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BICOMPLEX NUMBER AND TENSOR PRODUCT SURFACES IN $\mathbb{R}^4_2$

ПОВЕРХНІ ДОБУТКУ БІКОМПЛЕКСНИХ ЧИСЕЛ ТА ТЕНЗОРНОГО ДОБУТКУ В $\mathbb{R}^4_2$

We show that a hyperquadric M in $\mathbb{R}^4_2$ is a Lie group by using the bicomplex number product. For our purpose, we change the definition of tensor product. We define a new tensor product by considering the tensor product surface in the hyperquadric M. By using this new tensor product, we classify totally real tensor product surfaces and complex tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve.

By means of the tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve, we determine a special subgroup of the Lie group M. Thus, we obtain the Lie group structure of tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve. Morever, we obtain left invariant vector fields of these Lie groups. We consider the left invariant vector fields on these groups, which constitute a pseudo-Hermitian structure. By using this, we characterize these Lie groups as totally real or slant in $\mathbb{R}^4_2$.

Із використанням добутку бікомплексних чисел показано, що гіперквадрика M у $\mathbb{R}^4_2$ є групою Лі. Для досягнення нашої мети модифіковано означення тензорного добутку. Новий тензорний добуток означено шляхом розгляду поверхні тензорного добутку в гіперквадриці M. За допомогою цього нового добутку класифіковано тотально дійсні поверхні тензорного добутку та комплексні поверхні тензорного добутку плоскої кривої Лоренца та евклідової плоскої кривої.

За допомогою поверхонь тензорного добутку плоскої кривої Лоренца та евклідової плоскої кривої отримано спеціальну підгрупу групи Лі M. Таким чином, отримано структуру групи Лі для поверхонь тензорного добутку плоскої кривої Лоренца та евклідової плоскої кривої. Крім того, отримано лівоінваріантні векторні поля цих груп Лі. Розглянуто лівоінваріантні векторні поля на цих групах, які утворюють псевдоермітову структуру. Це дає змогу охарактеризувати групи Лі як тотально дійсні або скісні в $\mathbb{R}^4_2$.

1. Introduction. In the Euclidean space $\mathbb{E}^n$, the tensor product immersion of two immersions of a given Riemannian manifold was firstly defined and studied by Chen in [3]. In particular, the direct sum and the tensor product maps of two immersions of two different Riemannian manifolds are defined by Decruyenaere and coauthors in [4] in the following way:

Let M and N be two differentiable manifolds and assume that $f\colon M \to \mathbb{E}^m$ and $h\colon N \to \mathbb{E}^n$ are two immersions. The direct sum map and tensor product map are defined respectively by

$$f \oplus h\colon M \times N \to \mathbb{E}^{m+n}, \qquad (f \oplus h)(p,q) = \big(f_1(p), \ldots, f_m(p), h_1(q), \ldots, h_n(q)\big),$$

and

$$f \otimes h\colon M \times N \to \mathbb{E}^{mn}, \qquad (f \otimes h)(p,q) = \big(f_1(p)h_1(q), \ldots, f_1(p)h_n(q), \ldots, f_m(p)h_n(q)\big).$$

Under certain conditions obtained in [4], the tensor product map $f \otimes h$ is an immersion in the space $\mathbb{E}^{mn}$.

The simplest examples of the tensor product immersions are tensor product surfaces. In the Euclidean space $\mathbb{E}^n$, the tensor product surfaces of two Euclidean planar curves, as well as of a Euclidean space curve and a Euclidean plane curve are investigated in [8] and [1], respectively. Morever, in the semi-Euclidean space $\mathbb{E}^n_v$, the tensor product surfaces of two Lorentzian planar curves, as well as of a Lorentzian plane curve and a Euclidean plane curve are studied in [9] and [10], respectively. Also, the tensor product surfaces of a Lorentzian space curve and a Euclidean plane curve as well as of a Euclidean space curve and a Lorentzian plane curve are studied in [5] and [6], respectively.

It is often quite diffucult to decide if a manifold is paralelizable. S^n is paralelizable if and only if $n = 1, 3, 7$. Is it possible to make paralelization of any surface? The answer is yes. If M is a Lie group then M is paralelizable.

In [12], the authors showed that a hyperquadric M in $\mathbb{R}^4$ is a Lie group by using bicomplex number product. Also, in the same paper, Lie group structure of tensor product surfaces of Euclidean planar curves was obtained .

In this paper, we obtain Lie group structure of some special hypersurface in $\mathbb{R}^4_2$. By changing the tensor product rule given in $\mathbb{R}^4_2$ in [3] we give a new tensor product definition. As a result, the tensor product surface is obtained as a subset of the hyperquadric M. Hence, we investigate the tensor product surface as a Lie group. For our aim, if we change the definition of tensor product given in above as;

$$(\alpha \otimes \beta)(t, s) = \big(\alpha_1(t)\beta_1(s), \alpha_1(t)\beta_2(s), -\alpha_2(t)\beta_2(s), \alpha_2(t)\beta_1(s)\big)$$

we can easily obtain the same results which given in [10]. By using the new tensor product, we classify totally real tensor product surfaces and complex tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve. Furthermore we give some theorems for tensor product surfaces to be Lie groups and one parameter Lie subgroups. Finally, we give the necessary conditions for Lie group structures of tensor product surfaces to be totally real or slant in $\mathbb{R}^4_2$, respectively.

At the beginning, we recall notions of bicomplex numbers.

2. Preliminary. A bicomplex number is defined by the basis $\{1, i, j, ij\}$ where i, j, ij satisfy $i^2 = -1$, $j^2 = -1$, $ij = ji$. Thus any bicomplex number x can be expressed as $x = x_1 1 + x_2 i + x_3 j + x_4 ij \ \ \forall x_1, x_2, x_3, x_4 \in \mathbb{R}$. We denote the set of bicomplex numbers by C_2. For any $x = x_1 1 + x_2 i + x_3 j + x_4 ij$ and $y = y_1 1 + y_2 i + y_3 j + y_4 ij$ in C_2, the bicomplex number addition is defined as

$$x + y = (x_1 + y_1)\,1 + (x_2 + y_2)\,i + (x_3 + y_3)\,j + (x_4 + y_4)\,ij.$$

The multiplication of a bicomplex number $x = x_1 1 + x_2 i + x_3 j + x_4 ij$ by a real scalar λ is defined as

$$\lambda x = \lambda x_1 1 + \lambda x_2 i + \lambda x_3 j + \lambda x_4 ij.$$

With this addition and scalar multiplication operations, C_2 is a real vector space.

Bicomplex number product, denoted by $\times$, over the set of bicomplex numbers C_2 is given by the following table:

$\times$	1	i	j	ij
1	1	i	j	ij
i	i	-1	ij	$-j$
j	j	ij	-1	$-i$
ij	ij	$-j$	$-i$	1

The vector space C_2 together with the bicomplex product $\times$ is an real algebra [13].

Since bicomplex number algebra is associative it can be considered in terms of matrices. Consider the set of matrices

$$Q = \left\{ \begin{bmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{bmatrix} : \quad x_i \in \mathbb{R}, \ 1 \leq i \leq 4 \right\}.$$

The set Q together with matrix addition and scalar matrix multiplication is a real vector space. Furthermore, the vector space together with matrix product is an algebra.

The transformation

$$h \colon C_2 \to Q$$

given by

$$h\left(x = x_1 1 + x_2 i + x_3 j + x_4 ij\right) = \begin{bmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{bmatrix}$$

is one-to-one and onto. Moreover, $\forall x, y \in C_2$ and $\forall \lambda \in \mathbb{R}$, we have

$$h\left(x + y\right) = h\left(x\right) \oplus h\left(y\right),$$

$$h\left(\lambda x\right) = \lambda h\left(x\right),$$

$$h\left(x \times y\right) = h\left(x\right) h\left(y\right).$$

Thus the algebras C_2 and Q are isomorphic.

For further bicomplex number concepts see [13].

3. Tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve. In this section, we change the definition of tensor product as follows:

Let $\alpha \colon \mathbb{R} \to \mathbb{R}^2_1\,(+-)$ and $\beta \colon \mathbb{R} \to \mathbb{R}^2$ be respectively a Lorentzian planar curve and a Euclidean planar curve. Put $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ and $\beta(s) = (\beta_1(s), \beta_2(s))$. Let us define their tensor product as

$$\begin{gathered} f = \alpha \otimes \beta \colon \ \mathbb{R}^2 \to \mathbb{R}^4_2(++--), \\ f\left(t, s\right) = \big(\alpha_1(t)\beta_1(s), \alpha_1(t)\beta_2(s), -\alpha_2(t)\beta_2(s), \alpha_2(t)\beta_1(s)\big). \end{gathered} \tag{3.1}$$

By using equation (3.1), the canonical tangent vectors of $f\left(t, s\right)$ can be easily computed as

$$\begin{gathered} \frac{\partial f}{\partial t} = \big(\alpha_1'(t)\beta_1(s), \alpha_1'(t)\beta_2(s), -\alpha_2'(t)\beta_2(s), \alpha_2'(t)\beta_1(s)\big), \\ \frac{\partial f}{\partial s} = \big(\alpha_1(t)\beta_1'(s), \alpha_1(t)\beta_2'(s), -\alpha_2(t)\beta_2'(s), \alpha_2(t)\beta_1'(s)\big), \end{gathered} \tag{3.2}$$

where α' means the derivative of α.

In the following, we will assume that α is a spacelike or a timelike curve with spacelike or a timelike position vector and we will assume that β is a regular curve. We shall also assume that the tensor product surface $f\,(t,s)$ is a regular surface, i.e., $g_{11}g_{22}-g_{12}^2\neq 0$.

Hence relations (3.1) and (3.2) imply that the coefficients of the pseudo-Riemannian metric, induced on $f\,(t,s)$ by the pseudo-Euclidian metric g of $\mathbb{R}_2^4$ is given $g=dx_1^2+dx_2^2-dx_3^2-dx_4^2$, are

$$g_{11}=g\left(\frac{\partial f}{\partial t},\frac{\partial f}{\partial t}\right)=g_1\left(\alpha',\alpha'\ \right)g_2\left(\beta,\beta\right),$$

$$g_{12}=g\left(\frac{\partial f}{\partial t},\frac{\partial f}{\partial s}\right)=g_1\left(\alpha,\alpha'\right)g_2\left(\beta,\beta'\right),$$

$$g_{22}=g\left(\frac{\partial f}{\partial s},\frac{\partial f}{\partial s}\right)=g_1\left(\alpha,\alpha\right)g_2\left(\beta',\beta'\right),$$

where $g_1=dx_1^2-dx_2^2$ and $g_2=dx_1^2+dx_2^2$ are the metrics of $\mathbb{R}_1^2$ and $\mathbb{R}^2$, respectively. Consequently, an orthonormal basis for the tangent space of $f\,(t,s)$ is given by

$$e_1=\frac{1}{\sqrt{|g_{11}|}}\frac{\partial f}{\partial t},$$

$$e_2=\frac{1}{\sqrt{\left|g_{11}\left(g_{11}g_{22}-g_{12}^2\right)\right|}}\left(g_{11}\frac{\partial f}{\partial s}-g_{12}\frac{\partial f}{\partial t}\right).$$

Recall that the mean curvature vector field H is defined by

$$H=\frac{1}{2}\big(\varepsilon_1 h(e_1,e_1)+\varepsilon_2 h(e_2,e_2)\big),$$

where h is a second fundamental form of $\alpha\otimes\beta$ and $\varepsilon_i=g(e_i,e_i)$, $i=1,2$. In particular by Beltrami's formula we have

$$H=-\frac{1}{2}\Delta f.$$

Next, recall that a surface M in $\mathbb{R}_2^4$ is said to be minimal, if its mean curvature vector field H vanishes identically.

A basis of the normal space of $f(t,s)$ can ce calculated as follows. Let $J_1\colon\mathbb{R}_1^2\to\mathbb{R}_1^2$ and $J_2\colon\mathbb{R}^2\to\mathbb{R}^2$ be the folowing maps:

$$J_1(x,y)=(y,x),$$

$$J_2(x,y)=(-y,x).$$

Observe that $g_1\left(X,J_1(X)\right)=0$ for $X\in\mathbb{R}_1^2$ and $g_2\left(Y,J_2(Y)\right)=0$ for $Y\in\mathbb{R}^2$.

Then the normal space is spanned by

$$n_1(t,s)=J_1(\alpha(t))\otimes J_2\left(\beta(s)\right)=(\alpha_2(t),\alpha_1(t))\otimes(-\beta_2(s),\beta_1(s))=$$

$$=\Big(-\alpha_2(t)\beta_2(s),\alpha_2(t)\beta_1(s),-\alpha_1(t)\beta_1(s),-\alpha_1(t)\beta_2(s)\Big),$$

$$n_2(t,s) = J_1(\alpha'(t)) \otimes J_2(\beta'(s)) = (\alpha_2'(t), \alpha_1'(t)) \otimes (-\beta_2'(s), \beta_1'(s)) =$$

$$= \Big(-\alpha_2'(t)\beta_2'(s), \alpha_2'(t)\beta_1'(s), -\alpha_1'(t)\beta_1'(s), -\alpha_1'(t)\beta_2'(s) \Big).$$

4. Totally real and complex Lorentzian immersion and slant tensor product surface. Let $\alpha\colon \mathbb{R} \to \mathbb{R}^2_1(+-)$ and $\beta\colon \mathbb{R} \to \mathbb{R}^2$ be respectively a Lorentzian planar curve and a Euclidean planar curve and let $f = \alpha \otimes \beta$ be their tensor product. We consider the pseudo-Hermitian structure J given by

$$J(u, v, z, w) = (-v, u, -w, z), \quad u, v, z, w \in \mathbb{R}.$$

In the next theorem by using the new product we classify totally real tensor product surface in the semi-Euclidean space $\mathbb{R}^4_2$, i.e., the pseudo-Hermitian structure J at each point transforms the tangent space to the surface into the normal space.

Theorem 1. *The tensor product immersion $f = \alpha \otimes \beta$ of a Lorentzian plane curve and a Euclidean plane curve is a totally real Lorentzian immersion with respect to the pseudo-Hermitian structure J on $\mathbb{R}^4_2$ if and only if α is a Lorentzian circle centered at 0 or β is a straight line passing through origin.*

Proof. $\operatorname{Im} f$ is a totally real surface if and only if $J\left(\dfrac{\partial f}{\partial t}\right)$ is orthogonal to $\dfrac{\partial f}{\partial s}$ and $J\left(\dfrac{\partial f}{\partial s}\right)$ is orthogonal to $\dfrac{\partial f}{\partial t}$.

We have

$$J\left(\frac{\partial f}{\partial t}\right) = \Big(-\alpha_1'(t)\beta_2(s), \alpha_1'(t)\beta_1(s), -\alpha_2'(t)\beta_1(s), -\alpha_2'(t)\beta_2(s)\Big).$$

By a straightforward calculation, we obtain

$$g\left(J\left(\frac{\partial f}{\partial t}\right), \frac{\partial f}{\partial s}\right) = -g\left(J\left(\frac{\partial f}{\partial s}\right), \frac{\partial f}{\partial t}\right),$$

$$g\left(J\left(\frac{\partial f}{\partial t}\right), \frac{\partial f}{\partial s}\right) = 0$$

if and only if $(\alpha_1\alpha_1' - \alpha_2\alpha_2') = 0$ or $(\beta_1\beta_2' - \beta_1'\beta_2) = 0$. Integrating these equations, we find that either β is a straight line passing through origin, or α is a Lorentzian circle centered at 0.

Theorem 1 is proved.

Theorem 2. *The tensor product immersion $f = \alpha \otimes \beta$ of a Lorentzian plane curve and a Euclidean plane curve is a complex Lorentzian immersion with respect to the pseudo-Hermitian structure J on $\mathbb{R}^4_2$ if and only if α is a straight line passing through origin and β is an Euclidean planar curve.*

Proof. By definition, the following equations are satisfied:

$$g\left(J\left(\frac{\partial f}{\partial t}\right), n_i\right) = 0, \qquad g\left(J\left(\frac{\partial f}{\partial s}\right), n_i\right) = 0, \quad i = 1, 2. \tag{4.1}$$

We have

$$J\left(\frac{\partial f}{\partial t}\right) = \Big(-\alpha_1'(t)\beta_2(s), \alpha_1'(t)\beta_1(s), -\alpha_2'(t)\beta_1(s), -\alpha_2'(t)\beta_2(s)\Big),$$

$$J\left(\frac{\partial f}{\partial s}\right) = \Big(-\alpha_1(t)\beta_2'(s), \alpha_1(t)\beta_1'(s), -\alpha_2(t)\beta_1'(s), -\alpha_2(t)\beta_2'(s)\Big).$$

By a straightforward calculation, we obtain

$$g\left(J\left(\frac{\partial f}{\partial t}\right),\ n_2\right) = g\left(J\left(\frac{\partial f}{\partial s}\right),\ n_1\right) = 0,$$

$$g\left(J\left(\frac{\partial f}{\partial t}\right),\ n_1\right) = \left(\alpha_1(t)\alpha_2'(t) - \alpha_1'(t)\alpha_2(t)\right)\left(\beta_1^2(s) + \beta_2^2(s)\right), \tag{4.2}$$

$$g\left(J\left(\frac{\partial f}{\partial s}\right),\ n_2\right) = \left(\alpha_1'(t)\alpha_2(t) - \alpha_1(t)\alpha_2'(t)\right)\left(\beta_1'^2(s) + \beta_2'^2(s)\right).$$

From equations (4.1) and (4.2) we have

$$\alpha_1'(t)\alpha_2(t) - \alpha_1(t)\alpha_2'(t) = 0.$$

It follows that α is straight line passing through the origin.

Theorem 2 is proved.

Recall the definition of a slant surface with respect to pseudo-Hermitian structure J on $\mathbb{R}^4_2$. Let M be a surface with respect to the pseudo-Hermitian structure J on $\mathbb{R}^4_2$. Then M is said to be a proper slant if

$$g\left(J\left(e_1\right), e_2\right) = \lambda, \quad \lambda \in \mathbb{R},$$

along M for a given orthonormal basis $\{e_1, e_2\}$ of T_pM $(p \in M)$ which is independent of the choice of $\{e_1, e_2\}$ [3].

Let $\alpha\colon \mathbb{R} \to \mathbb{R}^2_1\,(+-)$ and $\beta\colon \mathbb{R} \to \mathbb{R}^2$ be respectively a Lorentzian planar curve and a Euclidean planar curve. We consider polar coordinates on α and β. Then,

$$\alpha(t) = \rho_1(t)\left(\cosh t, \sinh t\right),$$

$$\beta\left(s\right) = \rho_2\left(s\right)\left(\cos s, \sin s\right).$$

A straightforward computation leads to

$$g\left(J\left(e_1\right), e_2\right) = \frac{\rho_1'\rho_2}{\sqrt{\left|\left(\rho_1'^2 - \rho_1^2\right)\left(\rho_2'^2 + \rho_2^2\right) - \rho_1'^2\rho_2'^2\right|}}.$$

If $\rho_2 = \text{constant}$, it follows that $\rho_1 = a_1 e^{b_1 t}$, $a_1 \in \mathbb{R}$, $b_1 \in \mathbb{R}$. Hence α is a hyperbolic spiral and β is a circle centered at the origin. If $\rho_2 \neq$constant, let us put $\dfrac{\rho_k}{\rho_k'} = c_k$, $k = 1, 2$. Then

$$g\left(J\left(e_1\right), e_2\right) = \frac{c_2}{\sqrt{\left|\left(c_2^2 + 1\right)\left(1 - c_1^2\right) - 1\right|}}.$$

Therefore $\operatorname{Im} f$ is a proper slant surface if and only if

$$\frac{c_2^2}{\left(c_2^2+1\right)\left(1-c_1^2\right)-1}=\lambda^2.$$

It follows that

$$\frac{c_2^2+\lambda^2}{c_2^2+1}=\lambda^2\left(1-c_1^2\right).$$

This means that $c_1(t)$ and $c_2(s)$ must be constant functions, which implies that $\rho_1(t)=a_1e^{b_1t}$, $\rho_2(s)=a_2e^{b_2s}$, $a_1,b_1,a_2,b_2\in\mathbb{R}$. Consequently, α is a hyperbolic spiral and β is a logarithmic spiral. In this way, we proved the following theorem.

Theorem 3. *The tensor product immersion $f=\alpha\otimes\beta$ of a Lorentzian plane curve and a Euclidean plane curve is a slant surface with respect to pseudo-Hermitian structure J on $\mathbb{R}^4_2$ if and only if α is a hyperbolic spiral and β is either a circle centered at O or a spiral curve.*

5. Lie groups and some special subgroup. In this section, we deal with the hyperquadric

$$M=\big\{x=\left(x_1,x_2,x_3,x_4\right):\ x_1x_3+x_2x_4=0,\ g\left(x,x\right)\neq 0\big\},$$

$$M=\big\{x=\left(x_1,x_2,x_3,x_4\right):\ x_1x_3+x_2x_4=0,\ x_1^2+x_2^2-x_3^2-x_4^2\neq 0\big\}.$$

We consider M as the set of bicomplex numbers,

$$M=\big\{x=x_11+x_2i+x_3j+x_4ij\in C_2:\ x_1x_3+x_2x_4=0,\quad g\left(x,x\right)\neq 0\big\}.$$

The components of M are easily obtained by representing bicomplex number multiplication in matrix form:

$$\widetilde{M}=\left\{x=\begin{bmatrix}x_1 & -x_2 & -x_3 & x_4\\ x_2 & x_1 & -x_4 & -x_3\\ x_3 & -x_4 & x_1 & -x_2\\ x_4 & x_3 & x_2 & x_1\end{bmatrix}:\ x_1x_3+x_2x_4=0,\quad g\left(x,x\right)\neq 0\right\}.$$

Theorem 4. *The set of M together with the bicomplex number product is a Lie group.*

Proof. $\widetilde{M}$ is a differentiable manifold and at the same time a group with group operation given by matrix multiplication. The group function

$$.:\widetilde{M}\times\widetilde{M}\to\widetilde{M}$$

defined by $(x,y)\to x.y$ is differentiable. So, $(M,\times)$ can be made a Lie group so that h is a isomorphism.

Theorem 4 is proved.

Consider the group M_1 of all unit bicomplex numbers $x=x_11+x_2i+x_3j+x_4ij$ on M with the group operation of bicomplex multiplication. That is

$$M_1=\left\{x\in M:\ g\left(x,x\right)=1\right\},$$

$$M_1=\big\{x\in M:\ x_1^2+x_2^2-x_3^2-x_4^2=1\big\}.$$

Lemma 1. *M_1 is 2-dimensional Lie subgroup of M.*

6. Lie algebra of Lie group M and M_1. M is a Lie group of dimension three. Let us find its Lie algebra. Thus, let

$$\alpha(t) = \alpha_1(t)1 + \alpha_2(t)i + \alpha_3(t)j + \alpha_4(t)ij$$

be a curve on M such that $\alpha(0) = 1$, i.e., $\alpha_1(0) = 1$, $\alpha_m(0) = 0$ for $m = 2, 3, 4$. Differentiation of the equation $\alpha_1(t)\alpha_3(t)+\alpha_2(t)\alpha_4(t) = 0$ yields the equation $\alpha_1'(t)\alpha_3(t)+\alpha_1(t)\alpha_3'(t)+\alpha_2'(t)\alpha_4(t)+ + \alpha_2(t)\alpha_4'(t) = 0$. Substituting $t = 0$, we obtain $\alpha_3'(0) = 0$. The Lie algebra is thus constituted by vectors of the form $\zeta = \zeta_m \left(\dfrac{\partial}{\partial \alpha_m}\right) |_{\alpha=1}$, where $m = 1, 2, 4$. The vector ζ is formally written in the form $\zeta = \zeta_1 + \zeta_2 i + \zeta_4 ij$. Let us find the left invariant vector field X on M for which $X |_{\alpha=1} = \zeta$. Let $\beta(t)$ be a curve on M such that $\beta(0) = 1$, $\beta'(0) = \zeta$. Then $L_x(\beta(t)) = x\beta(t)$ is the left translation of the curve $\beta(t)$ by the bicomplex number x, its tangent vector is $x\beta'(0) = x\zeta$. In particular, denote by X_m those left invariant vector fields on M for which,

$$X_m |_{\alpha=1} = \left.\frac{\partial}{\partial \alpha_m}\right|_{\alpha=1},$$

where $m = 1, 2, 4$. These three vector fields are represented at the point $\alpha = 1$, in bicomplex notation, by the bicomplex units $1, i, ij$. For the components of these vector fields at the point $x = x_1 1 + x_2 i + x_3 j + x_4 ij$ we have $(X_1)_x = x1$, $(X_2)_x = xi$, $(X_3)_x = xij$:

$$X_1 = (x_1, x_2, x_3, x_4),$$

$$X_2 = (-x_2, x_1, -x_4, x_3),$$

$$X_4 = (x_4, -x_3, -x_2, x_1),$$

where all the partial derivaties are at the point x.

M_1 is a Lie group of dimension two. Its Lie algebra can be easily found that

$$X_2 = (-x_2, x_1, -x_4, x_3),$$

$$X_4 = (x_4, -x_3, -x_2, x_1).$$

Theorem 5. *M is paralelizable.*

Proof. If we put

$$x_1 = \rho_1 \cos\phi,$$

$$x_2 = \rho_1 \sin\phi,$$

$$x_3 = \rho_2 \cos\theta,$$

$$x_4 = \rho_2 \sin\theta,$$

then from $x_1 x_3 + x_2 x_4 = 0$ we have

$$\cos\phi = \sin\theta, \qquad \sin\phi = -\cos\theta,$$

or

$$\cos\phi = -\sin\theta, \qquad \sin\phi = \cos\theta.$$

Besides, the condition $x_1^2+x_2^2-x_3^2-x_4^2 \neq 0$ gives $\rho_1^2-\rho_2^2 \neq 0$. We have the parametric representation of one component of M with vector position $r(\rho_1, \rho_2, \phi)$

$$r = (\rho_1\cos\phi, \rho_1\sin\phi, -\rho_2\sin\phi, \rho_2\cos\phi).$$

Hence, we have three vectors tangent to coordinate curves

$$r_{\rho_1} = (\cos\phi, \sin\phi, 0, 0),$$

$$r_{\rho_2} = (0, 0, -\sin\phi, \cos\phi),$$

$$r_\phi = (-\rho_1\sin\phi, \rho_1\cos\phi, -\rho_2\cos\phi, -\rho_2\sin\phi).$$

Evidently these vectors are ortogonal each other and put together a paralelization of M.

Theorem 5 is proved.

7. Tensor product surfaces and Lie groups. In this section, by using the tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve, we determine some special subgroup of this Lie group M. Thus, we obtain Lie group structure of tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve. Also, we obtain left invariant vector fields of these Lie groups.

Theorem 6. *Let $\alpha\colon \mathbb{R} \to \mathbb{R}^2_1$ be a hyperbolic spiral, and $\beta\colon \mathbb{R} \to \mathbb{R}^2$ be a spiral with the same parameter, i.e., $\alpha(t) = e^{at}(\cosh t, \sinh t)$ and $\beta(t) = e^{bt}(\cos t, \sin t)$, $a, b \in \mathbb{R}$. Then their tensor product is a one-parameter subgroup in a Lie group M.*

Proof. We obtain

$$\gamma(t) = \alpha(t) \otimes \beta(t) = e^{(a+b)t}(\cosh t\cos t, \cosh t\sin t, -\sinh t\sin t, \sinh t\cos t).$$

It can be easily seen that

$$\gamma(t_1) \times \gamma(t_2) = \gamma(t_1 + t_2)$$

for all t_1, t_2. Hence, $(\gamma(t), \times)$ is a one-parameter Lie subgroup of $(M, \times)$.

Theorem 6 is proved.

***Corollary* 1.** *Let $\alpha\colon \mathbb{R} \to \mathbb{R}^2_1$ be a hyperbolic spiral and $\beta\colon \mathbb{R} \to \mathbb{R}^2$ be a circle centered at O with the same parameter, i.e., $\alpha(t) = e^{at}(\cosh t, \sinh t)$, $a \in \mathbb{R}$, and $\beta(t) = (\cos t, \sin t)$. Then their tensor product is a one-parameter subgroup in a Lie group M.*

Proof. In Theorem 6 taking $b = 0$, we find that β is a circle centered at O. Then their tensor product is a one-parameter subgroup in a Lie group M.

***Corollary* 2.** *Let $\alpha\colon \mathbb{R} \to \mathbb{R}^2_1$ be a Lorentzian circle centered at O and $\beta\colon \mathbb{R} \to \mathbb{R}^2$ be circle centered at O with the same parameter, i.e., $\alpha(t) = (\cosh t, \sinh t)$ and $\beta(t) = (\cos t, \sin t)$. Then their tensor product is a one-parameter subgroup in a Lie group M_1.*

Proof. Since $\|\alpha(t) \otimes \beta(t)\|_L = 1$, it follows that $\alpha(t) \otimes \beta(t) \subset M_1$. By taking $a = b = 0$, in Theorem 6, we find that α is a Lorentzian circle centered at O and β is a circle centered at O. Then their tensor product is a one-parameter subgroup in a Lie group M_1.

Theorem 7. *Let $\alpha\colon \mathbb{R} \to \mathbb{R}^2_1$ be a Lorentzian circle centered at O and $\beta\colon \mathbb{R} \to \mathbb{R}^2$ be circle centered at O with the same parameter, i.e., $\alpha(t) = (\cosh t, \sinh t)$, $\beta(t) = (\cos t, \sin t)$, and $\gamma(t) = \alpha(t) \otimes \beta(t)$ be their tensor product. Then, the left invariant vector field on $\gamma(t)$ is $X = X_2 + X_4$, where X_2 and X_4 are left invariant vector fields on M_1.*

Proof. Let us find the left invariant vector field on $\gamma(t)$ to the vector,

$$u = \left.\frac{d}{dt}\right|_{e=0}$$

$\eta(t) = (1, t, 0, t)$ is a curve with tangent vector u. Its image under L_g is the curve,

$$L_g(\eta(t)) = g\eta(t) = (x_1 1 + x_2 i + x_3 j + x_4 ij) \times (1 + ti + tij) =$$

$$= (x_1 - x_2 t + x_4 t) + i\,(x_1 t + x_2 - x_3 t) + j\,(-x_2 t + x_3 - x_4 t) + ij\,(x_1 t + x_3 t + x_4)\,.$$

Its tangent vector is,

$$L_g(\eta(t))(t) = (-x_2 + x_4) + i\,(x_1 - x_3) + j\,(-x_2 - x_4) + ij\,(x_1 + x_3)\,.$$

For the left invariant vector X we have

$$X = (-x_2 + x_4)\frac{\partial}{\partial x_1} + (x_1 - x_3)\frac{\partial}{\partial x_2} + (-x_2 - x_4)\frac{\partial}{\partial x_3} + (x_1 + x_3)\frac{\partial}{\partial x_4}.$$

Theorem 7 is proved.

Conclusion 1. *Let $\alpha\colon \mathbb{R} \to \mathbb{R}^2_1$ be a hyperbolic spiral* (*or a Lorentzian circle centered at O*) *and $\beta\colon \mathbb{R} \to \mathbb{R}^2$ be a spiral* (*or circle centered at O*) *with the same parameter. Then their tensor product is the maximal integral curve.*

Now, we want to classify these Lie groups as totally real or slant in semi-Euclidean space $\mathbb{R}^4_2$. In order to do so, consider the left invariant vector field on these groups which constitute pseudo-Hermitian structure which is given by $J = X_2$.

***Corollary* 3.** *Let $\alpha\colon \mathbb{R} \to \mathbb{R}^2_1$ be a Lorentzian circle centered at O, $\beta\colon \mathbb{R} \to \mathbb{R}^2$ be either a spiral or a circle centered at O, and $f = \alpha \otimes \beta$ be their tensor product immersion. Then the Lie group $f(t, s)$ is totally real Lorentzian immersion with respect to the pseudo-Hermitian structure J.*

Proof. From Theorem 1 we know that, if α is a Lorentzian circle centered at O then $f = \alpha \otimes \beta$ is totally real surface with respect to the pseudo-Hermitian structure J.

***Corollary* 4.** *Let $\alpha\colon \mathbb{R} \to \mathbb{R}^2_1$ be a hyperbolic spiral and $\beta\colon \mathbb{R} \to \mathbb{R}^2$ be either a circle centered at O or a spiral and $f = \alpha \otimes \beta$ be their tensor product. Then the Lie group $f\,(t, s)$ is a proper slant surface with respect to pseudo-Hermitian structure J on $\mathbb{R}^4_2$.*

Proof. From Theorem 3 we know that, if α is a hyperbolic spiral and β is either a circle centered at O or a spiral curve then $f = \alpha \otimes \beta$ is proper slant surface with respect to the pseudo-Hermitian structure J.

1. *Arslan K., Ezentas R., Mihai I., Murathan C., Özgür C.* Tensor product surfaces of a Euclidean space curve and a Euclidean plane curve // Beiträge Algebra Geom. – 2001. – **42**, № 2. – P. 523 – 530.

2. *Brickell F., Clark R. S.* Differentiable manifolds. – London: Van Nostrand Reinhold Comp., 1970.
3. *Chen B.* Geometry of slant submanifolds. – Katholieke Univ. Leuven, 1990.
4. *Decruyenaere F., Dillen F., Verstraelen L., Vrancken L.* The semiring of immersions of manifolds // Beitrage Algebra Geom. – 1993. – **34**. – P. 209 – 215.
5. *Ilarslan K., Nesovic E.* Tensor product surfaces of a Lorentzian space curve and a Euclidean plane curve // Kuwait J. Sci. Engrg. – 2007. – **34**, № 2A. – P. 41 – 55.
6. *Ilarslan K., Nesovic E.* Tensor product surfaces of a Euclidean space curve and a Lorentzian plane curve // Different. Geometry-Dynamical Systems. – 2007. – **9**. – P. 47 – 57.
7. *Karger A., Novak J.* Space kinematics and Lie groups. – Gordan and Breach Publ., 1985.
8. *Mihai I., Rosca R., Verstraelen L., Vrancken L.* Tensor product surfaces of Euclidean planar curves // Rend. Semin. Mat. Messina. Ser. II. – 1993. – **18**, № 3. – P. 173 – 185.
9. *Mihai I., Van De Woestyne I., Verstraelen L., Walrave J.* Tensor product surfaces of a Lorentzian planar curves // Bull. Inst. Math. Acad. Sinica. – 1995. – **23**. – P. 357 – 363.
10. *Mihai I., Van De Woestyne I., Verstraelen L., Walrave J.* Tensor product surfaces of Lorentzian plane curve and a Euclidean plane curve // Rend. Semin. Mat. Messina. Ser. II. – 1994–1995. – **3**, № 18. – P. 147 – 185.
11. *O'Neill B.* Semi-Riemannian geometry with applicaions to relativity. – New York: Acad. Press, 1983.
12. *Özkaldı S., Yaylı Y.* Tensor product surfaces and Lie groups // Bull. Malays. Math. Sci. Soc. (2). – 2010. – **33**, № 1. – P. 69 – 77.
13. *Price G. B.* An introduction to multicomplex spaces and functions. – Marcel Dekker Inc., 1990.

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