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D. J. S. Robinson (Univ. Illinois, USA)

MINIMALITY AND SYLOW-PERMUTABILITY IN LOCALLY FINITE GROUPS

МІНІМАЛЬНІСТЬ І СИЛОВСЬКА КОМУТАТИВНІСТЬ У ЛОКАЛЬНО СКІНЧЕННИХ ГРУПАХ

We give a complete classification of the locally finite groups that are minimal with respect to Sylow-permutability being intransitive.

Наведено повну класифікацію локально скінченних груп, мінімальних із нетранзитивною властивістю силовської комутативності.

1. Introduction and results. A subgroup H of a group G is called *Sylow-permutable*, or *S-permutable*, if $HP = PH$ for every Sylow subgroup P of G. If S-permutability is transitive in G, i.e., if the S-permutability of H in K and of K in G always imply that H is S-permutable in G, then G is called a *PST-group*. Now it was shown by Kegel [1] that an S-permutable subgroup of a finite group is subnormal. From this it follows easily that a finite group is a PST-group if and only if every subnormal subgroup is S-permutable.

In recent years there has been an upsurge of interest in finite PST-groups (see [2–9]). For example, Agrawal [2] characterized finite soluble PST-groups, proving the following basic result.

Theorem. *A finite group G is a soluble PST-group if and only if it has an abelian normal subgroup L of odd order such that G/L is nilpotent, $\pi(L) \cap \pi(G/L)$ is empty, and elements of G induce power automorphisms in L.*

Notice the consequence: all finite soluble PST-groups are supersoluble. The structure of finite insoluble PST-groups has recently been described in Robinson [9]. Also, in [4] Ballester-Bolinches et al. gave an interesting property of a finite PST-group G: the p-chief factors occuring below G' are G-isomorphic, as are those occuring above G'.

Our aim in the present work is to determine the minimal non-PST-groups which are locally finite. The motivation for investigating such minimal classes is that detailed knowledge of groups that just fail to have a group theoretic property is likely to give some insight into just what makes a group have a property. The minimal classes have been determined for a number of properties, for example T, the property that normality is transitive [10].

It turns out that all the locally finite, minimal non-PST-groups are finite, and they fall into five distinct types, as delineated below.

Type I. Let p and q be primes such that $p \equiv 1 \pmod{q^f}$ where $q^f > 1$. Let i be the least positive primitive q^f-th root of unity modulo p. Put $j = 1 + kq^{f-1}$ where $0 < k < q$. Define

$$G_1 = X \ltimes A$$

where $X = \langle x \rangle$ has order q^r with $r \geq f$, $A = \langle a, b \rangle$ is elementary abelian of order p^2, and $a^x = a^i$, $b^x = b^{ij}$.

Type II. Let p and q be distinct primes such that $p \not\equiv 1 \pmod{q}$. Let z be a primitive q-th root of unity modulo p and denote by F the field $\mathbb{Z}_p(z)$. Define

$$G_2 = X \ltimes F^+$$

where $X = \langle x \rangle$ has order $q^r > 1$ and x acts on F^+ via multiplication by z.

Type III. Let p and q be primes such that $p \equiv 1 \pmod q$, and write q^f for the highest power of q dividing $p-1$. Let A be an elementary abelian p-group with basis $\{a_0, a_1, \ldots, a_{q-1}\}$ and let $X = \langle x \rangle$ have order q^r where $r > f$. Set i equal to the least positive primitive q^f-th root of unity modulo p. Define

$$G_3 = \langle x \rangle \ltimes A$$

where $a_i^x = a_{i+1}$ for $0 \le i < q-1$ and $a_{q-1}^x = a_0^i$.

Type IV. $G_4 = X \ltimes Q$ where Q is a quaternion group of order 8, $X = \langle x \rangle$ has order 3^r and x permutes cyclically the three subgroups of Q with order 4.

Type V. Let p and q be distinct primes such that the exponent of p modulo q is even, say $2m$. Let P be a (non-abelian) special p-group of rank $2m$ which can be generated by elements of order p. Take $X = \langle x \rangle$ to have order $q^r > 1$ and let x induce an automorphism in P such that P/P' is a simple $\mathbb{Z}_p X$-module and $[P', X] = 1$. Define

$$G_5 = X \ltimes P.$$

Our main conclusion is the following theorem.

Theorem 1. *The locally finite, minimal non-PST-groups are precisely the groups of Types* I *to* V.

We shall make some comments on the classification. First it is easy to check that each of the five types of group is 2-generator and nilpotent-by-cyclic, and involves just two primes in its order. On the basis of this observation we can establish:

Corollary. *Let G be a locally finite group such that each 2-generator, nilpotent-by-cyclic subgroup with order divisible by two primes is a PST-group. Then G is a PST-group.*

For if G is finite but non-*PST*, it must contain a minimal non-*PST*-subgroup, which is impossible by the preceding remark. So G is infinite and every finite subgroup is *PST*, whence G is *PST* by Proposition 1 below. More generally, Theorem 1 can be used to establish sufficient conditions for a finite group to be a soluble *PST*-group: for example, the results in [7], Theorem C can be proved in this manner.

The proof of Theorem 1 provides additional srtuctural information about the groups of Type V, namely

$$|P'| \le p^m.$$

In addition a scheme for constructing all groups of Type V is given in Section 6 below.

Finally, it is a routine matter to verify with the aid of Agrawal's theorem that any group of Types I – V is a minimal non-*PST*-group. We therefore concentrate on demonstrating that only this groups can arise.

Notation: $\pi(G)$ — the set of primes dividing the orders of elements of G, $\mathrm{Syl}_p(G)$ — the set of Sylow p-subgroups of G, $\varphi_\infty(G)$ — the limit of the lower central series of a finite group G.

2. Finiteness of the groups. The first step in the proof of Theorem 1 is to show that a locally finite, minimal non-*PST*-group is finite. This is an immediate consequence of the following result.

Proposition 1. *Let G be a locally finite group in which every finite subgroup is a PST-group. Then the following hold.*

(i) *There is an abelian normal subgroup L containing no involutions such that G/L is locally nilpotent, $\pi(L) \cap \pi(G/L)$ is empty, and elements of G induce power automorphisms in L.*

(ii) *The serial subgroups and the S-permutable subgroups of G are the same. Hence G is a PST-group.*

During the proof we will make use of an auxiliary result (cf. [11], proposition 2.3.10).

Lemma 1. *Let G be a locally finite group with an abelian normal subgroup L such that G/L is locally nilpotent and $\pi(L)\cap\pi(G/L)=\varnothing$. Assume that elements of G induce power automorphisms in L. If $L \triangleleft G$, then*

$$\mathrm{Syl}_p(G/N) = \{PN/N \mid P \in \mathrm{Syl}_p(G)\}$$

for all primes p.

Proof. We may assume that $O_p(G)=1$, so that $p \notin \pi(L)$. Put $C=C_G(L)$. Then G/C, being a torsion group of power automorphisms of L, is finite. Also, since $C=L\times O_{\pi(C/L)}(C)$ and $O_{\pi(C/L)}(C)$ is locally nilpotent, we see that C is a p'-group. Hence Sylow p-subgroups of G are finite.

Let $P\in \mathrm{Syl}_p(G)$; then PN/N is contained in some $P_1/N \in \mathrm{Syl}_p(G/N)$ and P_1/N is finite. Hence $P_1 = FN$ where F is a finite subgroup containing P. Clearly $|F:P|$ is a p'-number, as must be $|P_1:PN|$. Therefore $P_1=PN$.

Conversely, let $Q/N\in \mathrm{Syl}_p(G/N)$. Since Q/N is finite, $Q=FN$ with F a finite subgroup. Since $F/F\cap N$ is a p-group, $F\le P(F\cap N)$ for some $P\in \mathrm{Syl}_p(G)$. Thus $Q=FN\le PN$ and $Q=PN$.

***Proof of Proposition* 1.** First we show that finite subgroups of G are soluble. Indeed suppose that this is false and let F be a smallest insoluble finite subgroup. Every proper subgroup of F is a soluble PST-group and hence is supersoluble. It now follows from results of Doerk [12] that F is soluble.

Next write

$$L = \bigcup \gamma_\infty(F),$$

where the union is formed over all finite subgroups F of G. By Agrawal's theorem each $\gamma_\infty(F)$ is abelian of odd order. If F_1 and F_2 are finite subgroups, $F=\langle F_1, F_2\rangle$ is a finite soluble PST-group and $\gamma_\infty(F_1)\gamma_\infty(F_2)\subseteq\gamma_\infty(F)\subseteq L$. This implies that L is an abelian subgroup containing no involutions. In addition $L\triangleleft G$ and G/L is locally nilpotent, being the union of all FL/L with F finite.

Let $a\in L_p$ and suppose that $x\in G$ is a q-element where q is a prime different from p. Since $F=\langle x,a\rangle$ is a finite soluble PST-group and $\langle a\rangle$ is subnormal in F, we have $\langle a\rangle\langle x\rangle=\langle x\rangle\langle a\rangle$, which implies that $a^{\langle x\rangle}=\langle a\rangle$ since $a^{\langle x\rangle}\le L_p$. Therefore x induces a power automorphism in G.

Next suppose that $p\in\pi(L)\cap\pi(G/L)$, and let $a\in L$ and $bL\in G/L$ both have order p. Then there is a finite subgroup F such that $b\in F$ and $a\in\gamma_\infty(F)$. Since $\pi(\gamma_\infty(F))\cap\pi(F/\gamma_\infty(F))=\varnothing$, we must have $b^m\in\gamma_\infty(F)\le L$ for some positive p'-number m. But this implies that $b\in L$. Hence $\pi(L)\cap\pi(G/L)=\varnothing$. If now follows from the previous paragraph that arbitrary elements of G induce power automorphisms in L.

It remains to prove that the serial subgroups and the S-permutable subgroups of G are one and the same. Suppose that H is serial in G: we show H is S-permutable. Since $H\cap L\triangleleft G$, it is enough by Lemma 1 to prove that $H/H\cap L$ is S-permutable in $G/H\cap L$: so assume that $H\cap L=1$. Thus H is locally nilpotent and we can suppose it is a q-group. If $q\in\pi(L)$, then $H\le L$ and $H\triangleleft G$, so all is clear. Assume that $q\notin\pi(L)$; we claim that H^G is a q-group. Indeed, if $h\in H$ and $g_1, g_2, \ldots, g_n \in$ $\in G$, then $F\in\langle h, g_1, g_2, \ldots, g_n\rangle$ is finite and $H\cap F$ is a subnormal q-subgroup of F. Hence h^F is a q-group, and it follows that H^G is a q-group, as is claimed.

Let $P\in \mathrm{Syl}_p(G)$. If $p=q$, then $H\le P$, so assume that $p\ne q$. If $p\in\pi(L)$, we shall have $P \leq L$ and $P \triangleleft G$. If on the other hand $p \notin \pi(L)$, then $H^G P \cap L = 1$ since $H^G P$ is a $\{p, q\}$-group. It follows that $H^G P$ is locally nilpotent and $[H, P] = 1$. Hence H is S-permutable in G.

Conversely, assume that H is an S-permutable subgroup of G. We need to show that H is serial. As before we may assume $H \cap L = 1$ and H is a q-group. Now it is sufficient to deal with the case where $\pi(L)$ contains a single prime. For suppose this case has been settled, and let $\pi(L) = \{p_1, p_2, \ldots\}$. By Lemma 1 each $HL_{p_i'}$ is S-permutable, and hence serial, in G, from which it follows that $HL_{\{p_1, \ldots, p_{i+1}\}'}$ is serial in $HL_{\{p_1, \ldots, p_i\}'}$ for $i = 1, 2, \ldots$, since $H \cap L = 1$. Therefore $H = \bigcap_{i=1,2,\ldots} HL_{\{p_1, \ldots, p_i\}'}$ is serial in G.

Now assume that $\pi(L) = \{p\}$. Let $Q \in \mathrm{Syl}_q(G)$ and put $C = C_Q(L)$; then Q/C is a finite q-group. Next QL/L is the unique Sylow q-subgroup of G/L by Lemma 1, so $H \leq QL$. Since $C \triangleleft QL$, it follows that HC is a q-group. Also H is S-permutable, so HQ is a subgroup and $|HQ : HC|$ is a power of q. Consequently HQ is a q-group and $H \leq Q$. It follows that $H \leq O_q(G)$. Since $O_q(G)$ is locally nilpotent, H is serial in G.

3. Finite minimal non-*PST*-groups. By Proposition 1 locally finite, minimal non-*PST*-groups are finite. From now on in the proof of Theorem 1 *all groups will be finite.* Initial insight into the structure of the groups is given by the following lemma.

Lemma 2. *Let G be a minimal non-PST-group. Then G has a nontrivial normal Sylow p-subgroup P for some p. In addition*

(i) *$G = X \ltimes P$ where X is a cyclic group of order $q^r > 1$ and q is a prime different from p;*

(ii) *either P is abelian or $[P, X^q] = 1$.*

Proof. Assume that G has no nontrivial normal Sylow subgroups and let G_1 be a subgroup of G which is minimal with respect to this property. Then G_1 cannot be supersoluble since otherwise there would be a normal Sylow subgroup associated with the largest prime divisor of $|G_1|$.

We claim that G_1 is soluble. If this is not true, G_1 contains a minimal insoluble subgroup G_2. But each proper subgroup of the group G_2 is supersoluble, so G_2 is soluble. It follows that G_2 must be soluble and thus all of its proper subgroups are supersoluble. In short G_1 is a minimal nonsupersoluble group. But by another result of Doerk [12] this implies that G_1 has a nontrivial normal Sylow subgroup, a contradiction which establishes our original claim.

We now have $1 \neq P = O_p(G)$, a Sylow p-subgroup, for some prime p. Hence $G = X \ltimes P$ where X is a p'-group. Suppose that X is not cyclic of prime power order. If $\langle x \rangle$ is a subgroup of X with prime power order, then $\langle x, P \rangle \neq G$, so that $\langle x, P \rangle$ is a soluble *PST*-group. It follows that x induces a power automorphism in P, whence X induces a nontrivial p'-group of power automorphisms in P. By a result of Huppert [13] this means that P is abelian, and hence G is a *PST*-group by Agrawal's theorem. We conclude that $X = \langle x \rangle$ where $|x| = q^r > 1$ and $q \neq p$ is a prime.

Finally, assume that $[P, X^q] \neq 1$. Since $G \neq \langle x^q, P \rangle$, the latter is a *PST*-group and x^q induces a nontrivial p'-power automorphism in P. By [13] again P is abelian.

We can now complete the classification in the case where P is abelian.

Lemma 3. *Let G be a minimal non-PST-group. If G has a nontrivial normal Sylow p-subgroup P which is abelian, then G is of Type* I, II *or* III.

Proof. Write $G = X \ltimes P$ where X is a cyclic q-group. Let H be a proper subgroup of G. Then H is a PST-group, so $L = \gamma_\infty(H)$ is abelian and $\pi(L) \cap \pi(H/L) = \varnothing$. Since $L \le P \cap H$, it follows that either $L = 1$ or $L = P \cap H$. In the first case H is nilpotent and hence abelian. Otherwise H/L is a cyclic q-group and elements of H induce power automorphisms in L, which implies that H is a T-group by a result of Gaschütz [14]. Therefore every proper subgroup of G is a T-group and G is a minimal non-T-group. An examination of the list of minimal non-T-groups in [10] reveals that Types I, II and III are the only possibilities for G.

For the remainder of the proof of Theorem 1 we shall assume that P is *non-abelian*. Also we still have $G = X \ltimes P$, with $X = \langle x \rangle$ of order $q^r > 1$ and $[P, x^q] = 1$. Write

$$\overline{P} = P/\varphi(P)$$

and regard $\overline{P}$ as a $\mathbb{Z}_p X$-module in the obvious way. The structure of this module is critical to the investigation; our aim is to prove that it is simple.

4. Simplicity of the module $\overline{P}$. Assume that $\overline{P}$ is not a simple $\mathbb{Z}_p X$-module. Then by Maschke's Theorem $\overline{P} = \overline{P}_1 \oplus \ldots \oplus \overline{P}_n$ where $\overline{P}_i = P_i/\varphi(P)$ is a simple module and $n \ge 2$. The first step is to show that $n = 2$.

Suppose that $n \ge 3$. Since $[P, X] \ne 1$, there is an i for which $[P_i, X] \ne 1$. For any $j \ne i$ the subgroup $U = XP_iP_j$ is proper since $n \ge 3$. By Agrawal's theorem $T = \gamma_\infty(U)$ is abelian. Also $T \le P$ and $T \ne 1$ since $[P_i, X] \ne 1$. Hence U/T is a p'-group and $P_iP_j \le T$. This shows that each P_j is abelian and that x induces a uniform power automorphism in each P_iP_j; thus $[P_j, X] \ne 1$. If $k \ne j$, the same argument shows that P_jP_k is abelian. However this implies that P is abelian.

Thus far we have shown that $n = 2$ and $\overline{P} = \overline{P}_1 \oplus \overline{P}_2$. Also XP_1 and XP_2 are PST-groups. If $[P_1, x] \ne 1$, then P_1 is abelian and x induces a p'-power automorphism in P_1. Hence $C_{P_1}(x) = 1$, whence $[P_2, x] \ne 1$. It follows that P_2 is abelian and therefore P is nilpotent of class 2. In addition $|\overline{P}| = p$ since x induces a power automorphism in P_1, and for this reason $p \equiv 1 \pmod q$. Thus p is certainly odd.

Next assume that P/P' is not an elementary abelian p-group. Then x induces a p'-power automorphism in $\Omega_1(P/P')$ and hence in P/P'. Let $a \in P$; now $Xa^G \ne G$ since $a^G P'/P'$ is cyclic. Therefore Xa^G is a PST-group and consequently x induces a p'-power automorphism in P, a statement which implies that P is abelian. We conclude that P/P' is elementary, so that $P' = \varphi(P)$ and $|\overline{P}| = p^2$. Since $P' \le Z(P)$, it follows that $|\overline{P}| = p^3$ and $Z(P) = P'$.

Now we show this situation to be impossible. Write $P_1 = \langle a, P' \rangle$ and $P_2 = \langle b, P' \rangle$. Recall that x induces power automorphisms in P_1 and P_2, say $u \mapsto u^l$ and $u \mapsto u^m$ respectively. These must agree on P', so $l \equiv m \pmod p$ and we can assume $l = m$. Hence $[a, b]^l = [a, b]^x = [a^l, b^l]$, which yields $l^2 \equiv l \pmod p$ and $l \equiv 1 \pmod p$. By this contradiction the simplicity of the module $\overline{P}$ is established.

5. Completion of the proof of Theorem 1. Let G be a minimal non-PST-group with $G = X \ltimes P$, $X = \langle x \rangle$ of order $q^r > 1$ and $P = O_p(G)$ non-abelian. It is known that

$$\overline{P} = P/\varphi(P)$$

is a simple $\mathbb{Z}_p X$-module. A sequence of assertions about G will be proved, culminating in the proof of Theorem 1.

(i) *q is odd.*

For if $q = 2$, then $[P, x^2] = 1$ by Lemma 2, which implies that $|\overline{P}| = p$ and P is abelian.

(ii) *P is a special p-group. Also $P = [P, x]$ and $[P', x] = 1$.*

If P/P' is not elementary abelian, x induces a p'-power automorphism in $\Omega_1(P/P^p)$ and hence in P/P'. The simplicity of the module $\overline{P} = P/\varphi(P)$ implies that $\overline{P}$, and hence P, is cyclic. Therefore P/P' must be elementary and

$$\varphi(P) = P'.$$

Since x induces a power automorphism in P', we have $[P', [P, x]] = 1$. In addition, $P = [P, x]P'$ since $\overline{P}$ is a simple $\mathbb{Z}_p X$-module, so $P = [P, x]$. Hence $[P', P] = 1$ and P is nilpotent of class 2. Thus $P' \le Z(P)$, so that $P' = Z(P)$. Clearly P is a special p-group.

If $[P', x] \ne 1$, then x induces a power automorphism in P' of order q. This implies that $p \equiv 1 \pmod q$, which leads to the contradiction $|\overline{P}| = p$. Therefore $[P', x] = 1$.

(iii) *If p is odd, $P^p = 1$. If $p = 2$, then $P^4 = 1$.*

Suppose first that p is odd: then $(ab)^p = a^p b^p$ since P has class 2 and $(P')^p = 1$. Therefore $a \mapsto a^p$ is an X-operator homomorphism $\alpha: P \to P'$ and $P' \le \mathrm{Ker}(\alpha) \lhd G$. Thus $\mathrm{Ker}(\alpha) = P$ or P'. Since $[P', X] = 1$, the latter case cannot occur. Hence $P^p = 1$. If $p = 2$, the same argument for the map $a \mapsto a^4$ shows that $P^4 = 1$.

(iv) *If $p = 2$ and P cannot be generated by involutions, then P is a quaternion group of order 8, $q = 3$ and G is of Type IV.*

Suppose first that $a \in P \setminus P'$ has order 2. Then $P = a^G P'$, whence $P = a^G$ and P is generated by involutions. Hence every element of $P \setminus P'$ has order 4.

Let $a \in P \setminus P'$, so that a has order 4 and $a^2 \in P'$. Observe that $[a, x] \notin P'$ since otherwise $[P, x] \le P'$. Put $U = \langle a, a^x \rangle = \langle a, [a, x] \rangle$. Since $a^2 = (a^2)^x = (a^x)^2$, we have $|U| \le 8$. Also $1 = [a^2, x] = [a, x]^a [a, x]$, so $[a, x]^a = [a, x]^{-1}$. Now $[a, x]^2 \ne 1$ because $[a, x] \notin P'$. Hence U is a quaternion group of order 8 and $[a, x]^2 = a^2 = z$, say.

Next we have

$$[a, a^x] = [a, [a, x] = [a, x]^2 = z,$$

and in the same way $[a, a^{x^2}] = z$. Hence

$$(aa^x a^{x^2})^2 = a^2 (a^x)^2 (a^{x^2})^2 [a^x, a][a^{x^2}, a][a^{x^2} a^x] = z^6 = 1.$$

Therefore $aa^x a^{x^2} \in P'$ and $a^{x^2} \in UP'$. It follows that UP' is X-invariant. Thus $P = UP'$ and $P = U$ is quaternion of order 8. Clearly $q = 3$ and G is of Type IV.

From now on it will be assumed that P is generated by elements of order p.

(v) *$\dim_{\mathbb{Z}_p}(\overline{P})$ is even.*

Suppose this is false and let G be a counterexample of smallest order. Then $|P'| \ne p$ since otherwise P is extra-special, when $\dim(\overline{P})$ is known to be even.

Hence there is a proper nontrivial subgroup Z of P'. Of course $Z \triangleleft G$ since $[P', X] = 1$.

Put $G_1 = G/Z$. If G_1 is a PST-group, x will induce a p'-power automorphism in P/Z, and hence in P/P', which is impossible. Consequently, G_1 is a minimal non-PST-group. Clearly $O_p(G_1) = P/Z$, and $Z(P/Z) = P'/Z$ since otherwise P/Z would be abelian. By minimality of $|G|$ we deduce that $\dim(\overline{P})$ is even.

From now on we shall write

$$\dim(\overline{P}) = 2m.$$

(vi) *The exponent of* p *modulo* q *equals* $2m$.

This is because $\overline{P}$, being a nontrivial simple $\mathbb{Z}_p(X/X^q)$-module, has dimension equal to the exponent of p modulo q.

(vii) $|P'| \le p^m$.

Choose a basis $\{\overline{a}_1, \ldots, \overline{a}_{2m}\}$ of $\overline{P}$ and write $\overline{a}_i = a_i P'$. Then P' is generated by the elements $[a_i, a_j]$, $i < j = 1, 2, \ldots, 2m$. Let M denote the Schur multiplicator of $\overline{P}$. Thus

$$M = \overline{P} \wedge \overline{P},$$

the exterior square, which has the basis $\{\overline{a}_i \wedge \overline{a}_j \mid i < j = 1, 2, \ldots, 2m\}$. Further there is a surjective linear map

$$\theta \colon M \to P'$$

in which $(\overline{a} \wedge \overline{b})\theta = [a, b]$ where $\overline{a} = aP'$ and $\overline{b} = bP'$ are in $\overline{P}$. Now M is an X-module via the diagonal action $(\overline{a} \wedge \overline{b}) \cdot x = \overline{a}^x \wedge \overline{b}^x$, and θ is a $\mathbb{Z}_p X$-module homomorphism. Since P' is a trivial $\mathbb{Z}_p X$-module, $[M, X] \le \mathrm{Ker}(\theta)$. Now M is completely reducible, so

$$M = [M, X] \oplus M^X,$$

and M^X maps homomorphically onto P'. It is therefore sufficient to show that $\dim(M^X) \le m$. This will follow from the following result.

Proposition 2. *The multiplicator* M *of* $\overline{P}$ *is the direct sum of* $m-1$ *nontrivial simple* $\mathbb{Z}_p(X/X^q)$*-modules, each of dimension* $2m$, *and* m *copies of the trivial module* $\mathbb{Z}_p$. *Thus* $\dim(M^X) = m$.

Proof. Let x' denote the linear operator induced in $\overline{P}$ by x and let f be its minimum polynomial. Since $\overline{P}$ is a simple module, f is irreducible and its degree is $2m$. The roots of f, which are all different, are written $d_1, \ldots, d_{2m}$. Let C be the matrix representing x' with respect to some ordered basis of $\overline{P}$. Then C is similar over some extension field to the diagonal matrix

$$D = \begin{bmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ & & \ddots & \\ 0 & 0 & \ldots & d_{2m} \end{bmatrix},$$

say $D = U^{-1}CU$. The linear operator induced in $M = \overline{P} \wedge \overline{P}$ by x' is represented by $C \wedge C$, the exterior square of C. Also

$$(U \wedge U)^{-1}(C \wedge C)(U \wedge U) = (U^{-1}CU) \wedge (U^{-1}CU) = D \wedge D,$$

and $D \wedge D$ is the $\binom{2m}{2}$-square matrix

$$\begin{bmatrix} d_1d_2 & 0 & \dots & 0 \\ 0 & d_2d_3 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & d_{2m-1}d_{2m} \end{bmatrix}.$$

From this it follows that $\dim(M^X)$ equals the number of pairs of inverse roots (d_i, d_j) where $d_j = d_i^{-1}$ and $i < j$. The mapping $u \mapsto u^{p^m}$ is an automorphism of the splitting field of f, so it permutes the roots of f. Since $p^m \equiv -1 \pmod q$, it follows that the roots of f fall into m pairs of inverses. Therefore $\dim(M^X) = m$.

Finally, the number of nontrivial simple summands in the didect decomposition of M into simple modules is

$$\left(\binom{2m}{2} - m\right)\Big/ 2m = m - 1$$

since each nontrivial simple $\mathbb{Z}_p(X / X^q)$-module has dimension $2m$. It now follows that G is of Type V and this completes the proof of Theorem 1.

6. Consrtucting the groups of Type V. We now show to construct explicitly all the minimal non-PST-groups of Type V. Choose distinct primes p and q such that the exponent of p modulo q is even, say $2m$. Let f be an irreducible divisor of the cyclotomic polynomial $\Phi_q \in \mathbb{Z}_p[t]$, and write

$$f = t^{2m} + f_{2m-1}t^{2m-1} + \ldots + f_1 t + f_0.$$

Next form the special p-group

$$P_0 = \left\langle a_0, a_1, \ldots, a_{2m-1} \mid a_i^p = [a_i, a_j, a_k] = 1,\ i, j, k = 1, 2, \ldots, 2m-1 \right\rangle.$$

Thus $P_0' \simeq M(P_0 / P_0')$. Let $X = \langle x \rangle$ have order $q^r > 1$.

At this point a distinction between the cases $p = 2$ and p odd becomes necessery. First let p be odd. Then we can allow x to act on P as an automorphism of order q where

$$a_i^x = a_{i+1},\quad 0 \le i < 2m-1 \quad \text{and} \quad a_{2m-1}^x = a_0^{-f_0} a_1^{-f_1} \ldots a_{2m-1}^{-f_{2m-1}}.$$

Because $P_0^p = 1$, this is automorphism. Note that P_0 / P' is a simple $\mathbb{Z}_p X$-module since f is irreducible. Now form

$$P_1 = P_0 / [P_0', X].$$

Then $P_0' = Z(P_1) = P_0' / [P_0', X]$, which has dimension m by Proposition 2.

Choose a subgroup K such that $[P_0', X] \le K < P_0'$ and put $P = P_0 / K$. Then define

$$G = X \ltimes P,$$

which is a minimal non-PST-group with order

$$\frac{p^{3m}q}{|K:[P_0', X]|}.$$

Now let $p = 2$. While we would like x to act on P_0 by the same rule as above, this may not be an automorphism, so another approach is required. Let P_0 be the free nilpotent group of class 2 with basis $\{a_0, a_1, \ldots, a_{2m-1}\}$. An automorphism x of P_0 is defined by

$$a_i^x = a_{i+1} \quad \text{for} \quad 0 \le i < 2m-1, \qquad a_{2m-1}^x = a_0^{f_0} a_1^{f_1} \ldots a_{2m-1}^{f_{2m-1}}.$$

Now add the relations

$$a_i^2 = 1 \quad \text{and} \quad \left(\left(a_0^{f_0} a_1^{f_1} \ldots a_{2m-1}^{f_{2m-1}}\right)^2\right)^{x^j} = 1,$$

for $i = 0, 1, \ldots, 2m-1$ and $j = 1, 2, \ldots, 2q-1$, to get a special 2-group $\tilde{P}_0$. Choose a subgroup K such that $[\tilde{P}_0', x] \le K < \tilde{P}_0'$. Put $P = \tilde{P}_0 / K$ and let $G = X \ltimes P$ where $X = \langle x \rangle$ has order $q^r > 1$. Then G is a minimal non-PST-group.

7. Minimal non-*PT*-groups. A group G is called a *PT-group* if H permutable in K and K permutable in G always imply that H is permutable in G. For finite groups this is equivalent to all the subnormal subgroups being permutable. The structure of finite PT-groups has been studied intensively, in the soluble case by Zacher [15] and in general by Robinson [9]. The minimal non-PT-groups which are locally finite can be determined with the aid of Theorem 1. The definitive result is the following theorem.

Theorem 2. *The locally finite, minimal non-PT-groups are the minimal non-PST-groups of Types* I – IV, *together with the minimal nonmodular p-groups.*

We remark that the minimal nonmodular p-groups were determined by Napolitani [16] and fall into eleven classes. Two general properties of PT-groups, which may be of independent interest, precede the proof.

Lemma 4. *A group G is a PT-group if and only if every ascendant subgroup is permutable.*

Proof. In the first place, by a theorem of Stonehewer [17] permutable subgroups are always ascendant, and so the sufficiency of the condition follows.

Conversely let G be a PT-group and H an ascendant subgroup of G, with an ascending series

$$H = H_0 \lhd H_1 \lhd \ldots H_\gamma = G.$$

Should H not be permutable in G, there is a least ordinal α for wich H is not permutable in H_α. If α is not a limit ordinal, then H is permutable in $H_{\alpha-1}$ and $H_{\alpha-1} \lhd H_\alpha$. Hence H is permutable in H_α. By this contradiction α is a limit ordinal. Let $x \in H_\alpha$; then $x \in H_\beta$ where $\beta < \alpha$, and H is permutable in H_β. Therefore $H\langle x \rangle = \langle x \rangle H$ and H is permutable in H_α.

Lemma 5. *The property PT is a local property of groups.*

Proof. Suppose that G is locally a PT-group, but it is not a PT-group. By Lemma 4 there is an ascendant subgroup H which is not permutable in G. Let $h \in H$ and $g \in G$. Then there is a PT-subgroup F containing h and g. Now $H \cap F$ is ascendant in F, so Lemma 4 may be applied to show that $H \cap F$ is permutable in F. Hence $hg \in (H \cap F)\langle g \rangle = \langle g \rangle (H \cap F) \subseteq \langle g \rangle H$. It follows that $H\langle g \rangle = \langle g \rangle H$ and H is permutable in G.

Proof of Theorem 2. Let G be a locally finite, minimal non-PT-group. It follows immediately from Lemma 5 that G must be finite. Also G is soluble: for otherwise it has a minimal insoluble subgroup and all its subgroups, being soluble PT-groups, are supersoluble, which is impossible.

If G is not a PST-group, then it is minimal non-PST-group and must be on our list of groups. Type V can be excluded since the subgroup P is not modular. However Types I–IV qualify.

Now assume that G is a PST-group. Then $G = X \ltimes L$ where X is nilpotent, L is abelian, $\pi(L) \cap \pi(X) = \varnothing$, and elements of X induce power automorphisms in L. If $L = 1$, then G is nilpotent and hence is a minimal nonmodular p-group. If $L \neq 1$, then X is modular and G is a PT-group by [15]. The proof of Theorem 2 is now complete.

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