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SOME INVERSE PROBLEMS FOR STRONG PARABOLIC SYSTEMS

ДЕЯКІ ОБЕРНЕНІ ЗАДАЧІ ДЛЯ СИЛЬНО ПАРАБОЛІЧНИХ СИСТЕМ

The questions of correctness and approximate solution of the inverse problems of finding unknown functions on the right-hand side of the system of parabolic equations are investigated in the work. For the considered problems, the theorems on the uniqueness, existence, and stability of solution have been proved and examples, which show the exactness of the established theorems are given.

Moreover, on the set of correctness, the rate of convergence of the method of successive approximation, suggested for approximate solution of the given problems has been estimated.

Досліджено коректність та наближене розв'язуванняобернених задач визначення невідомих функцій у правій частині системи параболічних рівнянь. Для цих задач доведено теореми єдиності, існування та стабільності розв'язкуі наведено приклади, що показують точність встановлених теорем.

Також на множині коректності встановлено оцінку швидкості збіжності методу послідовного наближення, що запропонований для розв'язуванняданих задач.

1. Statement of the problems. Let $D' \subset R^{n-1}$, $D \subset R^n$, and $Q = D' \times (a,b) \subset R^n$ (a, b are some numbers) be bounded domains with boundaries $\partial D'$, ∂D, and $\partial Q \in C^{2+\alpha}$ let $x' = (x_1, \ldots, x_{n-1})$ and $x = (x', x_n)$ be arbitrary points of the domains D' and D or Q, respectively. The spaces $C^{l+\alpha,(l+\alpha)/2}(\cdot)$, $l = 0,1,2,\ 0 < \alpha < 1$, and norms in these spaces were defined in [1, p. 16], $\|v_k\|_l = \sum_{k=1}^{m} \|v_k\|_{C^l}$, $0 < T$, is a given number.

For simplicity, without loss of generality, the following system of parabolic equations is taken as a model:

$$u_{kt} - \Delta u_k = \Phi_k(x,t,u), \quad k = \overline{1,m}, \tag{1}$$
$$(x,t) \in D \times (0,T] \quad ((x,t) \in Q \times (0,T]),$$

where $u_{kt} = \dfrac{\partial u_k(x,t)}{\partial t}$, $\Delta \cdot = \sum_{i=1}^{n} \dfrac{\partial^2 \cdot}{\partial x_i^2}$ is a Laplace operator, $u = (u_1, \ldots, u_m)$,

$$u_k(x,0) = \varphi_k(x), \quad x \in \bar{D} = D \cup \partial D \quad \big(x \in \bar{Q} = Q \cup \partial Q\big), \tag{2}$$

$$u_k(x,t) = \psi_k(x,t), \quad (x,t) \in \partial D \times [0,T] \quad ((x,t) \in \partial Q \times [0,T]). \tag{3}$$

From system (1) – (3) under the corresponding conditions on the input data, it is possible to define exactly or approximately the functions $u_k(x,t)$, $k = \overline{1,m}$. The questions of solvability of problem (1) – (3) in more general statement were considered, for example, in the works [1, 2].

Let the right-hand side of equation (1) contain unknown functions and have one of the following forms: 1) $\Phi_k(\cdot) = f_k(t) g_k(x,t,u)$; 2) $\Phi_k(\cdot) = f_k(x',t)\, g_k(x,t,u)$; 3) $\Phi_k(\cdot) = f_k(u_k)\, g_k(x,t,u)$ ($g_k(x,t,u)$ are the given functions).

Then, in system (1) – (3), it is necessary to join some additional conditions. Depending on the structure of the right-hand side of $\Phi_k(\cdot)$, the following inverse problems are considered:

Problem I. *It is required to define the functions* $\left\{ f_k(t), u_k(x,t), k=\overline{1,m} \right\}$ *from conditions* (1) – (3) *and*

$$u_k\left(x^*,t\right)=p_k(t), \quad x^*\in D \quad \text{is fixed point}, \quad t\in[0,T]. \tag{4}$$

Problem II. *It is required to define the functions* $\left\{ f_k\left(x',t\right), u_k(x,t), k=\overline{1,m} \right\}$ *from conditions* (1) – (3) *and*

$$u_k\left(x',c,t\right)=q_k\left(x',t\right), \quad c\in(a,b) \quad \text{is fixed point,} \quad (x',t)\in D'\times(0,T]. \tag{5}$$

Problem III. *It is required to define the functions* $\left\{ f_k\left(u_k\right), u_k(x,t), k=\overline{1,m} \right\}$ *from conditions* (1) – (3) *and*

$$u_k\left(x^*,t\right)=h(t), \quad x^*\in D \quad \text{is fixed point,} \quad t\in[0,T]. \tag{6}$$

Relative to the input data of Problems I – III, we suppose that:

1^0) $g_k\left(x,t,w\right)\in \mathrm{Lip}_{(\mathrm{loc})}B$, $|g_k(\cdot)|\geq \nu_1>0$, $(x,t,w)\in B$ (in Problems I, III — $B=\bar{D}\times[0,T]\times R^1$, and in Problem II — $B=\bar{Q}\times[0,T]\times R^1$);

2^0) $\varphi_k(x)\in C^{2+\alpha}\left(\bar{D}\right)$, $\psi_k(x,t)\in C^{2+\alpha,1+\alpha/2}\left(\partial D\times[0,T]\right)$, $\varphi_k(x)=\psi_k\left(x,0\right)$, $x\in\partial D$ (in Problem II — $\varphi_k(x)\in C^{2+\alpha}\left(\bar{Q}\right)$, $\psi_k(x,t)\in C^{2+\alpha,1+\alpha/2}\left(\partial Q\times[0,T]\right)$, $\varphi_k(x)=\psi_k\left(x,0\right)$, $x\in\partial Q$);

3^0) $\left[\psi_{kt}\left(x,0\right)-\Delta\varphi_k(x)\right]g_k\left(x,T,\psi\left(x,T\right)\right)=\left[\psi_{kt}(x,T)-\Delta\psi_k(x,T)\right]g_k(x,0,\varphi(x))$, $x\in\partial D$;

4^0) $p_k(t)\in C^{1+\alpha}\left[0,T\right]$, $p_k(0)=\varphi_k\left(x^*\right)$;

5^0) $q_k\left(x',t\right)\in C^{2+\alpha,1+\alpha/2}\left(\bar{D}'\times[0,T]\right)$, $q_k\left(x',0\right)=\varphi_k\left(x',c\right)$, $x'\in\bar{D}'$;

6^0) $h_k(t)\in C^{1+\alpha}\left[0,T\right]$, $h_k(0)=\varphi_k\left(x^*\right)$, $\nu_2\leq h_k(t)\leq\nu_3$, $t\in[0,T]$;

7^0) $\left[\psi_{kt}\left(x,0\right)-\Delta\varphi_k(x)\right]g_k\left(x^*,0,\varphi(x)\right)=\left[p_{kt}(0)-\Delta\varphi_k(x)|_{x=x^*}\right]g_k(x,0,\varphi(x))$, $x\in\partial D$;

8^0) $\left[\psi_{kt}\left(x,0\right)-\Delta\varphi_k(x)\right]g_k\left(x,0,\varphi(x)\right)|_{x_n=c}=\left[q_{kt}(x',0)-\Delta\varphi_k(x)|_{x_n=c}\right]g_k(x,0,\varphi(x))$, $x\in\partial Q$;

9^0) $\left[\psi_{kt}\left(x,0\right)-\Delta\varphi_k(x)\right]g_k\left(x^*,0,h(t)\right)=\left[h_{kt}(0)-\Delta\varphi_k(x)|_{x=x^*}\right]g_k(x,0,\varphi(x))$, $x\in\partial D$.

Definition 1. *The functions* $\left\{ f_k(t), u_k(x,t),\ k=\overline{1,m} \right\}$, *are called the solution of Problem* I, *if*:

1) $f_k(t)\in C\left([0,T]\right)$, 2) $u_k(x,t)\in C^{2,1}\left(\bar{D}\times[0,T]\right)$, 3) *correlations* (1) – (3), (5) *are satisfied.*

Definition 2. *The functions* $\left\{ f_k\left(x',t\right),\ u_k(x,t),\ k=\overline{1,m} \right\}$ *are called the solution of Problem* II, *if:* 1) $f_k\left(x',t\right)\in C\left(\bar{D}'\times[0,T]\right)$, 2) $u_k(x,t)\in C^{2,1}\left(\bar{Q}\times[0,T]\right)$, 3) *correlations* (1) – (3), (6) *are satisfied.*

Definition 3. *The functions* $\left\{ f_k(u_k),\ u_k(x,t),\ k=\overline{1,m} \right\}$ *are called the solution of Problem* III, *if*: 1) $f_k\left(u_k\right)\in C(R^1)$, 2) $u_k(x,t)\in C^{2,1}\left(\bar{D}\times[0,T]\right)$, 3) *correlations* (1) – (3), (7) *are satisfied.*

Consider Problems I – III relatively the class of incorrect Hadamard's problems. The solution of these problems does not always exist, and if it exists, then it can be nonunique and unstable.

The inverse problems of finding the right-hand side of the scalar equation of parabolic type were considered earlier in the works [3 – 13] (see also the bibliography in these works).

2. Uniqueness and estimation of the solutions stability. It is know that the uniqueness theorem and also estimation of solution's stability of the inverse problems take central place in the investigation of their correctness questions [13]. Under the fairly general suppositions, the following theorems of solution's uniqueness of Problems I – III have been proved and the estimations of solution's stability have been established.

Theorem 1. *Let conditions* 1^0), 2^0), 4^0), 7^0) *be fulfilled. Then if the solution of Problem* I *exists and belongs to the set* $K_1 = \{(f_k, u_k,\ k=\overline{1,m}) / f_k(t) \in C^{\alpha}[0,T],$ $u_k(x,t) \in C^{2+\alpha,1+\alpha/2}(\bar{D}\times[0,T])\}$, *then it is unique and the estimation of stability is true:*

$$\|u-\bar{u}\|_0 + \left\|f-\bar{f}\right\|_0 \le M_1\left[\|g-\bar{g}\|_0 + \|\varphi-\bar{\varphi}\|_2 + \left\|\psi-\bar{\psi}\right\|_{2,1} + \|p-\bar{p}\|_1\right],$$

where $M_1 > 0$ *depends on data of Problem* I *and the set* K_1, $\left\{(\bar{f}_k(t), \bar{u}_k(x,t),\ k = \overline{1,m})\right\}$ *is a solution of Problem* I *with data* $\bar{g}_k(\cdot)$, $\bar{\varphi}_k(\cdot)$, $\bar{\psi}_k(\cdot)$, $\bar{p}_k(\cdot)$, *which satisfy conditions* 1^0), 2^0), 4^0), 7^0), *respectively.*

Theorem 2. *Let conditions* 1^0), 2^0), 5^0), 8^0) *be fulfilled. Then if the solution of Problem* II *exists and belongs to the set*

$$K_2 = \Big\{(f_k, u_k,\ k=\overline{1,m}) / f_k(x',t) \in C^{\alpha,\alpha/2}(\bar{D}'\times[0,T]),$$
$$u_k(x,t) \in C^{2+\alpha,1+\alpha/2}(\bar{Q}\times[0,T])\Big\},$$

then it is unique and the estimation of stability is true:

$$\|u-\bar{u}\|_0 + \left\|f-\bar{f}\right\|_0 \le M_2\left[\|g-\bar{g}\|_0 + \|\varphi-\bar{\varphi}\|_2 + \left\|\psi-\bar{\psi}\right\|_{2,1} + \|q-\bar{q}\|_{0,1}\right],$$

where $M_2 > 0$ *depends on data of Problem* II *and the set* K_2, $\{\bar{f}_k(x',t),\ \bar{u}_k(x,t),$ $k = \overline{1,m}\}$ *is a solution of Problem* II *with data* $\bar{g}_k(\cdot)$, $\bar{\varphi}_k(\cdot)$, $\bar{\psi}_k(\cdot)$, $\bar{q}_k(\cdot)$, *which satisfy conditions* 1^0), 2^0, 5^0), 8^0), *respectively.*

Theorem 3. *Let conditions* 1^0), 2^0), 6^0), 9^0) *be fulfilled. Then if the solution of Problem* III *exists and belongs to the set*

$$K_3 = \Big\{(f_k, u_k, k=\overline{1,m})\, | f_k(\cdot) \in C^{\alpha}(R^1),\ \|f\|_{C(R')} \le$$
$$\le \|f\|_{C[\nu_2,\nu_3]},\ u_k(x,t) \in C^{2+\alpha,1+\alpha/2}(\bar{D}\times[0,T])\Big\},$$

then it is unique and the estimation of stability is true:

$$\|u-\bar{u}\|_0 + \left\|f-\bar{f}\right\|_0 \le M_3\left[\|g-\bar{g}\|_0 + \|\varphi-\bar{\varphi}\|_2 + \left\|\psi-\bar{\psi}\right\|_{2,1} + \left\|h-\bar{h}\right\|_1\right], \quad (7)$$

where $M_3 > 0$ *depends on data of Problem* III *and the set* K_3, $\left\{\bar{f}_k(\cdot),\ \bar{u}_k(x,t),\ k = \overline{1,m}\right\}$ *is a solution of Problem* III *with data* $\bar{g}_k(\cdot)$, $\bar{\varphi}_k(\cdot)$, $\bar{\psi}_k(\cdot)$, $\bar{h}_k(\cdot)$, *which satisfy conditions* 1^0), 2^0), 6^0), 9^0), *respectively.*

Theorems 1 – 3 are proved with close method. Let us show the proof of Theorem 3.

Proof. From equation (1) as $x = x^*$ taking into account the conditions of Theorem 3 for the function $f_k(u_k)$, we obtain:

$$f_k(h_k(t)) = [h_{kt}(t) - \Delta u_k|_{x=x^*}]/g_k(x^*, t, h(t)), \qquad (8)$$
$$k = \overline{1, m}, \quad (x, t) \in \Omega = D \times (0, T],$$

Define the function [2, p. 87]

$$\rho_k(x,t) \in C^{2+\alpha, 1+\alpha/2}(\bar{\Omega}), \quad \rho_k(x, 0) = \varphi_k(x), \quad x \in \bar{D},$$
$$\rho_k(x,t) = \psi_k(x,t), \quad k = \overline{1,m}, \quad (x,t) \in S = \partial D \times [0, T]. \qquad (9)$$

Let $z_k(x,t) = u_k(x,t) - \bar{u}_k(x,t)$, $\lambda_k(u_k, \bar{u}_k) = f_k(u_k) - \bar{f}_k(u_k)$, $\delta_{1k}(x,t,u) = g_k(x,t,u) - \bar{g}_k(x,t,u)$, $\delta_{2k}(x) = \varphi_k(x) - \bar{\varphi}_k(x)$, $\delta_{3k}(x,t) = \psi_k(x,t) - \bar{\psi}_k(x,t)$, $\delta_{4k}(x) = h_k(t) - \bar{h}_k(t)$, $\delta_{5k}(x,t) = \rho_k(x,t) - \bar{\rho}_k(x,t)$, $k = \overline{1,m}$.

It is easy to check that the functions $\{\lambda_k(u_k\bar{u}_k), \vartheta_k(x,t) = z_k(x,t) - \delta_{5k}(x,t), k = \overline{1,m}\}$ satisfy the system:

$$\vartheta_{kt} - \Delta\vartheta_k = \lambda_k(u_k, \bar{u}_k)\, g_k(x,t,u) + F_k(x,t), \quad (x,t) \in \Omega, \qquad (10)$$

$$\vartheta_k(x, 0) = 0, \quad x \in \bar{D}; \quad \vartheta_k(x,t) = 0, \quad (x,t) \in S, \qquad (11)$$

$$\lambda_k(h_k, \bar{h}_k) = H_k(t) - \Delta z_k|_{x=x^*}/g_k(x^*, t, h(t)), \quad t \in [0, T], \qquad (12)$$

where

$$F_k(x,t) = \bar{f}_k(\bar{u}_k)[g_k(x,t,u) - g_k(x,t,\bar{u}) + \delta_{1k}(x,t,\bar{u})] - \delta_{5kt}(x,t) + \Delta\delta_{5k},$$
$$H_k(t) = \delta_{4k}(t)/g_k(x^*, t, h) -$$
$$-[\delta_{1k}(x^*, t, \bar{h}) + g_k(x^*, t, h) - g_k(x^*, t, \bar{h})]/\left[\bar{g}_k(x^*, t, h) \cdot g_k(x^*, t, \bar{h})\right].$$

Under the conditions of Theorem 3 and from the definition of the set K_3 it follows that coefficients and right-hand side of equation (10) satisfy the Hölder condition. It means that there exists classical solution of definition problem of $\vartheta_k(x,t)$ from conditions (10), (11) and it can be represented in the form [1, p. 468]

$$\vartheta_k(x,t) = \int\limits_0^t \int\limits_D G_k(x,t;\xi,\tau)\left[\lambda_k(u_k, \bar{u}_k)\, g_k(\xi, \tau, u) + F_k(\xi, \tau)\right] d\xi d\tau, \qquad (13)$$

where $d\xi = d\xi_1 \ldots d\xi_n$, $G_k(x,t;\xi,\tau)$ is Green's function of Problem (10), (11), for which the following estimations [1] (Chapter IV) are true:

$$|G_k(x,t;\xi,\tau)| \le N_1(t-\tau)^{-n/2} \exp\left(-N_2|x-\xi|^2/(t-\tau)\right),$$
$$\int\limits_D \left|D_x^l G_k(x,t;\xi,\tau)\right| d\xi \le N_3(t-\tau)^{-(l-\alpha)/2}, \quad l = 0, 1, 2, \qquad (14)$$

here, $D_x^l \cdot$ are various derivatives in x_i of order l and $N_i > 0$, $i = 1, 2, 3$, depend on data of Problem III.

Taking into account that $\vartheta_k(x,t) = z_k(x,t) - \delta_{5k}(x,t)$, $k = \overline{1,m}$, from (13) we obtain

$$z_k(x,t) = \delta_{5k}(x,t) + \int\limits_0^t \int\limits_D G_k\left(x,t;\xi,\tau\right)\left[\lambda_k\left(u_k,\bar{u}_k\right)g_k\left(\xi,\tau,u\right) + F_k\left(\xi,\tau\right)\right]d\xi d\tau. \tag{15}$$

Assume that

$$æ = \|u-\bar{u}\|_0 + \left\|f-\bar{f}\right\|_0.$$

Under the conditions of theorem and from definition of the set K_3, taking into account estimation (14), we obtain

$$|z_k(x,t)| \le M_4\left[\|\delta_5\|_{2,1} + \|\delta_1\|_0\right] + M_5 æ\, t, \quad (x,t)\in\bar{\Omega}, \tag{16}$$

$$\left|\lambda_k\left(h_k,\bar{h}_k\right)\right| \le M_6\left[\|\delta_1\|_0 + \|\delta_4\|_1 + \|\delta_5\|_{2,1}\right] + M_7 æ\, t^{\alpha/2}, \quad t\in[0,T]. \tag{17}$$

Inequalities (16), (17) are satisfied for any values $(x,t)\in\bar{\Omega}$. Therefore, they must be also satisfied for maximum values of the left parts.

Consequently,

$$æ \le M_8\left[\|\delta_1\|_0 + \|\delta_4\|_1 + \|\delta_5\|_{2,1}\right] + M_9 æ\, t^{\alpha/2}. \tag{18}$$

Let T_1, $0 < T_1 \le T$, be such number that $M_9 T_1^{\alpha/2} < 1$. Then we obtain from (17) that if $(x,t)\in\bar{D}\times[0,T_1]$, then the estimation of stability (18) for solution of Problem III is true.

By induction method, we show that estimation (18) is true for all $t\in[0,T]$.

So, it is proved that the estimation of stability (7) holds for all $(x,t)\in\bar{D}\times[0,T]$. The uniqueness of the solution of Problem III follows from estimation (7) if $g_k\left(x,t,w\right) = \bar{g}_k\left(x,t,w\right)$, $\varphi_k(x) = \bar{\varphi}_k(x)$, $\psi_k(x,t) = \bar{\psi}_k(x,t)$, $h_k(t) = \bar{h}_k(t)$.

So, Theorem 3 is completely proved.

3. The method of successive approximations. The method of successive approximations is applied for approximate solution of the considered inverse problems.

The method of successive approximations with reference to Problem I consists of the following:

Let $\left\{f_k^{(s)}(t), u_k^{(s)}(x,t),\, k=\overline{1,m}\right\}\in K_1$ be already found. Consider the problem on definition of $u_k^{(s+1)}(x,t)$, $k=\overline{1,m}$, from the conditions

$$u_{kt}^{(s+1)} - \Delta u_k^{(s+1)} = f_k^{(s)}(t)g_k\left(x,t,u^{(s)}\right), \quad (x,t)\in\Omega = D\times(0,T], \tag{19}$$

$$u_k^{(s+1)}\left(x,0\right) = \varphi_k(x), \quad x\in\bar{D}; \quad u_k^{(s+1)}(x,t) = \psi_k(x,t), \tag{20}$$

$$(x,t)\in S = \partial D\times(0,T].$$

This problem has a unique classical solution (if input data will satisfy conditions 1^0), 2^0), 4^0), 7^0)) belonging to $C^{2+\alpha,1+\alpha/2}\left(\bar{\Omega}\right)$ [1, p. 364].

Then, under the functions $u_k^{(s+1)}(x,t)$, $k=\overline{1,m}$, from the condition

$$f_k^{(s+1)}(t) = \left[p_{kt}(t) - \Delta u^{(s+1)}|_{x=x^*}\right]/g_k\left(x,t,u^{(s+1)}\right)|_{x=x^*}, \quad t\in[0,T], \tag{21}$$

$f_k^{(s+1)}(t)\in C^\alpha[0,T]$, $k=\overline{1,m}$, are defined and these functions are used for the next step of iteration. So, if we choose $f_k^{(0)}(t)\in C^\alpha[0,T]$, $u_k^{(0)}(x,t)\in C^{2+\alpha,1+\alpha/2}\left(\bar{Q}\right)$,

$k = \overline{1,m}$, from system (19) – (21) for $s = 0, 1, 2, \ldots$, we consequently find the functions $f_k^{(s)}(t) \in C^{\alpha}[0,T]$, $u_k^{(s)}(x,t) \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega})$, $k = \overline{1,m}$.

Let us show uniform boundedness of the sequences $\left\{f_k^{(s)}(t)\right\}$ and $\left\{u_k^{(s)}(x,t)\right\}$, $k = \overline{1,m}$, which we need below.

Lemma 1. *Let conditions* $1^0), 2^0), 4^0), 7^0)$ *be fulfilled. Then if* $f_k^{(0)}(t) \in C^{\alpha}[0,T]$, $u_k^{(0)} \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega})$, *then the functions* $\left\{f_k^{(s)}(t),\ u_k^{(s)}(x,t),\ k = \overline{1,m}\right\}$ *found from system* (19) – (21) *for* $s = 1, 2, \ldots$ *are uniformly bounded* (*by sup norm*) *at* $(x,t) \in \bar{\Omega}$.

Proof. As stated above, if we choose $\left\{f_k^{(0)}(t),\ u_k^{(0)}(x,t),\ k = \overline{1,m}\right\} \in K_1$, then under the conditions of Lemma 1 and by virtue of statement of the theorem proved in [1, p. 364] it follows that $\left\{f_k^{(s)}(t), u_k^{(s)}(x,t),\ k = \overline{1,m}\right\} \in K_1$ for any $s = 1, 2, \ldots$. Then, by Green's function [1, p. 468], we find the expressions for solution of the problem of definition of $u_k^{(s+1)}(x,t)$, $k = \overline{1,m}$, form (19), (20):

$$u_k^{(s+1)}(x,t) = \rho_k(x,t) +$$

$$+ \int_0^t \int_D G_k(x,t;\xi,\tau)\left[f_k^{(s)}(\tau)\, g_k\left(\xi,\tau,u^{(s)}\right) - \rho_{k\tau}(\xi,\tau) + \Delta\rho_k\right] d\xi d\tau, \quad (22)$$

where $\rho_k(x,t)$ is defined in (9), $G_k(x,t;\xi,\tau)$ is Green's function of problem (19), (20) for which estimations (14) are true.

Taking into account estimations (14) and the conditions of Lemma 1, from (22) and (21) we obtain

$$\left|D_x^l u_k^{(s+1)}(x,t)\right| \le N_4 \|\rho\|_{2,1} + N_5 \left|f_k^{(s)}(t)\right| t^{(2+\alpha-l)/2},$$

$$k = \overline{1,m}, \quad l = 0, 1, 2, \quad (x,t) \in \bar{\Omega},$$

$$\left|f_k^{(s+1)}(t)\right| \le N_6 \|p\|_1 + N_7 \left|D_x^2 u_k^{(s+1)}(x^*,t)\right| t^{\alpha/2}, \quad t \in [0,T],$$

or

$$\gamma^{(s+1)} \le N_8\left[\|\rho\|_{2,1} + \|p\|_1\right] + N_9 t^{\alpha/2}\gamma^{(0)},$$

where

$$\gamma^{(s)} = \sum_{l=0}^{2}\left\|D_x^l u^{(s)}\right\|_0 + \left\|f^{(s)}\right\|_0.$$

From the last inequality, we have

$$\gamma^{(s+1)} \le N_{10}\left[\|\rho\|_{2,1} + \|p\|_1\right](1-\sigma^s)/(1-\sigma) + \sigma^s\gamma^{(0)}, \quad \sigma = N_9 t^{\alpha/2}.$$

Let T_2, $0 < T_2 \le T$, be such a number that $N_9 T_2^{\alpha/2} < 1$. Then we obtain that the sequences $\left\{f_k^{(s)}(t)\right\}$, $\left\{D_x^l u_k^{(s)}(x,t)\right\}$, $l = 0, 1, 2$, $k = \overline{1,m}$, are uniformly (on sup norm) bounded if $(x,t) \in \bar{D} \times [0, T_2]$.

Considering problem (19) – (20) is turn for the intervals $(T_2, 2T_2)$, $(2T_2, 3T_2)$ and etc., for finite number of steps we shall obtain the uniform boundedness of the sequences $\left\{f_k^{(s)}(t)\right\}$, $\left\{D_x^l u_k^{(s)}(x,t)\right\}$, $l = 0, 1, 2$, $k = \overline{1,m}$, for all $(x,t) \in \bar{D} \times [0,T]$.

Theorem 4. *Let*: 1) *conditions* $1^0), 2^0), 4^0), 7^0)$ *be fulfilled*, 2) *Problem* I *have a unique solution belonging to the set* K_1. *Then the functions* $\left\{f_k^{(s)}(t),\ u_k^{(s)}(x,t),\ k =\right.$

$= \overline{1, m}\,\}$, *found from system* (19) – (21) *uniformly tend to the solution of Problem* I *with rate of geometric progression.*

Proof. From equation (1) with $x = x^*$, taking into account the conditions of Theorem 4 for the functions $f_k(t)$, we obtain

$$f_k(t) = [p_{kt}(t) - \Delta u_k|_{x=x^*}] / g_k(x^*, t, p(t)), \quad t \in [0, T]. \tag{23}$$

Subtracting from the correlations of system (1), (2), (23) the corresponding correlations of system (19) – (21), we obtain that the functions

$$\left\{\lambda_k^{(s)}(t) = f_k(t) - f_k^{(s)}(t), z_k^{(s)}(x,t) = u_k(x,t) - u_k^{(s)}(x,t), \quad k = \overline{1, m}\right\}$$

satisfy the conditions of the system

$$z_{kt}^{(s+1)} - \Delta z_k^{(s+1)} = \lambda_k^{(s)}(t) g_k(x,t,u) + f_k^{(s)}(t)\left[g_k(x,t,u) - g_k\left(x,t,u^{(s)}\right)^{(s)}\right], \tag{24}$$

$$(x,t) \in \Omega,$$

$$z_k^{(s+1)}(x,0) = 0, \quad x \in \bar{D}; \quad z_k^{(s+1)}(x,t) = 0, \quad (x,t) \in S, \tag{25}$$

$$\lambda_k^{(s+1)}(t) = -\Delta z_k^{(s+1)}|_{x=x^*} / g_k(x^*, t, p(t)) + [\left(p_{kt}(t) - \Delta u^{(s+1)}|_{x=x^*}\right) \times$$

$$\times (g_k\left(x^*, t, u^{(s+1)}(x^*, t)\right) - g_k(x^*, t, p(t)))]/$$

$$g_k\left(x^*, t, u^{(s+1)}(x^*, t)\right) g_k(x^*, t, p(t)). \tag{26}$$

It follows from the assumptions of Theorem 4 and statement of Lemma 1, that the right-hand side of (24) belongs to the class $C^{\alpha,\alpha/2}\left(\bar{\Omega}\right)$ and it is uniformly bounded. Therefore, there exists classical solution of problem of definition of $z_k^{(s+1)}(x,t)$ from conditions (24), (25) and it can be represented in the form [1, p. 468]

$$z_k^{(s+1)}(x,t) = \int_0^t \int_D G_k(x,t;\xi,\tau) \times$$

$$\times \left[\lambda_k^{(s)}(\tau) g_k(\xi,\tau,u) + f^{(s)}(\tau)\left(g_k(\xi,\tau,u) - g_k\left(\xi,\tau,u^{(s)}\right)\right)\right] d\xi d\tau. \tag{27}$$

For Green's function of problem (24), (25) $G_k(x,t;\xi,\tau)$, estimations (14) are true.

Acting as in the proof of Lemma 1, we obtain

$$\left|D_x^l z_k^{(s+1)}(x,t)\right| \le N_{11}\left[\left|\lambda_k^{(s)}(t)\right| + \left|z_k^{(s)}(x,t)\right|\right] t^{(2+\alpha-l)/2},$$

$$l = 0, 1, 2, \quad (x,t) \in \bar{\Omega},$$

$$\left|\lambda_k^{(s+1)}(t)\right| \le N_{12}\left[\left|z_k^{(s+1)}(x^*,t)\right| + \left|D_x^2 z_k^{(s+1)}(x^*,t)\right|\right] t^{\alpha/2}, \quad t \in [0,T],$$

or

$$æ^{(s+1)} \le N_{13} æ^{(s)} t^{\alpha/2},$$

where $æ^{(s)} = \sum_{l=0}^{2}\left\|D_x^l z^{(s)}\right\|_0 + \left\|\lambda^{(s)}\right\|_0$.

Applying the last inequality, we obtain

$$æ^{(s+1)} \le \theta^{s+1} æ^{(0)}, \quad \theta = N_{13} t^{\alpha/2}. \tag{28}$$

Let now T_3, $0 < T_3 \leq T$, be such a number that $N_{13}T_3^{\alpha/2} < 1$. Then it follows that the sequence $\left\{æ^{(s)}\right\}$ is majorized by decreased geometric progression. Acting as indicated above, we obtain that inequality (28) is true for all $t \in [0,T]$. It means that $æ^{(s)} \to 0$ as $s \to \infty$ not slower than geometric progression.

So, we obtain that the functions $\left\{f_k^{(s)}(t), u_k^{(s)}(x,t),\ k = \overline{1,m}\right\}$ found from (19)–(21) uniformly tend to the solution of problem (1)–(4) as $s \to \infty$ with rate of convergence, which is not slower than rate of convergence of the geometric progression.

As in Theorem 4, the following convergence theorems on the method of successive approximations used in Problems II are proved.

Theorem 5. *Let*: 1) *conditions* 1^0), 2^0), 5^0), 8^0) *be fulfilled*; 2) *Problem* II *have a unique solution belonging to the set* K_2. *Then the functions* $\left\{f_k^{(s)}(x',t),\ u_k^{(s)}(x,t),\ k = \overline{1,m}\right\}$ *found from the system*

$$u_{kt}^{(s+1)} - \Delta u_k^{(s+1)} = f_k^{(s)}(x',t)\, g_k\left(x,t,u^{(s)}\right), \quad (x,t) \in Q \times (0,T], \tag{29}$$

$$\begin{gathered} u_k^{(s+1)}(x,0) = \varphi_k(x), \quad x \in \bar{Q}; \\ u_k^{(s+1)}(x,t) = \psi_k(x,t), \quad (x,t) \in \partial Q \times [0,T], \end{gathered} \tag{30}$$

$$\begin{gathered} f_k^{(s+1)}(x',t) = \left[q_{kt}(x',t) - \Delta u^{(s+1)}|_{x=c}\right] / g\left(x,t,u^{(s+1)}\right)|_{x_n=c}, \\ (x',t) \in \bar{D}' \times [0,T], \end{gathered} \tag{31}$$

uniformly tend to the solution of Problem II *with rate of geometric progression.*

4. Existence of solution. The existence of solution of Problems I, II is proved by the method of successive approximations used in Section 3.

Theorem 6. *Let conditions* 1^0), 2^0), 4^0), 7^0) *be fulfilled. Then Problem* I *has at least one solution in the sense of Definition* 1.

Theorem 7. *Let conditions* 1^0), 2^0), 5^0), 8^0) *be fulfilled. Then Problem* II *has at least one solution in the sense of Definition* 2.

Theorems 6, 7 are proved with close method. Let prove below Theorem 7.

Proof. Note that if we choose $f_k^{(0)}(x',t) \in C^{\alpha,\alpha/2}\left(\bar{D}' \times [0,T]\right)$, $u_k^{(0)}(x,t) \in C^{2+\alpha,1+\alpha/2}\left(\bar{Q} \times [0,T]\right)$, $k = \overline{1,m}$, then, under the conditions of Theorem 7, $u^{(s)}(x,t) \in C^{2+\alpha,1+\alpha/2}\left(\bar{Q} \times [0,T]\right)$ for all $s = 1,2,\ldots$ [1, p. 364]. Then, under the conditions of Theorem 7, it follows from (29) that $f_k^{(s)}(x',t) \in C^{\alpha,\alpha/2}\left(\bar{D}' \times [0,T]\right)$, $k = \overline{1,m}$. Using the functions $\rho_k(x,t)$, $k = \overline{1,m}$, defined in (9) and representations of solution through Green's function [1, p. 468], let us find the expressions for solution of problem of definition $u_k^{(s+1)}(x,t)$ from conditions (29), (30):

$$u_k^{(s+1)}(x,t) = \rho_k(x,t) + \int_0^t \int_Q G_k(x,t;\xi,\tau)\, F_k^{(s)}(\xi,\tau)\, d\xi d\tau,$$

where $F_k^{(s)}(x,t) = f_k^{(s)}(x',t)\, g_k\left(x,t,u^{(s)}\right) - \rho_{kt} + \Delta\rho_k$.

By analogy with Lemma 1, we prove the following lemma:

Lemma 2. *Let the conditions of Theorem* 7 *be fulfilled. Then the sequences* $\left\{f_k^{(s)}(x',t)\right\}$, $\left\{D_x^l u_k^{(s)}(x,t)\right\}$, $l = 0,1,2$, $k = \overline{1,m}$, *are uniformly bounded* (*on sup norm*) *if* $(x,t) \in \bar{Q} \times [0,T]$.

The equipotential continuity of the sequences $\left\{D_x^l u_k^{(s)}(x,t)\right\}$, $l = 0,1,2$, $k = \overline{1,m}$, follows from the inequality

$$\left|D_x^l u_k^{(s+1)}(x,t) - D_x^l u_k^{(s)}\left(\bar{x},\bar{t}\right)\right| \le \left|D_x^l u_k^{(s+1)}(x,t) - D_x^l u_k^{(s+1)}\left(\bar{x},t\right)\right| +$$

$$+\left|D_x^l u_k^{(s+1)}\left(\bar{x},t\right) - D_x^l u_k^{(s+1)}\left(\bar{x},\bar{t}\right)\right| \le \left|D_x^l \rho_k(x,t) - D_x^l \rho_k\left(\bar{x},t\right)\right| +$$

$$+\left|D_x^l \rho_k\left(\bar{x},t\right) - D_x^l \rho_k\left(\bar{x},\bar{t}\right)\right| + \int\limits_0^t \int\limits_Q \left|D_x^l G_k\left(x,t;\xi,\tau\right) - D_x^l G_k\left(\bar{x},t;\xi,\tau\right)\right| \times$$

$$\times\left|F_k^{(s)}\left(\xi,\tau\right)\right| d\xi d\tau + \int\limits_0^{\bar{t}} \int\limits_Q \left|D_x^l G_k\left(\bar{x},t;\xi,\tau\right) - D_x^l G_k\left(\bar{x},\bar{t};\xi,\tau\right)\right| \times$$

$$\times\left|F_k^{(s)}\left(\xi,\tau\right)\right| d\xi d\tau + \int\limits_{\bar{t}}^t \int\limits_Q \left|D_x^l G_k\left(\bar{x},t;\xi,\tau\right)\right| \left|F_k^{(s)}\left(\xi,\tau\right)\right| d\xi d\tau$$

taking into account the uniform boundedness of $\left\{f_k^{(s)}\left(x',t\right)\right\}$, the continuity and boundedness of input data, estimations (14), and the following [1, p. 469] relations:

$$\left|D_x^l G\left(x,t;\xi,\tau\right) - D_x^l G\left(\bar{x},t;\xi,\tau\right)\right| \le$$

$$\le N_{14}\left|x-\bar{x}\right|^{\alpha}\left|t-\tau\right|^{-(n+2+\alpha)/2} \exp\left(-N_{15}\left|x-\xi\right|^2/\left(t-\tau\right)\right),$$

$$\left|D_x^l G\left(x,t;\xi,\tau\right) - D_x^l G\left(\bar{x},\bar{t};\xi,\tau\right)\right| \le$$

$$\le N_{16}\left|t-\bar{t}\right|^{(2+\alpha-l)/2}\left(\bar{t}-\tau\right)^{-(n+2+\alpha)/2} \exp\left(-N_{17}\left|x-\xi\right|^2/\left(t-\tau\right)\right).$$

The equipotential continuity of the sequence $\left\{f_k^{(s)}\left(x',t\right)\right\}$, $k=\overline{1,m}$, follows from the inequality

$$\left|f_k^{(s)}(x',t) - f_k^{(s)}(\bar{x}',\bar{t})\right| \le$$

$$\le \left|f_k^{(s)}(x',t) - f_k^{(s)}(\bar{x}',t)\right| + \left|f_k^{(s)}(\bar{x}',t) - f_k^{(s)}(\bar{x}',\bar{t})\right| \le$$

$$\le \left[\left|q_{kt}(x',t) - q_{kt}(\bar{x}',t)\right| + \left|\Delta u_k^{(s+1)}(x,t)\right|_{x_n=c} -\right.$$

$$\left.-\left|\Delta u_k^{(s+1)}(\bar{x},t)\right|_{\bar{x}_n=c}\right] / \left|g_k(x,t,u^{(s)})|_{x_n=c}\right| +$$

$$+\left|g_k(x,t,u^{(s)})|_{x_n=c} - g_k(\bar{x},t,u^{(s)})|_{\bar{x}_n=c}\right|\left(\left|q_{kt}(\bar{x}',t)\right| +\right.$$

$$\left.+\left|\Delta u_k^{(s+1)}(\bar{x},t)\big|_{\bar{x}_n=c}\right|\right) / \left|g_k(x,t,u^{(s)})\big|_{x_n=c} g_k(\bar{x},t,u^{(s)})\big|_{\bar{x}_n=c}\right| +$$

$$+\left[\left|q_{kt}(\bar{x}',t) - q_{kt}\left(\bar{x}',\bar{t}\right)\right| +\right.$$

$$\left.+\left|\Delta u_k^{(s)}(\bar{x},t)\big|_{\bar{x}_n=c} - \Delta u_k^{(s)}(\bar{x},\bar{t})\big|_{\bar{x}_n=c}\right|\right] / \left|g_k(\bar{x},t,u^{(s)})\big|_{\bar{x}_n=c}\right| +$$

$$+\left|g_k(\bar{x},t,u^{(s)})\big|_{\bar{x}_n=c} - g_k(\bar{x},\bar{t},u^{(s)})\big|_{\bar{x}_n=c}\right| \times$$

$$\times\Big(\left|q_{kt}(\bar{x}',\bar{t})\right| + \left|\Delta u_k^{(s+1)}(\bar{x},\bar{t})\right|_{\bar{x}_n=c}\Big)\Big/\Big|q_k(\bar{x},t,u^{(s)})\big|_{\bar{x}_n=c}\,g_k(\bar{x},\bar{t},u^{(s)})\big|_{\bar{x}_n=c}\Big|,$$

taking into account the uniform boundedness and equipotential continuity of the sequence $\left\{D_x^l u_k^{(s)}(x,t)\right\}$, $l=0,1,2,$ the continuity and boundedness of input datas.

The uniform boundedness and equipotential continuity of the sequence $\left\{u_{kt}^{(s)}(x,t)\right\}$ follows from (29).

By Arcela's theorem [2, p. 84], from the sequences $\left\{u_{kt}^{(s)}\right\}$, $\left\{D_x^l u_k^{(s)}\right\}$, $\left\{f_k^{(s)}\right\}$, $l=0,1,2, k=\overline{1,m}$, it is possible to choose sequences convergent to some functions $\{u_{kt}^*\}$, $\left\{D_x^l u_k^*\right\}$, $l=0,1,2,$ $\{f_k^*\}$, respectively, and $u_k^*(x,t) \in C^{2,1}\left(\bar{Q}\times[0,T]\right)$, $f_k^*(x',t) \in C\left(\bar{D}'\times[0,T]\right)$.

Then, passing to the limit as $s\to\infty$ in correlations (29) – (31), we obtain relations

$$u_{kt}^* - \Delta u_k^* = f_k^*(x',t)\,g_k(x,t,u^*), \quad (x,t)\in Q\times(0,T],$$

$$u_k^*(x,0) = \varphi_k(x), \quad x\in\bar{Q} \quad u_k^*(x,t) = \psi_k(x,t), \quad (x,t)\in\partial Q\times[0,T],$$

$$f_k^*(x',t) = \left[q_{kt}(x',t) - \Delta u_k^*|_{x_n=c}\right]/g_k(x,t,u^*)\,|_{x_n=c}, \quad (x',t)\in\bar{D}'\times[0,T].$$

This implies that

$$u_{kt}^* - \Delta u_k^* = g_k(x,t,u^*)\left[q_{kt}(x',t) - \Delta u_k^*|_{x_n=c}\right]/g_k(x,t,u^*)\,|_{x_n=c}.$$

By using the last equality with $x_n=c$ and taking into account conditions $\varphi_k(x',c,0)=q(x',0)$, we obtain $u_k^*(x',c,t)=q_k(x',t)$.

Thus, the existence of solution of Problem II is proved in terms of Definition 2.

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