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GENERALIZED MOMENT REPRESENTATIONS AND PADÉ APPROXIMANTS ASSOCIATED WITH BILINEAR TRANSFORMATIONS

УЗАГАЛЬНЕНІ МОМЕНТНІ ЗОБРАЖЕННЯ ТА АПРОКСИМАЦІЇ ПАДЕ, ПОВ'ЯЗАНІ З ДРОБОВО-ЛІНІЙНИМИ ПЕРЕТВОРЕННЯМИ

By using the method of generalized moment representations with an operator of bilinear transformation of an independent variable, we construct elements of the first subdiagonal of the Padé table for certain special power series.

З використанням методу узагальнених моментних зображень з оператором дробово-лінійного перетворення незалежної змінної побудовано елементи першої піддіагоналі таблиці Паде для деяких спеціальних степеневих рядів.

1^0. Introduction. V. K. Dzyadyk [1] in 1981 proposed the method of generalized moment representations allowing one to construct and to investigate rational Padé approximants for a number of elementary and special functions.

Definition 1. *We shall call by generalized moment representation of the numerical sequence* $\{s_k\}_{k=0}^{\infty}$ *on the product of linear spaces* $\mathcal{X}$ *and* $\mathcal{Y}$ *the two-parameter collection of equalities*

$$s_{k+j} = \langle x_k, y_j \rangle, \quad k,\ j = \overline{0,\infty}, \tag{1}$$

where $x_k \in \mathcal{X}$, $k = \overline{0,\infty}$, $y_j \in \mathcal{Y}$, $j = \overline{0,\infty}$, *and* $\langle \cdot, \cdot \rangle$ *— bilinear form defined on* $\mathcal{X} \times \times \mathcal{Y}$.

In the case when a linear operator $A: \mathcal{X} \to \mathcal{Y}$ exists such that

$$Ax_k = x_{k+1}, \quad k = \overline{0,\infty},$$

and in the space $\mathcal{Y}$ a linear operator $A^*: \mathcal{Y} \to \mathcal{Y}$ exists such that

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \quad \forall x \in \mathcal{X}, \quad \forall y \in \mathcal{Y}$$

(we shall call operator A^* as adjoint to operator A with respect to bilinear form $\langle \cdot, \cdot \rangle$), the representation (1) as it was shown in [2] is equivalent to the representation

$$s_k = \langle A^k x_0, y_0 \rangle, \quad k = \overline{0,\infty}. \tag{2}$$

In this paper the representation of the form (2) will be considered with operator A defined by bilinear transformation of independent variable.

Let us introduce some necessary definitions. We shall denote by $\mathcal{R}[M/N]$ a class of rational functions with numerators of degree $\le M$ and denominators of degree $\le N$

$$\mathcal{R}[M/N] = \left\{ r(z) = \frac{p(z)}{q(z)},\ \deg p(z) \le M,\ \deg p(z) \le N \right\}.$$

Definition 2 [3] (Part 1, Chapter 1, Paragraph B). *We shall call by Padé approximant of the order* $[M/N]$, $M, N = \overline{0,\infty}$, *for power series*

$$f(z) = \sum_{k=0}^{\infty} s_k z^k$$

the rational function

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$$[M/N]_f(z) = \frac{P_M(z)}{Q_N(z)} \in \mathcal{R}[M/N]$$

such that

$$f(z) - [M/N]_f(z) = O(z^{M+N+1})$$

in the neighbourhood of $z = 0$.

2⁰. Compositions of bilinear transformations. Let as consider for some $\gamma \in (0, +\infty) \setminus \{1\}$ bilinear transformation

$$\sigma(t) = \frac{t}{(1-\gamma)t + \gamma}.$$

It is easily seen that transformation σ maps the segment $[0, 1]$ onto itself, and in addition $\sigma(0) = 0$ as well $\sigma(1) = 1$. Let us define in the space $\mathcal{X} = C[0, 1]$ of continuous on $[0, 1]$ functions linear bounded operator

$$(A\varphi) = \varphi(\sigma(t)) = \varphi\left(\frac{t}{(1-\gamma)t + \gamma}\right).$$

It is simple to calculate its powers

$$(A^k\varphi) = \varphi\left(\frac{t}{(1-\gamma^k)t + \gamma^k}\right).$$

Let us assume for some $\delta \in (0, +\infty) \setminus \{1\}$

$$x_0(t) = \frac{t}{(1-\delta)t + \delta},$$

and construct a system of functions

$$x_k(t) = (A^k x_0)(t) = \frac{t}{(1-\delta\gamma^k)t + \delta\gamma^k}, \quad k = \overline{0,\infty}. \tag{3}$$

For arbitrary system of points

$$0 < t_0 < t_1 < \ldots < t_N < 1, \quad N = \overline{0,\infty},$$

let us consider determinants

$$\Delta_N = \Delta_N(t_0, t_1, \ldots, t_N) = \det \| x_k(t_j) \|_{k,j=0}^N = \det \left\| \frac{t}{(1-\delta\gamma^k)t_j + \delta\gamma^k} \right\|_{k,j=0}^N =$$

$$= \prod_{j=0}^N t_j \prod_{k=0}^N \frac{1}{1-\delta\gamma^k} \det \left\| \frac{1}{t_j + \kappa_k} \right\|_{k,j=0}^N,$$

where $\kappa_k = \dfrac{\delta\gamma^k}{1-\delta\gamma^k}$, $k = \overline{0,N}$, $N = \overline{0,\infty}$. The last determinant is determinant of Cauchy matrix (see [4], Chapter I, §3, Example 4) which is equal

$$\det \left\| \frac{1}{t_j + \kappa_k} \right\|_{k,j=0}^N = \frac{\prod_{j<k}(t_k - t_j)(\kappa_k - \kappa_j)}{\prod_{j,k}(t_j + \kappa_k)}.$$

Because as easily seen $\kappa_k \neq \kappa_j$ for $k \neq j$ then last determinant as well as determinant Δ_N is different from zero, hence, system of functions $\{x_k(t)\}_{k=0}^N$ for any $N = \overline{0,\infty}$ is Tchebycheff on segment $[0, 1]$ (see [4], Chapter I, § 1, Definition 1.1).

Let us consider on the product of spaces $\mathscr{X} \times \mathscr{X}$ bilinear form

$$\langle x, y \rangle = \int_0^1 x(t) y(t)\, dt. \tag{4}$$

Simple calculations give expressions for the powers of operator A^* adjoint to operator A with respect to bilinear form (4)

$$(A^*\psi)(t) = \frac{\gamma}{(1-(1-\gamma)t)^2}\, \psi\left(\frac{\gamma t}{1-(1-\gamma)t}\right).$$

Let us assume now that $y_0(t) \equiv 1$, and construct system of functions

$$y_j(t) = (A^{*j} y_0)(t) = \frac{\gamma^j}{(1-(1-\gamma^j)t)^2}. \tag{5}$$

Let us verify that system of functions (5) is also Tchebycheff. It is easily seen that

$$\frac{d^m}{dt^m} y_j(t) = \frac{(m+1)!\, \gamma^j (1-\gamma^j)^m}{(1-(1-\gamma^j)t)^{m+2}}.$$

Therefore Wronskian of system of functions (5) will have a form

$$W_N = \det\left\| \frac{d^m}{dt^m} y_j(t) \right\|_{j,m=0}^{N} = \det\left\| \frac{(m+1)!\, \gamma^j (1-\gamma^j)^m}{(1-(1-\gamma^j)t)^{m+2}} \right\|_{j,m=0}^{N} =$$

$$= \prod_{m=0}^{N} (m+1)! \prod_{j=0}^{N} \gamma^j \prod_{j=0}^{N} \frac{\gamma^j}{(1-(1-\gamma^j)t)^2} \det\left\| \frac{1}{\left(1/(1-\gamma^j)-t\right)^m} \right\|_{j,m=0}^{N}.$$

The last determinant is Vandermonde determimant (see [4], Chapter I, § 1)

$$\det\left\| \frac{1}{\left(1/(1-\gamma^j)-t\right)^m} \right\|_{j,m=0}^{N} = \prod_{k<j} \left(\frac{1}{1/(1-\gamma^k)-t} - \frac{1}{1/(1-\gamma^j)-t} \right) =$$

$$= \prod_{k<j} \frac{\gamma^j - \gamma^k}{(1-t(1-\gamma^k))(1-t(1-\gamma^j))} \neq 0.$$

It implies that system of functions (5) is Tchebycheff on $[0, 1]$ for any $N = \overline{0,\infty}$ (see [4], Chapter XI, § 1, Theorem 1.1).

3⁰. Generalized moment representations associated with bilinear transformations. Previous considerations may be summarized in the following results.

Theorem 1. *For the sequence*

$$s_k = \frac{t_0}{(1-\delta\gamma^k)t_0 + \delta\gamma^k}, \quad k = \overline{0,\infty},$$

where $\gamma \in (0, \infty) \setminus \{1\}$, $\delta \in (0, \infty) \setminus \{1\}$, $t_0 \in (0, 1)$ *the generalized moment representation holds in Banach space* $\mathscr{X} = C[0, 1]$

$$s_{k+j} = y_j(x_k), \quad k, j = \overline{0,\infty},$$

where

$$x_k(t) = \frac{t}{(1-\delta\gamma^k)t + \delta\gamma^k}, \quad k = \overline{0,\infty},$$

and functionals $y_j(x)$, $j = \overline{0,\infty}$, *are defined by formulae*

$$y_j(x) = x(t_j) = x\left(\frac{t_0}{(1-\gamma^j)t_0 + \gamma^j}\right), \quad j = \overline{0,\infty}. \tag{6}$$

Theorem 2. *For the sequence*

$$s_k = \frac{1}{1-\delta\gamma^k} + \frac{(\ln\delta + k\ln\gamma)\delta\gamma^k}{(1-\delta\gamma^k)^2}, \quad k = \overline{0,\infty},$$

where $\gamma \in (0,\infty)\setminus\{1\}$, $\delta \in (0,\infty)\setminus\{1\}$, *the generalized moment representation holds on the product of spaces* $C[0,1]\times C[0,1]$

$$s_{k+j} = \int_0^1 \frac{t}{(1-\delta\gamma^k)t + \delta\gamma^k}\,\frac{\gamma^j}{(1-(1-\gamma^j)t)^2}\,dt, \quad k, j = \overline{0,\infty}.$$

4⁰. Applications to Padé approximants. Using the main result by V. K. Dzyadyk [1] on application of generalized moment representations to the problem of Padé approximation we can receive the following results.

Theorem 3. *Padé approximants for the power series*

$$f(z) = \sum_{k=0}^{\infty} \frac{t_0 z^k}{(1-\delta\gamma^k)t_0 + \delta\gamma^k},$$

where $\gamma \in (0,\infty)\setminus\{1\}$, $\delta \in (0,\infty)\setminus\{1\}$, $t_0 \in (0,1)$ *of orders* $[N-1/N]$, $N \geq 1$, *exist and are nondegenerate and may be represented in the form*

$$[N-1/N]_f(z) = \frac{P_{N-1}(z)}{Q_N(z)},$$

where

$$Q_N(z) = \sum_{k=0}^{N} c_k^{(N)} z^{N-k},$$

$$P_{N-1}(z) = \sum_{m=0}^{N-1} z^m \sum_{k=0}^{N} c_{N-k}^{(N)} \frac{t_0}{(1-\delta\gamma^{m-k})t_0 + \gamma^{m-k}},$$

and $c_k^{(N)}$, $k = \overline{0,N}$, *— coefficients of biorthogonal polynomial*

$$Y_N = \sum_{j=0}^{N} c_j^{(N)} y_j,$$

defined by the relations

$$Y_N(x_k) = 0, \quad k = \overline{0,N-1}$$

(functions $x_k(t)$, $k = \overline{0,\infty}$, *are defined by formulae* (3), *and functionals* y_j, $j = \overline{0,\infty}$, *are defined by formulae* (6)).

Theorem 4. *Padé approximants for the power series*

$$f(z) = \sum_{k=0}^{\infty} \left\{ \frac{1}{1-\delta\gamma^k} + \frac{(\ln\delta + k\ln\gamma)\delta\gamma^k}{(1-\delta\gamma^k)^2} \right\} z^k,$$

where $\gamma \in (0, \infty) \setminus \{1\}$, $\delta \in (0, \infty) \setminus \{1\}$, *of orders* $[N-1/N]$, $N \geq 1$, *exist and are nondegenerate and may be represented in the form*

$$[N-1/N]_f(z) = \frac{P_{N-1}(z)}{Q_N(z)},$$

where

$$Q_N(z) = \sum_{k=0}^{N} c_k^{(N)} z^{N-k},$$

$$P_{N-1}(z) = \sum_{j=0}^{N-1} z^j \sum_{m=0}^{j} c_{N-m}^{(N)} \left\{ \frac{1}{1-\delta\gamma^{j-m}} + \frac{(\ln\delta + (j-m)\ln\gamma)\delta\gamma^{j-m}}{(1-\delta\gamma^{j-m})^2} \right\},$$

and $c_k^{(N)}$, $k = \overline{0, N}$, *are coefficients of generalized polynomial*

$$X_N(t) = \sum_{k=0}^{N} c_k^{(N)} \frac{t}{(1-\delta\gamma^k)t + \delta\gamma^k},$$

for which biorthogonality conditions

$$\int_0^1 X_N(t) \frac{\gamma^j}{(1-(1-\gamma^j)t)^2} dt = 0, \quad j = \overline{0, N-1},$$

are satisfied.

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