nu(du)\,ds\,dz +$$

$$+2E\int_{[0,t]^2}\int_{|u|<1}\int_{|v|\ge 1}\Big(\square_{uv}^{x}f-\Big(f'\big(x_{s\vee z}+v+x\big)-$$

$$--f'\big(x_{s\vee z}+x\big)\Big)\,u\Big)\,I\{s\wedge z\}\,\nu(du)\,\nu(dv)\,ds\,dz + E\int_{[0,t]^2}\int_{|u|,|v|<1}\Big(\square_{uv}^{x}f-$$

$$-\Big(f'\big(x_{s\vee z}+u+x\big)-f'\big(x_{s\vee z}+x\big)\Big)v-\Big(f'\big(x_{s\vee z}+v+x\big)-$$

$$f'\big(x_{s\vee z}+x\big)\Big)u + f''\big(x_{s\vee z}+x\big)\,uv\Big)\,I\{s\wedge z\}\,\nu(du)\,\nu(dv)\,ds\,dz, \qquad (7)$$

$$\square_{uv}^{x}f = f\big(x_{s\vee z}+u+v+x\big)-f\big(x_{s\vee z}+u+x\big)-$$

$$-f\big(x_{s\vee z}+v+x\big)+f\big(x_{s\vee z}+x\big).$$

Note that

$$\lim_{t_1\vee t_2\to 0}(t_1t_2)^{-1}E\int_{[0,t]} f^{(i)}\big(x_s+x\big)\,ds = f^{(i)}\big(x_s\big), \quad i=1,2;$$

$$\lim_{t_1\vee t_2\to 0}(t_1t_2)^{-1}E\int_{[0,t]}\int_{R_0}\Big(f\big(x_s+u+x\big)-f\big(x_s+x\big)-$$

$$-f'\big(x_s+x\big)\,u\,I\{|u|<1\}\Big)\,\nu(du)\,ds =$$

$$=\int_{R_0}\Big(f(x+u)-f(x)-f'(x)\,u\,I\{|u|<1\}\Big)\,\nu(du).$$

Further,

$$(t_1t_2)^{-1}\left|E\int_{[0,t]^2} f''\big(x_{s\vee z}+x\big)\,I\{s\wedge z\}\,ds\,dz\right| \le$$

$$\leq C(t_1t_2)^{-1} \int\limits_{[0,t]} s_1 z_2 \, ds_1 \, dz_2 = C_1 t_1 t_2 \to 0, \quad t_1 \vee t_2 \to 0. \tag{8}$$

Other integrals in (7) can be estimated similarly to (8) and therefore they are zero, whence we obtained our assertion for $f \in C_b^{(4)}(R)$.

Let $f \in C_b^{(2)}(R)$, and suppose there exists such $N > 0$ that $f = 0$ if $|x| \geq N$. Then there exist $\{f_n, n \geq 1\} \subset C_b^{(4)}(R)$ such that

$$\sup_{x \in R} \left| f_n^{(i)}(x) - f^{(i)}(x) \right| \to 0, \quad i = 0, 1, 2.$$

Note that, from the generalized one-parameter Ito formula and Lemma 1,

$$|Af(x) - Af_n(x)| \leq \lim_{t_1 \vee t_2 \to 0} (t_1t_2)^{-1} \left(\int_0^{t_1} |a| |g_n'(x_{s_1t_2}^{-+})| t_2 \, ds_1 + \right.$$

$$\left. + E \left| \sum_{s_1 \leq t_1} \left(g_n(x_{s_1t_2}) - g_n(x_{s_1t_2}^{-+}) - g_n'(x_{s_1t_2}^{-+}) I\{|\Delta^1 x_{s_1t_2}| < 1\} \right) \right| \right) \leq$$

$$\leq |a| \sup_{x \in R} \left| g_n'(x) \right| + \lim_{t_1 \vee t_2 \to 0} (t_1t_2)^{-1} \left(E \int\limits_{|u|<1} u^2 \mu(t, du) \sup_{x \in R} \left| g_n''(x) \right| + \right.$$

$$\left. + E \int\limits_{|u| \geq 1} \mu(t, du) \, 2 \sup_{x \in R} \left| g_n(x) \right| \right) =$$

$$= |a| \sup_{x \in R} \left| g_n'(x) \right| + \int\limits_{|u|<1} u^2 \nu(du) \sup_{x \in R} \left| g_n''(x) \right| + \int\limits_{|u| \geq 1} \nu(du) \, 2 \sup_{x \in R} \left| g_n(x) \right| \to 0,$$

$$n \to \infty, \quad g_n(x) = f(x) - f_n(x).$$

Therefore,

$$Af(x) = \lim_{n \to \infty} \left(a f_n'(x) + \frac{1}{2} \sigma^2 f_n''(x) + \right.$$

$$\left. + \int\limits_{R_0} \left(f_n(x+u) - f_n(x) - f_n'(x) u I\{|u|<1\} \right) \right) \nu(du) =$$

$$= a f'(x) + \frac{1}{2} \sigma^2 f''(x) + \int\limits_{R_0} \left(f(x+u) - f(x) - f'(x) u I\{|u|<1\} \right) \nu(du)$$

$$. \tag{9}$$

At last, let $f \in C_b^{(2)}(R)$. Then there exist such $\{\tilde{f}_n, n \geq 1\} \subset C_b^{(2)}(R)$ that $\tilde{f}_n(x) = f(x)$ for $|x| \leq n$, $\tilde{f}_n(x) = 0$ for $|x| \geq 2n$,

$$\sup_{x \in R} |f_n(x)| \leq \sup_{x \in R} |f(x)| \leq C.$$

Then, for sufficiently large n,

$$|Af(x) - A\tilde{f}_n(x)| \leq \lim_{t_1 \vee t_2 \to 0} (t_1t_2)^{-1} \left| Ef\left(x_t + x\right) - E\tilde{f}_n\left(x_t + x\right) \right| \leq$$

$$\le C \lim_{t_1 \vee t_2 \to 0} (t_1 t_2)^{-1} P\{|x_t| > n - x\} \le$$

$$\le 9C(n-x)^{-2} \lim_{t_1 \vee t_2 \to 0} (t_1 t_2)^{-1} E|B_t|^2 +$$

$$+ 6C(n-x)^{-1} \lim_{t_1 \vee t_2 \to 0} (t_1 t_2)^{-1} E \int_{|u|<1} |u| \mu(t, du) +$$

$$+ 3C(n-x)^{-1} \lim_{t_1 \vee t_2 \to 0} (t_1 t_2)^{-1} E \int_{|u|\ge 1} |u| \mu(t, du) =$$

$$= 9C(n-x)^{-2}\sigma^2 + 6C(n-x)^{-1} \int_{|u|<1} |u| \nu(du) +$$

$$+ 3C(n-x)^{-1} \int_{|u|>1} |u| \nu(du) \to 0, \quad n \to \infty,$$

and

$$A f(x) = \lim_{n \to \infty} A \tilde{f}_n(x). \tag{10}$$

From (9) and (10) we obtain the proof.

Remark. Some properties of two-parameter Levy processes, for example, existence and smoothness of local time, were considered in [11].

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