

TO THE PROBLEM ON CANONICAL FACTORIZATION FOR MARKOV ADDITIVE PROCESSES

ДО ЗАДАЧІ ПРО КАНОНІЧНУ ФАКТОРИЗАЦІЮ ДЛЯ МАРКОВСЬКОГО АДІТИВНОГО ПРОЦЕСУ

The well-known results on canonical factorization for Markov additive process with finite a Markov chain are extended to the case when this chain is countable. Some corollaries of these results are stated as well.

Відомі результати про канонічну факторизацію для марковського адитивного процесу зі скінченним ланцюгом Маркова переносяться на випадок, коли цей ланцюг злічений. Наводяться деякі наслідки цих результатів.

Suppose, on some probability space (Ω, \mathcal{F}, P) , we have:

1. A Markov chain x_n with a countable state space N .

2. The sequence of R -valued random variables $\{\xi_n; n = 0, 1, \dots\}$ such that $\{(x_n, \xi_n); n = 0, 1, \dots\}$ is a Markov chain on $N \times R$ having the transition function

$$P\{x_{n+1}=j, \xi_{n+1} \leq x/x_n=i, \mathcal{F}_n\} = p_{ij}F_{ij}(x),$$

where \mathcal{F}_n is the σ -algebra generated by $(x_0, \dots, x_{n-1}, \xi_1, \dots, \xi_n)$, $p_{ij} = P\{x_{n+1}=j/x_n=i\}$, $F_{ij}(x) = P\{\xi_{n+1} \leq x/x_n=i, x_{n+1}=j\}$.

Set $S_n = \xi_1 + \dots + \xi_n$, $n \geq 1$, $S_0 = 0$. The sequence $\{(x_n, S_n); n \geq 0\}$ is a Markov additive process [1].

Let $S_n^+ = \max_{0 < m < n} S_m$, $n \geq 1$, $(a_{ij}), i, j \in N$, denote the matrix with elements a_{ij} and P_j, E_j denote, respectively, conditional probability and expectation, provided that $x_0 = i$. For $|z| < 1$ put

$$c_-(z, x) = -I\{x < 0\} \sum_{n=1}^{\infty} z^n (P_i\{S_n^+ \leq S_n, x < S_n < 0, x_n = j\}),$$

$$c_+(z, x) = I\{x \geq 0\} \sum_{n=1}^{\infty} z^n (P_i\{S_n^+ < 0, 0 \leq S_n \leq x, x_n = j\}),$$

$$a_-(z, x) = -I\{x < 0\} \sum_{n=1}^{\infty} z^n (P_i\{S_n^+ < 0, x < S_n < 0, x_n = j\}),$$

$$a_+(z, x) = I\{x \geq 0\} \sum_{n=1}^{\infty} z^n (P_i\{S_n^+ \leq S_n, 0 \leq S_n \leq x, x_n = j\}),$$

$$A_+(z, s) = \int_{-0}^{\infty} e^{-sx} dc_+(z, x), \quad \operatorname{Re} s \geq 0,$$

$$A_-(z, s) = \int_{-\infty}^{-0} e^{-sx} dc_-(z, x), \quad \operatorname{Re} s \leq 0,$$

where $I\{A\}$ stands for the characteristic function of the set A .

The next result, a so-called right canonical factorization identity for the matrix

$$A(s) = (\rho_{ij} f_{ij}(s)), \quad f_{ij}(s) = \int_{-\infty}^{\infty} e^{-sx} dF_{ij}(x),$$

is known due to Presman E. L. [2], and here we state it in a slightly different form.

Theorem 1. For $|z| < 1$,

$$I - zA(s) = (I - A_-(z, s))(I - A_+(z, s)), \quad \operatorname{Re} s = 0, \quad (1)$$

with

$$(I - A_-(z, s))^{-1} = I + \int_{-\infty}^{-0} e^{-sx} da_-(z, x), \quad \operatorname{Re} s < 0, \quad (2)$$

$$(I - A_+(z, s))^{-1} = I + \int_{-0}^{\infty} e^{-sx} da_+(z, x). \quad \operatorname{Re} s > 0. \quad (3)$$

Here $I = (\delta_{ij})$ is the unit matrix.

It's well-known that the representations for the characteristic functions (or distributions) of the boundary functionals on the process (x_n, S_n) are expressed in terms of components of factorization (1), or, more precisely, in terms of functions $a_{\pm}(z, x)$. But in reality these representation have more theoretical interest as we don't know convenient formulas for the functions a_{\pm} . (It's obvious that the formulas provided by the above mentioned representations can hardly be regarded as convenient). In this respect the situation differs from the one-dimensional case (that is if $P\{x_n = x_0, n > 0\} = 1$), when we can write the representation for the factorizations components in the form of contour integrals. It enables us to produce an acceptable algorithm (at least from the mathematical point of view) for calculation of the distributions of the boundary functionals or some of their characteristics, for example, such as mean values. In the case of Markov additive process with a countable (or finite) Markov chain, we are prevented from such a possibility. In this connection the asymptotical analysis of the distributions in question is urgent when we are more interested in the qualitative results, such as the existence of the limit distributions, their form, than in numerical values of the limit characteristics. But to do such an analysis we need to extend factorization (1) to a more wide domain of change of the parameter z . For example, we can't be sure that factorization (1) is true for $|z| \leq 1$, since the function $c_-(1, x)$, $a_{\pm}(1, x)$ may be undefined and we also need a new interpretation for the formulas (2), (3). For the case of a finite chain x_n , this problem had been solved by Presman E. L. [2]. He used the dual Markov additive process, defined by the matrix $A^*(s) = \Pi^{-1} A^T(s) \Pi$, where $\Pi = (\pi_i \delta_{ij})$, π_i are stationary probabilities for x_n , and T denotes the conjugation. But in our case such an approach doesn't lead to the required results even in the case when the probabilities $\pi_i, i \in N$, exist, since now the matrix Π^{-1} is unbounded. Hence we must develop another approach and the present note is devoted to this problem.

First of all we introduce necessary notations, state conditions about the initial walks, and some auxiliary results from [3, 4].

Let l_{∞} be a Banach space of infinite sequences with the supremum norm. Its dual space is l_1 — the Banach space of the absolutely summable sequences. We will write elements of the space l_{∞} as column-vectors and elements of l_1 as row-vectors. A matrix $A(s)$ generates two operators: the first one acting from l_{∞} to l_{∞} and the second one acting from l_1 to l_1 . The results of their actions are denoted, respectively, as $A(s)x, x \in l_{\infty}$, and $yA(s), y \in l_1$. Using the same letter A for different operators

can't obviously lead to an ambiguity and, moreover, in what follows, we'll identify the notions "matrix" and "operator": $\vec{0}$ denotes the null-vector in the spaces l_1, l_∞ ; \bar{K} denotes the closure of the set K .

All relation between matrices are understood in an elementary sense, that is, for example, the notation $(a_{ij}) \leq (b_{ij})$ or $(a_{ij}(n)) - (b_{ij}(n))$ as $n \rightarrow \infty$ means, respectively, that $a_{ij} \leq b_{ij}$ or

$$\lim_{n \rightarrow \infty} \frac{a_{ij}(n)}{b_{ij}(n)} = 1$$

for all i, j .

The letters ε, δ will denote some sufficiently small (s.s.) positive numbers and C will denote some absolute positive constant, and all these number may be different in different contents.

Introduce the following conditions:

A.1. There exist numbers $s_- < 0 < s_+$ such that

$$\sup_{ij} f_{ij}(s_\pm) < \infty.$$

A.2. The chain x_n is irreducible, nonperiodical and uniformly ergodic.

(π_i) will denote ergodic distribution of this chain.

A.3. The walk (x_n, S_n) is nondegenerate. (The walk is called degenerate if there exist numbers β_i such that $P\{\xi_1 = \beta_j - \beta_i / x_0 = i, x_1 = j\} = 1$, then $S_n = \beta_{x_0} - \beta_{x_n}$ for all $n > 0$, so in fact we must deal the chain (x_n)).

A.4. The walk (x_n, S_n) is nonlattice. (The walk (x_n, S_n) is called lattice with the step $\Delta > 0$ if there exist numbers n_i such that $P\{S_1 = n_j - n_i + k\Delta, k = \pm 0, \pm 1, \dots, \dots / x_0 = i, x_1 = j\} = 1$).

Let $R(s)$ be the spectral radius of the operator $A(s)$, that is

$$R(s) = \lim_{n \rightarrow \infty} \sqrt[n]{\|A^n(s)\|}.$$

Then $R(s)$, for $s_- \leq s \leq s_+$, in the maximal eigenvalue of the operator $A(s)$ and, if $\pi(s) = (\pi_i(s)) \in l_1, y(s) = (y_i(s)) \in l_\infty$ are, respectively, the left and the right eigenvectors corresponding to the eigenvalue $R(s)$, then these vectors can be chosen such that $\pi_i(s) > 0, y_i(s) > 0, \pi(s)y(s) = 1, s_- \leq s \leq s_+, \pi_i(0) = \pi, y_i(0) = 1, i \in \mathbb{N}$. The eigenvalue $R(s)$, as a function in s , has the following properties: $R(s) < R(\text{Re } s), s_- \leq \text{Re } s \leq s_+, R(s)$ is strictly convex in the interval $[s_-, s_+]$ and $R'(s) = \pi(s)A_1(s)y(s)$, where $A_1(s) = (p_{ij}f'_{ij}(s))$. The later yields that the equation

$$R(s) = z^{-1} \tag{4}$$

has no more than two solutions in the interval $[s_-, s_+]$ for

$$\min_{s_- \leq s \leq s_+} R(s) \leq z^{-1} \leq \max(R(s_+), R(s_-)).$$

We'll assume that this equation has only two solutions. It will be granted by the condition.

A.5. There exists a number $s_0 \in (s_-, s_+)$ such that

$$\min_{s_- \leq s \leq s_+} R(s) = R(s_0).$$

If we denote

$$q_0 = R^{-1}(s_0), \quad q = (\min(R(s_-), R(s_+)))^{-1},$$

then equation (4) with $q \leq z \leq q_0$ has only two solutions $s_{\pm}(z)$ such that $s_- \leq s_-(z) \leq s_0$, $s_0 \leq s_+(z) \leq s_+$, and $s_{\pm}(q_0) = s_0$.

Remark 1. Since

$$R'(0) = \sum_{i \in N} \pi_i \sum_{j \in N} p_{ij} m_{ij} = m,$$

where $m_{ij} = E\{S_1 | x_0 = i, x_1 = j\}$, in the case when $m_0 = 0$, we have that $s_0 = 0$, and condition A.5 is fulfilled automatically.

A.6. For some $k > 0$,

$$\int_{|v|>1} |iv + a|^{-1} \|A^k(iv + a)\| dv < \infty, \quad s_- \leq a \leq s_+.$$

Denote

$$\Pi(s) = y(s)\pi(s), \quad \beta_{\pm}(z) = R'(s_{\pm}(z)),$$

$$K(\varepsilon_1, \varepsilon_2) = \{z : q + \varepsilon_1 < |z| < q_0 + \varepsilon_2, |\operatorname{Im} z| < \varepsilon_2, \operatorname{Re} z > 0\},$$

$$S(\varepsilon) = \{s : \operatorname{Re} s_-(z) - \varepsilon < \operatorname{Re} s < \operatorname{Re} s_+(z) + \varepsilon\}.$$

For some s.s. $\varepsilon_1, \varepsilon_2$, the functions $s_{\pm}(z)$ can be analytically extended in the domain $K(\varepsilon_1, \varepsilon_2)$ such that they remain solutions of equation (4).

Theorem 2 [3]. For $(z, s) \in K(\varepsilon_1, \varepsilon_2) \times S(\varepsilon)$, the function

$$A(z, s) = (I - zA(s))^{-1} - \frac{\Pi(s_+(z))}{(s_+(z) - s)\beta_+(z)} \frac{\Pi(s_-(z))}{(s_-(z) - s)\beta_-(z)} \quad (5)$$

is regular in (z, s) which belong to any bounded subdomain of the domain $K(\varepsilon_1, \varepsilon_2) \times S(\varepsilon)$. If conditions A.6 hold as well, then

$$A(s, z) = I + \int_{-\infty}^{+\infty} e^{-sx} da(z, x), \quad \operatorname{Re} s_-(z) - \varepsilon < \operatorname{Re} s < \operatorname{Re} s_+(z) + \varepsilon, \quad (6)$$

where the function $a(z, x)$ is regular in $z \in K(\varepsilon_1, \varepsilon_2)$ and

$$\|a(z, x)\| \leq Ce^{-(\operatorname{Re} s_{\pm}(z) \pm \varepsilon)x}, \quad \pm x \geq 0, \quad z \in \bar{K}(\varepsilon_1, \varepsilon_2). \quad (7)$$

The next theorem is the main result of this note.

Theorem 3. Let conditions A.1 – A.6 hold. Then, for $|z| \leq q_0$, the factorization (1) and formulas (2), (3) take place with $\|A_{\pm}(z, s)\| < \infty$ for $\pm \operatorname{Re} s \geq \pm s_{\pm}$. For $z \in \bar{K}(\delta, 0)$,

$$(I - A_+(z, s))^{-1} = I + \frac{\Pi(s_-(z))(I - A_-(z, s_-(z)))}{(s_-(z) - s)\beta_-(z)} + \int_{-0}^{+\infty} e^{-sx} dk_+(z, x), \quad (8)$$

$$\operatorname{Re} s > \operatorname{Re} s_-(z) - \varepsilon, \quad s \neq s_+(z).$$

$$(I - A_-(z, s))^{-1} = I + \frac{(I - A_+(z, s_+(z)))\Pi(s_+(z))}{(s_+(z) - s)\beta_+(z)} + \int_{-\infty}^{-0} e^{-sx} dk_-(z, x), \quad (9)$$

$$\operatorname{Re} s < \operatorname{Re} s_+(z) + \varepsilon, \quad s \neq s_-(z).$$

where the functions $k_{\pm}(z, x)$ are regular in $z \in \bar{K}(\delta, 0)$ for all x and

$$\|k_{\pm}(z, x)\| \leq C e^{(\operatorname{Re} s_{\pm}(z) \mp \varepsilon)x}, \quad \pm x > 0, z \in \bar{K}(\delta, 0). \quad (10)$$

We prove this theorem only for the case $q_0 = 1$, that is when $m = 0$, since the general case, as it will be shown below, can easily be reduced to this one. The proof itself is divided into several steps stated in the form of lemmas.

Lemma 1. *Let conditions A.1 – A.6 hold. Then, for $z \in \bar{K}(\delta, 0)$, equality (9) and estimate (10) hold for $k_{\pm}(z, x)$.*

Proof. Let $z \in K(\delta, 0)$. From (1), (5) we have

$$(I - A(z, s))^{-1} = (I - A_{+}(z, s)) \left(\frac{\Pi(s_{+}(z))}{(s_{+}(z) - s)\beta_{+}(z)} + \frac{\Pi(s_{-}(z))}{(s_{-}(z) - s)\beta_{-}(z)} + A(s, z) \right),$$

$$\operatorname{Re} s_{+}(z) < \operatorname{Re} s < \operatorname{Re} s_{+}(z) + \varepsilon.$$

From here, by using representation (6) and a standard argument (see, for example, [3]), we deduce (9), where $k_{-}(z, x) = a(z, x) - \int_0^{\infty} dv_{+}(z, y)a(z, x - y)$. Estimate (10), for $k_{-}(z, x)$, follows from (7).

Corollary 1. *For $\operatorname{Re} s \geq s_{-}$, $|z| \leq 1$, $\|A_{+}(z, s)\| < \infty$.*

Indeed, it's sufficient to show that $\|A_{+}(1, s_{-})\| < \infty$. For $|z| < 1$ we have

$$I - A_{+}(z, s) = (I - A_{-}(z, s))^{-1}(I - zA(s)).$$

Putting in this equality $|z| \uparrow 1$ and using Lemma 1, we obtain

$$A_{+}(1, s_{-}) = I - (I - A_{-}(1, s_{-}))^{-1}(I - A(s_{-})).$$

It proves Corollary 1.

Lemma 2. *For any $s, s, \varepsilon > 0$,*

$$\overline{\mathcal{R}}(I - A_{+}(1, \varepsilon)) = l_{\infty}, \quad (11)$$

where $\overline{\mathcal{R}}(\cdot)$ denotes the range of values of the operator $I - A_{+}(1, s)$ acting from l_{∞} to l_{∞} .

Proof. By using the same arguments as in [2] one can show that, for $|z| < 1$,

$$A_{-}(z, s) = \Pi^{-1} \left(E_j \{ z^{r_j^*} - e^{-s\eta_j^*}, s_{\tau_j^*} = j, \tau_j^* < \infty \} \right)^{\top} \Pi, \quad \operatorname{Re} s < s_{+}, \quad (12)$$

where $\tau_j^* = \inf \{ n : S_n^* < 0 \}$, $\eta_j^* = S_{\tau_j^*}$ and (x_n^*, S_n^*) is the dual Markov additive process, which is defined by the matrix $A^*(s) = \Pi^{-1} A^{\top}(s) \Pi$. It's easy to note that the process (x_n^*, S_n^*) satisfies conditions A.1 – A.6, and now it follows from (12) that the matrix $A_{-}(1, s)$ has bounded elements for $\operatorname{Re} s \leq s_{+}$, although it may be that $\|A_{-}(1, s)\| = \infty$ for some s . But, nevertheless, we can state that the equality

$$I - A(s) = (I - A_{-}(1, s))(I - A_{+}(1, s)), \quad s_{-} \leq \operatorname{Re} s \leq s_{+}, \quad (13)$$

takes place.

Let $\varepsilon > 0$ be fixed and such that $\|(I - A(\varepsilon))^{-1}\| < \infty$, $\|(I - A_{-}(1, \varepsilon))^{-1}\| < \infty$. To prove (11), it's sufficient to show that the equalities

$$(I - A_{+}(1, \varepsilon))y = \bar{0}, \quad y \in l_{\infty}, \quad (14)$$

$$x(I - A_{+}(1, \varepsilon)) = \bar{0}, \quad y \in l_1, \quad (15)$$

are possible only for $y = \bar{0}$, $x = \bar{0}$. These equalities follow from (13). Indeed, if (14)

takes place, then $(I - A(\varepsilon))y = \bar{0}$ and hence $y = \bar{0}$. If now (15) takes place, then $\bar{0} = x(I - A_-(1, \varepsilon))^{-1}(I - A(s))$, that is $\bar{0} = x(I - A_-(1, \varepsilon))^{-1}$ and hence $x = \bar{0}$. Lemma 2 is proved.

Lemma 3. For any $\operatorname{Re} s > 0$, $|z| \leq 1$, we have $\|(I - A_+(z, s))^{-1}\| < \infty$.

Proof. Since (see formula (3)) for $\operatorname{Re} s > 0$, $|z| \leq 1$,

$$\|(I - A_+(z, s))^{-1}\| \leq 1 + \left\| \int_{-0}^{\infty} e^{-s \operatorname{Re} s} da_+(1, x) \right\| = \|(I - A_+(1, \operatorname{Re} s))^{-1}\|$$

and

$$\|(I - A_+(1, \varepsilon_1))^{-1}\| \geq \|(I - A_+(1, \varepsilon_2))^{-1}\|$$

if $\varepsilon_1 \leq \varepsilon_2$, it's sufficient to show that

$$\|(I - A_+(1, \varepsilon))^{-1}\| < \infty \quad (16)$$

for any s.s. $\varepsilon > 0$.

Let $\varepsilon > 0$ be a small number such that Lemma 2 holds. Then $\|A_-(1, \varepsilon)\| < \infty$. Indeed, suppose that $\|A_-(1, \varepsilon)\| = \infty$. Lemma 2 yields that we can choose a vector $y_0 \in l_\infty$ such that, if $(y_i) = y = (I - A_+(1, \varepsilon))y_0$, then $\inf_i |y_i| \geq 1/2$.

By using (13) we have

$$\|A_-(1, \varepsilon)(I - A_+(1, \varepsilon))y_0\| < \|(I - A(\varepsilon))y_0\| + \|(I - A_+(1, \varepsilon))y_0\| < \infty.$$

But from the other side

$$\|A_-(1, \varepsilon)(I - A_+(1, \varepsilon))y_0\| = \|A_-(1, \varepsilon)y\| \geq \frac{1}{2}\|A_-(1, \varepsilon)\| = \infty.$$

So $\|A_-(1, \varepsilon)\| < \infty$ for any s.s. $\varepsilon > 0$, but then the equality

$$(I - A_+(1, \varepsilon))^{-1} = (I - A(\varepsilon))^{-1}(I - A_-(1, \varepsilon)),$$

which holds for any s.s. $\varepsilon > 0$, proves (16) and hence Lemma 3 is proved.

Proof of Theorem 3. The identity

$$A_-(z, s) = I - (I - zA(s))(I - A_+(z, s))^{-1}, \quad 0 < \operatorname{Re} s \leq s_+, \quad |z| < 1,$$

and Lemma 3 yield that $\|A_-(z, s)\| < \infty$ for $\operatorname{Re} s \leq s_+$, $|z| \leq 1$ and hence

$$I - zA(s) = (A_-(z, s))(I - A_+(z, s)), \quad s_- \leq \operatorname{Re} s \leq s_+, \quad |z| \leq 1.$$

If $0 < z < 1$, then

$$(I - A_+(z, s))^{-1} = I + s \int_{-0}^{\infty} e^{-s x} a_+(z, x) dx, \quad s > 0,$$

$$(I - A_-(z, s))^{-1} = I + s \int_{-\infty}^{-0} e^{-s x} a_-(z, x) dx, \quad s < 0.$$

Since all elements of the matrices $s a_{\pm}(z, x)$ are nonnegative for $\pm s > 0$, we have $\|a_{\pm}(1, x)\| < \infty$, $\pm x > 0$, and, hence, $\|a_{\pm}(z, x)\| < \infty$ for $|z| \leq 1$, and formulas (2), (3) take place.

By the same arguments we can show that $\|c_-(z, x)\| < \infty$ for $|z| \leq 1$.

For $z \in \bar{K}(\delta, 0)$, the representation (8) and estimate (10), for $k_+(z, x)$, can be proved by the same arguments as in Lemma 2.

Theorem 3 in the case $m = 0$ is proved.

Let now $m \neq 0$. Introduce the Markov additive process $(x_n^{(0)}, S_n^{(0)})$ which is defined by the matrix

$$A^{(0)} = (p_{ij}^0 f_{ij}^0(s)),$$

where

$$p_{ij}^{(0)} = \frac{p_{ij} y_j(s_0) f_{ij}(s)}{R(s_0) y_i(s_0)}, \quad f_{ij}^{(0)}(s) = \frac{f_{ij}(s + s_0)}{f_{ij}(s_0)}.$$

It is easy to check that

$$A^{(0)}(s) = R^{-1}(s_0) Y^{-1}(s_0) A(s + s_0) Y(s_0),$$

where $Y(s_0) = (y(s_0) \delta_{ij})$, and similar "transition formulas" can be written for another functions connected with the process $(x_n^{(0)}, S_n^{(0)})$. For example,

$$R^{(0)}(s) = \frac{R(s + s_0)}{R(s)}$$

$$A_{\pm}^{(0)}(z, s) = Y^{-1}(s_0) A_{\pm}(zR(s_0), s - s_0) Y(s_0).$$

Since

$$\frac{d}{ds} R^{(0)}(s) \Big|_{s=0} = \frac{R'(s_0)}{R(s)} = 0,$$

we can assert that Theorem 3 takes place for the process $(x_n^{(0)}, S_n^{(0)})$ and, by using the "transition formulas", one can complete proof of Theorem 3 for the case $m \neq 0$.

Corollary 2. *There exist*

$$\lim_{|z| \uparrow q_0} \frac{\Pi(s_-(z))(I - A_-(z, s_-(z)))}{\beta_-(z)} = A_-,$$

$$\lim_{|z| \uparrow q_0} \frac{I - A_+(z, s_-(z))\Pi(s_+(z))}{\beta_+(z)} = A_+,$$

and $\|A_{\pm}\| < \infty$.

Indeed, by putting in formulas (8), (9) $s = \varepsilon > 0$, $s = -\varepsilon < 0$, respectively, and then $|z| \uparrow q_0$, we obtain the statements of Corollary 2.

Now we state the result, which follows from equality (9) and Tauberian's theorems.

Theorem 4. *Let $m = 0$. Then, as $n \rightarrow \infty$,*

$$(P_j \{ \tau_+ > n, x_n = j \}) \sim \sqrt{R''(0)/2\pi n} A_+,$$

where $\tau_+ = \inf \{ k > 0 : S_k \geq 0 \}$.

1. Ney P., Nummelin E. Markov additive processes. I. Eigenvalue properties and limit theorems // Ann. Probab. - 1987. - 15, № 2. - P. 561-592.
2. Пресман Е. Л. Методы факторизации и граничная задача для сумм случайных величин, заданных на цепи Маркова // Изв. АН СССР, Сер. мат. - 1969. - 33, № 4. - С. 861-900.
3. Братичук Н. С. Спектральные свойства операторов, связанных с блужданиями на счетно-значной цепи Маркова. - Киев, 1987. - 39 с. - (Препринт / АН СССР. Ин-т математики; 87.12).
4. Братичук Н. С. Момент достижения и величина перескока уровня блужданием на цепи Маркова // Теория вероятностей и ее приложения. - 1990. - 35, № 2. - С. 231-241.

Received 02.03.94