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A RESULT ON GENERALIZED DERIVATIONS ON RIGHT IDEALS OF PRIME RINGS

(ОДИН РЕЗУЛЬТАТ) ПРО УЗАГАЛЬНЕНЕ ДИФЕРЕНЦІЮВАННЯ НА ПРАВИХ ІДЕАЛАХ ПРОСТИХ КІЛЕЦЬ

Let R be a prime ring of characteristic not 2 and let I be a nonzero right ideal of R. Let U be the right Utumi quotient ring of R and let C be the center of U. If G is a generalized derivation of R such that $[[G(x),x],G(x)]=0$ for all $x\in I$, then R is commutative or there exist $a,b\in U$ such that $G(x)=ax+xb$ for all $x\in R$ and one of the following assertions is true:

(1) $(a-\lambda)I=(0)=(b+\lambda)I$ for some $\lambda\in C$,

(2) $(a-\lambda)I=(0)$ for some $\lambda\in C$ and $b\in C$.

Нехай R — просте кільце, характеристика якого не дорівнює 2, а I — ненульовий правий ідеал R. Нехай U — праве фактор-кільце Утумі кільця R, а C — центр U. Якщо G є узагальненим диференціюванням R таким, що $[[G(x),x],G(x)]=0$ для всіх $x\in I$, то R є комутативним або існують $a,b\in U$ такі, що $G(x)=ax+xb$ для всіх $x\in R$ і виконується одне з наступних тверджень:

(1) $(a-\lambda)I=(0)=(b+\lambda)I$ для деякого $\lambda\in C$,

(2) $(a-\lambda)I=(0)$ для деяких $\lambda\in C$ та $b\in C$.

1. Introduction. Throughout this paper R will always denote a prime ring with center $Z(R)$, extended centroid C, right Utumi quotient ring U (sometimes, as in [2], U is called the maximal right ring of quotients), and two-sided Martindale quotient ring Q (see [2] for the definitions). For any $x,y\in R$, the commutator of x and y is denoted by $[x,y]$ and defined to be $xy-yx$.

An additive mapping d from R into itself is called a derivation of R if $d(xy)=d(x)y+xd(y)$ holds for all $x,y\in R$. An additive mapping $g\colon R\to R$ is called a generalized derivation of R if there exists a derivation d of R such that $g(xy)=g(x)y+xd(y)$ for all $x,y\in R$ [10]. Obviously any derivation is a generalized derivation. Moreover, other basic examples of generalized derivations are the mappings of the form $x\mapsto ax+xb$, for $a,b\in R$. A generalized derivation in this form is called (generalized) inner. Many authors have studied generalized derivations in the context of prime and semiprime rings (see [1, 10, 13, 14]).

In [13], T. K. Lee extended the definition of a generalized derivation as follows. By a generalized derivation he means an additive mapping $g\colon I\to U$ such that $g(xy)=g(x)y+xd(y)$ for all $x,y\in I$, where I is a dense right ideal of the prime ring R and d is a derivation from I into U. He also proved that every generalized derivation can be uniquely extended to a generalized derivation of U, and moreover, there exist $a\in U$ and a derivation d of U such that $g(x)=ax+d(x)$ for all $x\in U$ [13] (Theorem 3).

In [7], De Filippis proved that if R is a prime ring of characteristic not 2 and G is a generalized derivation of R such that $[[G(x),x],G(x)]=0$ for all $x\in R$, then either R is commutative or there exists $\lambda\in C$ such that $G(x)=\lambda x$ for all $x\in R$. In the same paper, he uses his result to prove a theorem concerning noncommutative Banach algebras. More precisely, he proves the following:

Let R be a noncommutative Banach algebra with a continuous generalized derivation $G = L_a + d$, where L_a denotes the left multiplication by $a \in R$ and d is a derivation of R. If $[[G(x), x], G(x)] \in \in \mathrm{rad}(R)$ (the Jacobson radical of R) for all $x \in R$, then $[a, R] \subseteq \mathrm{rad}(R)$ and $d(R) \subseteq \mathrm{rad}(R)$.

In [6], V. De Filippis and M. S. Tammam El-Sayiad considered this time a similar problem on a non-central Lie ideal L of a prime ring R of characteristic not 2. It was proved that if G is a generalized derivation of R such that $[[G(u), u], G(u)] \in Z(R)$ for all $u \in L$, a non-central Lie ideal of R, then either there exists $\lambda \in C$ such that $G(x) = \lambda x$ for all $x \in R$ or $G(x) = ax + xa + \lambda x$ for all $x \in R$ and for some $a \in U$, $\lambda \in C$ and R satisfies the standard identity s_4.

The aim of the present paper is to extend Filippis' main result in [7] to the right ideals in prime rings. Precisely, we will prove the following theorem.

Main theorem. *Let R be a prime ring of characteristic different from 2 with the extended centroid C and I be a nonzero right ideal of R. If G is a generalized derivation of R such that*

$$[[G(x), x], G(x)] = 0$$

for all $x \in I$, then R is commutative or there exist $a, b \in U$ such that $G(x) = ax + xb$ for all $x \in R$ and one of the following holds:

(i) $(a - \lambda)I = (0) = (b + \lambda)I$ *for some* $\lambda \in C$,

(ii) $(a - \lambda)I = (0)$ *for some* $\lambda \in C$ *and* $b \in C$.

Before we proceed, we give some illustrative examples.

***Example* 1.** Let $R = M_n(F)$ be the ring of all $(n \times n)$-matrices over a field F, and I be the right ideal of R generated by the matrix unit e_{11}, that is $I = e_{11}R$. We note that the extended centroid C of R coincides with its center $Z(R) = F$ which consists of all scalar matrices (here we identify F with the set of all scalar matrices up to isomorphism).

1. Let $a, b \in R$ be such that $a_{i1} = 0 = b_{i1}$ for all $2 \leq i \leq n$ and $a_{11} = \lambda = -b_{11}$. Then $(a - \lambda)I = (0) = (b + \lambda)I$ (here of course we identify λ with the scalar matrix $\lambda \cdot 1$). Define the generalized derivation of R by $G(r) = ar + rb$ for all $r \in R$. Then

$$[[G(x), x], G(x)] = [[ax + xb, x], ax + xb] =$$

$$= [[x(b + \lambda), x], x(b + \lambda)] = [-x^2(b + \lambda), x(b + \lambda)] = 0$$

for all $x \in I$.

2. Let $c, d \in R$ with $d \in Z(R)$ and $c_{i1} = 0$ for all $2 \leq i \leq n$, $c_{11} = \lambda$. Define now $G(r) = = cr + rd = (c + d)r$ for all $r \in R$. Then since $(c - \lambda)I = (0)$ and $d \in Z(R)$, it is readily verified that

$$[G(x), x] = [cx + xd, x] = [\lambda x, x] = 0$$

for all $x \in I$, and hence $[[G(x), x], G(x)] = 0$ follows.

2. Preliminaries. In what follows, R will be a prime ring. The related object we need to mention is the right Utumi quotient ring U of R. The definitions, the axiomatic formulations and the properties of this quotient ring U can be found in [2].

In any case, when R is a prime ring, all we need to know about U is that

(1) $R \subseteq U$;

(2) U is a prime ring;

(3) The center of U, denoted by C, is a field which is called the extended centroid of R.

We will make a frequent use of the theory of generalized polynomial identities and differential identities (see [2, 11, 12, 15]). In particular we need to recall the following:

***Remark* 1** [4]**.** If R is a prime ring and I is a non-zero right ideal of R, then I, IR and IU satisfy the same generalized polynomial identities with coefficients in U.

***Remark* 2** [11]**.** Let R be a prime ring, d a nonzero derivation of R and I a nonzero two-sided ideal of R. Let $f(x_1,\ldots,x_n,d(x_1),\ldots,d(x_n))$ be a differential identity in I, that is

$$f(r_1,\ldots,r_n,d(r_1),\ldots,d(r_n))=0$$

for all $r_1,\ldots,r_n\in I$. Then one of the following holds:

(i) d is an inner derivation of Q, in the sense that there exists $q\in Q$ such that $d(x)=[q,x]$ for all $x\in R$, and I satisfies the generalized polynomial identity

$$f(r_1,\ldots,r_n,[q,r_1],\ldots,[q,r_n]),$$

(ii) I satisfies the generalized polynomial identity

$$f(x_1,\ldots,x_n,y_1,\ldots,y_n).$$

We also need to mention the following fact about generalized polynomials. It enables us to decide whether a given generalized identity of a prime ring is a trivial identity or not.

***Remark* 3.** Denote by $T=U*_C C\{X\}$ the free product over C of the C-algebra U and the free C-algebra $C\{X\}$, with X a countable set consisting of non-commuting indeterminates $\{x_1,\ldots,x_n,\ldots\}$. The elements of T are called generalized polynomials with coefficients in U. Let $a_1,\ldots,a_k\in U$ be linearly C-independent, and

$$a_1g_1(x_1,\ldots,x_n)+\ldots+a_kg_k(x_1,\ldots,x_n)=0\in T$$

for some $g_1,\ldots,g_k\in T$. If $g_i(x_1,\ldots,x_n)=\sum_{j=1}^n x_jh_j(x_1,\ldots,x_n)$ and $h_j\in T$, then $g_1,\ldots,g_k$ are the zero element of T. The conclusion holds if

$$g_1(x_1,\ldots,x_n)a_1+\ldots+g_k(x_1,\ldots,x_n)a_k=0\in T$$

and $g_i(x_1,\ldots,x_n)=\sum_{j=1}^n h_j(x_1,\ldots,x_n)x_j$ for $h_j\in T$ (see [4]).

2. Results. We start with an easy lemma that will be used in the sequel.

Lemma 1. *Let R be a prime ring, I a nonzero right ideal of R. If $a\in R$ is such that $[ax,x]=0$ for all $x\in I$, then $(a-\lambda)I=(0)$ for some $\lambda\in C$.*

Proof. Linearizing $[ax,x]=0$, one gets

$$[a,x]y+[a,y]x=0 \tag{1}$$

for all $x,y\in I$. Letting $y=yr$ in (1) with $r\in R$ and using (1) again, it follows

$$[a,y][x,r]=y[a,r]x. \tag{2}$$

Letting now $x=xs$ in (2) with $s\in R$, we get $[a,I]I[R,R]=(0)$. Hence $[a,I]I=(0)$ or R is commutative. Of course $[a,I]I=(0)$ if R is commutative. Then $(a-\lambda)I=(0)$ for some $\lambda\in C$ by [3] (Lemma).

The following lemma is crucial and will be used in the proof of the inner case.

Lemma 2. *Let R be a prime ring of characteristic different from* 2, *I a nonzero right ideal of R and $a, b \in R$.*

(i) *If $[[ax, x], ax] = 0$ for all $x \in I$, then $(a - \lambda)I = (0)$ for some $\lambda \in C$.*

(ii) *If $[[xb, x], xb] = 0$ for all $x \in I$, then $bI = (0)$ or $b \in Z(R)$.*

Proof. (i) By the hypothesis

$$[[ax, x], ax] = 0 \tag{3}$$

for all $x \in I$. By Theorem 2 in [4] we see that (1) holds for all $x \in IU$. Replacing R and I with U and IU respectively, we may assume that $IC = I$ and R is centrally closed over its center C. In case C is infinite, set $\overline{R} = R \otimes_C \overline{C}$ and $\overline{I} = I \otimes_C \overline{C}$ where $\overline{C}$ is the algebraic closure of C. Then $\overline{R}$ is centrally closed over its center $\overline{C}$ by [8], and (3) holds for all $x \in \overline{I}$ by a standard argument. Thus, replacing R, I and C with $\overline{R}$, $\overline{I}$ and $\overline{C}$ respectively, we may assume further that C is either finite or algebraically closed. We proceed to show that $(a - \lambda)I = (0)$ for some $\lambda \in C$.

Let $u \in I$, then

$$[[aux, ux], aux] = 0$$

for all $x \in R$. Assume on the contrary that au and u are C-independent for some $u \in I$. We claim that

$$[[auX, uX], auX] \tag{4}$$

is a non-trivial generalized polynomial identity (GPI for short) for R. For otherwise,

$$au(XuXauX - XauXuX + XuXauX) - u(XauXauX)$$

is the zero element of $T = U *_C C\{X\}$. Then by Remark 3

$$uXauXauX = 0 \in T = U *_C C\{X\}$$

implying $au = 0$, contrary to our assumption on au and u. Therefore (4) is a nontrivial GPI for R. Thus R is a primitive ring with a nonzero socle $\mathrm{soc}(R) = H$ with C as the associated division ring by Martindale's theorem [15]. Now I and IH both satisfy (3), and so replacing I with IH, we may assume that $I \subseteq H$.

Let $e = e^2 \in I$ be any idempotent. Then

$$[[aere, ere], aere] = 0 \tag{5}$$

for all $r \in R$. Left multiplying (5) by e yields that

$$[[(eae)(ere), (ere)], (eae)(ere)] = 0$$

for all $r \in R$. Since eRe is a prime ring, $\mathrm{char}(eRe) = \mathrm{char}(R) \neq 2$ and $eae \in eRe$, we conclude that either eRe is commutative or $eae \in Z(eRe) = Ce$ by [7] (Proposition 1). In any case we have $eae \in Ce$. On the other hand,

$$[[aer(1 - e), er(1 - e)], aer(1 - e)] = 0$$

for all $r \in R$. Expanding the commutator we arrive at

$$er(1-e)aer(1-e)aer(1-e) = 0$$

for all $r \in R$. Therefore $((1-e)aer)^4 = 0$ for all $r \in R$, and so $(1-e)aeR$ is a nil right ideal of bounded index. Hence $(1-e)ae = 0$ by Levitzki's theorem [9] (Lemma 1.1). Now $ae = eae \in Ce$ for every idempotent $e \in I$. Since I is completely reducible right H-module, every element of I is contained in fH for some $f = f^2 \in I$. Then, for any $x \in I$, there exists an idempotent $f \in I$ such that $x = fx$, and so, it follows that

$$ax = afx = fafx \in Cfx = Cx.$$

Hence we see that $[ax, x] = 0$ for all $x \in I$, and then by Lemma 1 we have $(a - \lambda)I = (0)$ for some $\lambda \in C$.

(ii) Even if the proof of this part is very similar to the one in (i), we give its proof here for the sake of completeness.

We now have

$$[[xb, x], xb] = 0 \qquad (6)$$

for all $x \in I$ by the hypothesis. Again by Theorem 2 in [4] we see that (6) holds for all $x \in IU$. Replacing R and I with U and IU respectively, we may assume that $IC = I$ and R is centrally closed over its center C. As in (i) replacing R, I and C with $\overline{R}$, $\overline{I}$ and $\overline{C}$ respectively, when C is infinite, we may assume further that C is either finite or algebraically closed.

Let $u \in I$, then

$$[[uxb, ux], uxb] = 0 \qquad (7)$$

for all $x \in R$. Assume on the contrary that $b \notin C$ and $bI \neq (0)$. Then there exists $u \in I$ such that $bu \neq 0$. We claim that

$$[[uXb, uX], uXb]$$

is a non-trivial GPI for R. If not,

$$(uXbuXuX - uXuXbuX + uXbuXuX)b - (uXbuXbuX)$$

is the zero element of $T = U *_C C\{X\}$. Then by Remark 3 again,

$$uXbuXbuX = 0 \in T = U *_C C\{X\},$$

and hence $bu = 0$, contrary to our assumption. Therefore (7) is a non-trivial GPI for R. In the present case, R is a primitive ring with a nonzero socle $\mathrm{Soc}(R) = H$ [15]. Moreover, since (6) is also satisfied by IH, we may assume further that $I \subseteq H$ by replacing I with IH. Similar to above, let $e = e^2 \in I$ be an idempotent. Then

$$[[ereb, ere], ereb] = 0 \qquad (8)$$

for all $r \in R$. Right multiplying (8) by e yields that

$$[[(ere)(ebe),(ere)],(ere)(ebe)]=0$$

for all $r \in R$. Since eRe is a prime ring, $\text{char}(eRe) = \text{char}(R) \neq 2$ and $ebe \in eRe$, we conclude that either eRe is commutative or $ebe \in Z(eRe) = Ce$ by [7] (Proposition 1). In any case we have $ebe \in Ce$. On the other hand,

$$[[er(1-e)b, er(1-e)], er(1-e)b]=0$$

for all $r \in R$. Expanding the commutator we arrive at

$$er(1-e)ber(1-e)ber(1-e)=0$$

for all $r \in R$. Therefore $(1-e)beR$ is a nil right ideal of bounded index. Hence $(1-e)be = 0$ again by Levitzki's theorem [9] (Lemma 1.1). Thus, $be = ebe \in Ce$ for every idempotent $e \in I$. Since I is completely reducible right H-module, every element of I is contained in fH for some $f = f^2 \in I$. Then, for any $x \in I$, there exists an idempotent $f \in I$ such that $x = fx$. Therefore, it follows that

$$bx = bfx = fbfx \in Cfx = Cx$$

for all $x \in I$. Hence we see that $[bx, x] = 0$ for all $x \in I$, and so $(b-\mu)I = (0)$ for some $\mu \in C$ by Lemma 1. Now (6) reduces to

$$0 = [[xb,x],xb] = x^3\mu(b-\mu)$$

for all $x \in I$. In particular, $e\mu(b-\mu) = 0$ and $(e + er(1-e))\mu(b-\mu) = 0$ for all $e = e^2 \in I$ and $r \in R$. This implies $eR\mu(b-\mu) = 0$, that is to say $\mu = 0$ or $b = \mu \in C$. We must have $\mu = 0$ since $b \notin C$. But then $bI = (0)$, again a contradiction.

Lemma 3. *Let R be a prime ring of characteristic different from* 2, *I a nonzero right ideal of R and $a, b \in R$. If*

$$[[ax+xb,x],ax+xb]=0 \tag{9}$$

for all $x \in I$, then one of the following holds:

(i) $(a-\lambda)I = (0) = (b+\lambda)I$ *for some* $\lambda \in C$,

(ii) $(a-\lambda)I = (0)$ *for some* $\lambda \in C$ *and* $b \in Z(R)$.

Proof. Let $u \in I$. Then

$$[[aux+uxb,ux],aux+uxb]=0 \tag{10}$$

for all $x \in R$, and hence for all $x \in U$. Replacing R and I with U and IU, we may assume that C is just the center of R. We want to show that either R is a GPI-ring or the lemma holds. Therefore we assume that R is not a GPI-ring. Assume further that au and u are C-independent for some $u \in I$. Then R satisfies

$$[[auX+uXb,uX],auX+uXb].$$

Expansion of (10) yields that

$$auf(x)+ug(x)=0$$

for all $x \in R$, where

$$f(x) = 2xuxaux + 2xuxuxb - xauxux - xuxbux$$

and

$$g(x) = 2xbuxaux + 2xbuxuxb - xauxaux - xauxuxb - xuxbaux-$$

$$-xuxbuxb - xbauxux - xbuxbux.$$

Since R satisfies no non-trivial GPI, we must have

$$auf(X) = 0 \in T = U *_C C\{X\}$$

by Remark 3. Hence

$$2auXuXauX + 2auXuXuXb - auXauXuX - auXuXbuX \tag{11}$$

is the zero element of $T = U *_C C\{X\}$. If now 1 and b are C-dependent, that is $b \in C$, then (9) reduces to

$$[[(a+b)x, x], (a+b)x] = 0$$

for all $x \in I$. It follows from Lemma 2(i) that $(a+b-\alpha)I = (0)$ for some $\alpha \in C$. Set $\lambda = \alpha - b \in C$, and so $(a-\lambda)I = (0)$ for some $\lambda \in C$ and $b \in Z(R)$ (since $b \in R$). This gives (ii).

Therefore we may assume that 1 and b are C-independent. We rewrite (11) in the form

$$(2a(uX)^2auX - auXa(uX)^2 - a(uX)^2buX) + (2a(uX)^3)b = 0 \in T.$$

We conclude as above that $2a(uX)^3b = 0$ which is impossible unless $\operatorname{char} R = 2$ or $b = 0$ or $au = 0$, a contradiction. Until now we have shown that if au and u are C-independent for some $u \in I$, then either the lemma holds or R is a GPI-ring. So we may assume that au and u are C-dependent for all $u \in I$. Then $[au, u] = 0$ for all $u \in I$, and this implies $(a-\lambda)I = (0)$ for some $\lambda \in C$ by Lemma 1. Now (9) reduces to

$$[[x(b+\lambda), x], x(b+\lambda)] = 0$$

for all $x \in I$. Hence by Lemma 2(ii), we have $b \in C = Z(R)$ or $(b+\lambda)I = (0)$, giving (i) and (ii) simultaneously.

We are now in a position to consider the case when R is a GPI-ring. Then R is a primitive ring with a nonzero socle H with C as the associated division ring by Martindale's theorem [15]. Moreover, since I and IH both satisfy (9), after replacing I with IH we may assume that $I \subseteq H$. Let $e = e^2 \in I$ be any idempotent element. Then

$$[[aere + ereb, ere], aere + ereb] = 0 \tag{12}$$

for all $r \in R$. Now left and right multiplying (12) by $1 - e$ yields that

$$2(1-e)aererereb(1-e) = 0,$$

and so

$$(1-e)aererereb(1-e)=0$$

for all $r \in R$ since $\mathrm{char}(R) \neq 2$. It follows by the primeness of R that $(1-e)ae=0$ or $eb(1-e)=0$ by the Theorem in [16]. If $(1-e)ae=0$, then right multiplication of (12) by e yields

$$\Big[[(eae)(ere)+(ere)(ebe), ere], (eae)(ere)+(ere)(ebe)\Big]=0 \qquad (13)$$

for all $r \in R$. Similarly, if $eb(1-e)=0$, then the left multiplication of (12) by e gives us the same identity in (13). Thus in any case we have

$$\big[[a'x+xb', x], a'x+xb'\big]=0 \qquad (14)$$

for all $x \in eRe$, where $a'=eae$ and $b'=ebe$. Since eRe is a prime ring, $\mathrm{char}(eRe)=\mathrm{char}(R)\neq 2$ and $a', b' \in eRe$, (14) implies that either eRe is commutative or $a', b' \in Z(eRe)=Ce$ by [7] (Proposition 1). In any case we have $a', b' \in Ce$.

Now we claim that for a given $e=e^2 \in I$, if $eb(1-e)=0$, then we must have $(1-e)ae=0$, too. So assume on the contrary that $eb(1-e)=0$ but $(1-e)ae \neq 0$ for some $e=e^2 \in I$. Pick any $\alpha \in C$, $r \in R$ and set $q=\alpha er(1-e)$. Then $q^2=0$ and the mapping $\varphi(x)=(1+q)x(1-q)$, $x \in R$, defines a C-automorphism of R such that $\varphi(I)\subseteq I$. Thus

$$\Big[[\varphi(a)x+x\varphi(b), x], \varphi(a)x+x\varphi(b)\Big]=0 \qquad (15)$$

for all $x \in I$. As above (15) implies that $(1-e)\varphi(a)e=0$ or $e\varphi(b)(1-e)=0$. If $(1-e)\varphi(a)e=0$, then one gets that

$$0=(1-e)\varphi(a)e=(1-e)ae$$

which is a contradiction. So we must have $e\varphi(b)(1-e)=0$. By calculation we arrive at

$$\alpha^2 er(1-e)ber(1-e)+\alpha eber(1-e)-\alpha er(1-e)b(1-e)=0. \qquad (16)$$

In particular, taking $\alpha=1$ in (16) it follows that

$$er(1-e)ber(1-e)+eber(1-e)-er(1-e)b(1-e)=0.$$

In a similar fashion, taking this time $\alpha=-1$ in (16) one gets

$$er(1-e)ber(1-e)-eber(1-e)+er(1-e)b(1-e)=0.$$

Comparing these last two equations and using the fact that $\mathrm{char}(R)\neq 2$, we obtain

$$er(1-e)ber(1-e)=0$$

for all $r \in R$. Hence $(1-e)be=0$, and so

$$eb=ebe=be.$$

Let $s \in R$ and $f=e+es(1-e) \in I$. We note that $(1-f)af \neq 0$, and so we must have $fb(1-f)=0$. But this implies $bf=fb$ as above. Hence

$$\big[b, e + es(1-e)\big] = 0 \tag{17}$$

for all $s \in R$. Now (17) implies $b \in C$ by [5] (Lemma 1). So (9) reduces to

$$\big[[(a+b)x, x], (a+b)x\big] = 0$$

for all $x \in I$. Then for any $r \in R$, we have

$$0 = [[(a+b)er(1-e), er(1-e)], (a+b)er(1-e)],$$

that is

$$er(1-e)aer(1-e)aer(1-e) = 0.$$

Therefore $(1-e)ae = 0$ which is a contradiction. This proves our claim. So we have $(1-e)ae = 0$, that is $ae = eae \in Ce$ for all $e = e^2 \in I$. Then since I is completely reducible right H-module, every element of I is contained in fH for some idempotent $f \in I$. Let $x \in I$, then $fx = x$ for some $f = f^2 \in I$. Hence

$$ax = afx = fafx \in Cfx = Cx.$$

This means $[ax, x] = 0$ for all $x \in I$, and therefore $(a-\lambda)I = (0)$ for some $\lambda \in C$ by Lemma 1. From (9) we see that

$$\big[[x(b+\lambda), x], x(b+\lambda)\big] = 0$$

for all $x \in I$. Henceforth we have $(b+\lambda)I = (0)$ or $b \in Z(R)$ by Lemma 2(ii). This proves the lemma.

We are now ready to prove our main theorem.

Main theorem. *Let R be a prime ring of characteristic different from 2 with the extended centroid C and I be a nonzero right ideal of R. If G is a generalized derivation of R such that*

$$\big[[G(x), x], G(x)\big] = 0 \tag{18}$$

for all $x \in I$, then R is commutative or there exist $a, b \in U$ such that $G(x) = ax + xb$ for all $x \in R$ and one of the following holds:

(i) $(a-\lambda)I = (0) = (b+\lambda)I$ *for some* $\lambda \in C$,

(ii) $(a-\lambda)I = (0)$ *for some* $\lambda \in C$ *and* $b \in C$.

Proof. As we have already noted that every generalized derivation G on a dense right ideal of R can be uniquely extended to U and assumes form $G(r) = pr + d(r)$ for some $p \in U$ and a derivation d of U. Then

$$\big[[px + d(x), x], px + d(x)\big] = 0 \tag{19}$$

for all $x \in I$, and hence for all $x \in IU$ since I and IU satisfy the same differential identities [12]. If $d = 0$, then we get that

$$\big[[px, x], px\big] = 0$$

for all $x \in IU$. This last equation implies that $(p-\lambda)IU = (0)$ for some $\lambda \in C$ by Lemma 2(i). Therefore $g(r) = ar$ for all $r \in R$ and $(a-\lambda)I = (0)$ where $a = p$. So we may assume that $d \neq 0$.

In light of Kharchenko's theorem (Remark 2), we divide the proof into two cases:

Case 1. Let d be the X-inner derivation induced by the element $q \in U - C$. Then by (19) we see that

$$\big[[(p+q)x - xq, x], (p+q)x - xq\big] = 0 \tag{20}$$

for all $x \in I$. As we noted above (20) is also satisfied by IU. Therefore replacing R and I with U and IU respective, we may assume that $p, q \in R$. Set $a = p + q$ and $b = -q$ for simplicity. Now it follows from Lemma 3 that either $(a - \lambda)I = (0) = (b + \lambda)I$ for some $\lambda \in C$ or $(a - \lambda)I = (0)$ for some $\lambda \in C$ and $b \in C$.

Case 2. Let now d be an outer derivation of U. To continue the proof we first linearize (12). By replacing x with $x + y$ in (18) and using (18) again, we end up with

$$[[G(x), x], G(y)] + [[G(x), y], G(x)] + [[G(y), x], G(x)] +$$

$$+[[G(x), y], G(y)] + [[G(y), x], G(y)] + [[G(y), y], G(x)] = 0 \tag{21}$$

for all $x, y \in I$. Replacing x with $-x$ in (21) and adding up the resulting equation to (21) yields that

$$[[G(x), x], G(y)] + [[G(x), y], G(x)] + [[G(y), x], G(x)] = 0 \tag{22}$$

for all $x, y \in I$ since $\operatorname{char} R \neq 2$. Take xr instead of x in (22) with $r \in R$ to get

$$[[G(x)r + xd(r), xr], G(y)] + [[G(x)r + xd(r), y], G(x)r + xd(r)] +$$

$$+[[G(y), xr], G(x)r + xd(r)] = 0 \tag{23}$$

for all $x, y \in I$ and $r \in R$. By Kharchenko's theorem, since d is an outer derivation, R satisfies the identity:

$$[[G(x)r + xs, xr], G(y)] + [[G(x)r + xs, y], G(x)r + xs] + [[G(y), xr], G(x)r + xs] = 0$$

for all $x, y \in I$ and $r, s \in R$. In particular, R satisfies the blended component

$$[[xs, y], xs] = 0$$

for all $x, y \in I$ and $s \in R$ (and hence for all $s \in U$). So for $s = 1$ in this last equation we have $[[x, y], x] = 0$ for all $x, y \in I$. Then for any $x, y, z \in I$ we have

$$0 = [[x, yz], x] = 2[x, y][z, x],$$

and so

$$[x, y][x, z] = 0$$

since $\operatorname{char} R \neq 2$. Let now $z = zr$ in this last equation to get

$$[x, y]z[x, r] = 0$$

for all $x, y, z \in I$ and $r \in R$. Therefore for any $x \in I$, we see that $[x, I]I = (0)$ or $x \in Z(R)$. Thus we conclude that $[I, I]I = (0)$ or R is commutative. If the first possibility holds, then it follows from $[[x, y], x] = 0$, $x, y \in I$, that $x[x, y] = 0$. This clearly implies the commutativity of R, and so the theorem is proved.

We finish with an example which shows that the characteristic assumption in the theorem cannot be removed.

Example **2.** Let F be a field with $\operatorname{char} F = 2$, $R = M_2(F)$ and a be any element of R. Then for the mapping $G(x) = [a, x]$, $x \in R$, one can easily see that for every $x \in R$, $[[G(x), x], G(x)] = = [G(x)^2, x] = 0$ since $G(x)^2 \in Z(R)$ for all $x \in R$.

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